Consistency in House Allocation Problems

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Abstract

In *house allocation problems*, we look for a systematic way of assigning a set of indivisible objects, e.g. houses, to a group of individuals having preferences over these objects. Typical real life examples are graduate housing, assignment of offices and tasks. Once an allocation is decided upon, the actual assignments of the agents are not likely to take place simultaneously. Therefore, rules whose predictions are independent of the sequence in which the actual assignments are realized turn out to be very appealing. We model this property via the *consistency* principle and identify various classes of *consistent* rules and correspondences.

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1 Introduction

A *house allocation problem* is a one-sided matching problem, where a set of agents collectively own a set of indivisible goods, e.g. houses, and every agent has strict preferences over these indivisible goods. The number of agents and the number of houses are assumed to be finite and equal. An allocation is an assignment of the houses to the agents, such that each agent receives exactly one house. Assignment of dormitory rooms or offices at the beginning of the academic year are examples of house allocation problems.

The house allocation model is closely related to the *housing markets* introduced by Shapley and Scarf (1974). The only difference between the two classes is that, in the latter, each agent owns one house, whereas in the former, houses are owned collectively. The housing markets have been thoroughly investigated and many strong results have been obtained concerning the core (competitive) correspondence. Roth and Postlewaite (1977) show that the core correspondence is *singlevalued* and Roth (1982) shows that it is *strategyproof.* Ma (1994) shows that the core correspondence is the only correspondence that is *Pareto optimal*, *individually rational* and *strategyproof.* Abdulkadiroğlu and Sönmez (1998) introduce the core from random endowments as a lottery mechanism for house allocation problems. They show that the core from random endowments is equivalent to random serial dictatorship, which formally establishes the close relationship between the two models[1].

In the context of house allocation problems, a correspondence is a map that chooses a set of allocations for each problem. A rule is a *singlevalued* correspondence. In this paper, we identify various classes of *consistent* and *conversely consistent* correspondences. Informally, *consistency* requires that, if an allocation is chosen for a problem, then for any subgroup of agents, the restriction of that allocation should be chosen for the smaller problem consisting of that subgroup and their original assignments. *Consistent* rules are coherent in their suggestions for problems involving different groups of agents. For example, in n-person bargaining problems, a rule that selects the egalitarian outcome when n equals 2 and a dictatorial outcome when n is greater than 2, is quite implausible because it is not *consistent.* The *consistency*

[1] Also see Zhou (1991), Svensson (1997) and Bogomolnaia and Moulin (1999) for more exposition to the house allocation and the housing market models.

principle has been analyzed in many contexts, such as game theory, public finance, and fair allocation.[2] As we illustrate in the next paragraph, *consistent* rules also have a very practical appeal in classes of resource allocation problems where individuals are likely to receive their material allocations sequentially. Examples of such classes are two-sided matching, rationing and house allocation problems. In economies with indivisible goods and money, Tadenuma and Thomson (1991) identify the correspondences that satisfy *no-envy* and variants of *consistency*, *neutrality*, and *converse consistency*. In a large class of two-sided matching problems, Sasaki and Toda (1992) show that the stable correspondence (the core) is the only correspondence that satisfies *Pareto optimality*, *anonymity*, *consistency*, and *converse consistency*. Moulin (1999) investigates *consistent* rules in the context of rationing problems.

In the house allocation model, *consistency* requires that once an allocation is chosen and a group of agents take their assigned houses before the others, the allocation rule should not change the assignments of the remaining agents in the reduced problem involving the remaining agents and houses. For example, suppose that a rule assigning dormitory rooms to students is not *consistent*. Then, if some students occupy their rooms before the others, the rule may require a change in the assignments of the remaining students! Such a change would not only impose operational and transactional costs, but it would also lead the agents and the authorities to question the plausibility of the rule. *Consistent* rules are robust to non-simultaneous allocations of the houses. Therefore, we believe that *consistent* rules are more likely to emerge than *'inconsistent'* rules.

In a problem where every agent has the same preferences over the houses, *every allocation discriminates between agents*. Indeed, there is a one-to-one correspondence between allocations and priority orderings over the set of individuals, illustrating the impossibility of equal treatment of equals in this class of problems. For this reason, "sequential solutions" and "serial dictatorships" constitute a powerful class of rules when randomization or monetary compensations are not allowed. Given an exogenous priority ordering which may for example be based on seniority, a serial dictatorship rule sequentially assigns every agent his most preferred house while respecting earlier assignments. Sequential solutions are a more general class

[2] A comprehensive survey of *consistency* for resource allocation problems can be found in Thomson (1996).

of rules where certain agents receive their least preferred house when their turn comes. Simple sequential solutions are *consistent*, *conversely consistent*, and *neutral*. In Theorem 1, we show that simple sequential solutions are the only rules that satisfy a weak form of *consistency* and a weak form of *neutrality*, namely *pairwise consistency* and *pairwise neutrality*. Simple serial dictatorships are *Pareto optimal*, *strategyproof*, *consistent*, *conversely consistent*, and *neutral*. In Corollary 1, we show that simple serial dictatorships are the only rules that are *weakly Pareto optimal*, *pairwise consistent*, and *pairwise neutral*. Besides its descriptive nature, Theorem 1 can be interpreted as a negative finding, since dropping *efficiency* does not allow us to recover rules having other properties of normative interest. Then, we drop *singlevaluedness*. In Proposition 5, we show that *anonymous* correspondences are not very appealing even in the *multivalued* case. In Corollary 2, we characterize *Pareto optimal*, *consistent*, and *conversely consistent* correspondences via their behavior in two-person problems. Finally, in Theorem 2, we show that a correspondence is *nonempty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* if and only if it can be written as a union of serial dictatorships in a particular manner. Precise definitions of the above concepts are provided in the next section. The third section contains the results and the fourth section presents the concluding remarks. The independence of axioms and part of proofs are deferred to the appendix.

2 Environments

Let $\mathcal{N}$ be a ***set of potential agents*** and $\mathcal{H}$ a ***set of potential houses*** such that $|\mathcal{N}| \geq 3$ and $|\mathcal{H}| \geq 3$. A house allocation problem or simply a ***problem*** is a triplet $\mathcal{E} = (N, H, (R_i)_{i \in N})$ where $\emptyset \neq N \subset \mathcal{N}$, $\emptyset \neq H \subset \mathcal{H}$, $|N| = |H|$ is finite, and for each $i \in N$, R_i is a linear order on H representing agent i's preference over the houses in H.[3] For each $i \in N$, P_i denotes the asymmetric part of R_i.[4]

Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, an ***allocation*** $\mu \colon N \to H$ is a bijection, where $\mu(i)$

[3]A binary relation R_i on H is a ***linear order*** if it is ***reflexive*** ($\forall a \in H : aR_ia$), ***complete*** ($\forall a, b \in H : a \neq b \Longrightarrow aR_ib$ or bR_ia), ***transitive*** ($\forall a, b, c \in H : aR_ib$ and $bR_ic \Longrightarrow aR_ic$) and ***antisymmetric*** ($\forall a, b \in H : aR_ib$ and $bR_ia \Longrightarrow a = b$). Indifference between different houses is not allowed.

[4]For any $a, b \in H$, we say aP_ib if and only if aR_ib and not bR_ia. In general, a relation P_i on H is ***asymmetric*** if for any $a, b \in H$, aP_ib implies not bP_ia.

denotes the house assigned to agent i.

Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be any problem, μ any allocation for $\mathcal{E}$ and $i, j \in N$ any two agents. We say that ***i envies j under*** μ if $\mu(j)\ P_i\ \mu(i)$.

An allocation correspondence, or simply a ***correspondence***, is a map φ which associates with each problem a possibly empty set of allocations for that problem. An allocation rule, or simply a ***rule***, is a map φ which associates with each problem *exactly one* allocation for that problem. A rule is a *singlevalued* correspondence.

Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, an allocation μ' for $\mathcal{E}$ ***weakly Pareto dominates*** another allocation μ for $\mathcal{E}$ if every agent in N is weakly better off and at least one agent is strictly better off under μ' than under μ. The allocation μ' ***strongly Pareto dominates*** μ for $\mathcal{E}$ if every agent in N is strictly better off under μ' than under μ. The ***Pareto correspondence*** associates with each problem the set of allocations that are not weakly Pareto dominated. The ***weak Pareto correspondence*** associates with each problem the set of allocations that are not strongly Pareto dominated. A correspondence is ***Pareto optimal*** if it never chooses allocations that are weakly Pareto dominated. Similarly, a correspondence is ***weakly Pareto optimal*** if it never chooses allocations that are strongly Pareto dominated.

Abdulkadiroğlu and Sönmez (1998) show that serial dictatorships lead to *Pareto optimal* allocations. Serial dictatorships can be considered as the *Pareto optimal* subclass of a more general class of rules that we call sequential solutions. Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, a linear order $\succeq$ on N and a subset $M \subset N$, the ***sequential allocation induced by*** $\succeq$ ***and M for*** $\mathcal{E}$ is defined inductively as follows. Let i^k be the k^{th} person from the top in N w.r.t. $\succeq$. First, if $i^1 \in M$, then i^1 is allocated his top-ranked house in H, otherwise i^1 is allocated his bottom-ranked house in H. At the k^{th} step, if $i^k \in M$, then i^k is allocated his top-ranked house among those that are not already allocated in earlier steps, otherwise i^k is allocated his bottom-ranked house among the remaining ones. The set M identifies the set of agents whose welfares are maximized by the sequential solution. Let i^n be the bottom-ranked person in N w.r.t. $\succeq$. Note that the sequential allocation induced by $\succeq$ and M will be the same, whether $i^n \in M$ or not. Moreover, if $M \supset N \setminus \{i^n\}$, then the above sequential solution corresponds with the serial dictatorship induced by $\succeq$. Formally, given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$

and a linear order $\succeq$ on N, the ***serial dictatorship allocation induced by*** $\succeq$ ***for*** $\mathcal{E}$ is the sequential allocation induced by $\succeq$ and N for $\mathcal{E}$. Conversely, a sequential allocation coincides with the serial dictatorship allocation where the preferences of the agents in $N \setminus M$ are turned upside-down.

We next introduce natural extensions of sequential solutions to the variable population case. For any linear order $\succeq$ on $\mathcal{N}$ and any $\emptyset \neq N \subset \mathcal{N}$, let $\succeq|_N$ be the restriction of $\succeq$ to N. A rule is a ***simple sequential solution*** if there exists a linear order $\succeq$ on $\mathcal{N}$ and a subset $\mathcal{M} \subset \mathcal{N}$ such that for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, the rule selects the sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$. In this case, the rule is denoted by $\varphi^{\succeq,\mathcal{M}}$.[5] A rule is a ***simple serial dictatorship*** if it coincides with $\varphi^{\succeq,\mathcal{N}}$ for some linear order $\succeq$ on $\mathcal{N}$. For simplicity, we will denote such a rule by $\varphi^{\succeq}$.

For any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $\emptyset \neq N' \subset N$ and any allocation μ for $\mathcal{E}$, the ***reduced problem of*** $\mathcal{E}$ ***w.r.t.*** N' ***at*** μ is:

$$r^{\mu}_{N'}(\mathcal{E}) = \Big(N', \mu(N'), \big(R_i|_{\mu(N')}\big)_{i \in N'}\Big)$$

where $\mu(N')$ is the set of remaining houses after the agents in $N \setminus N'$ have left with their assigned houses, and $R_i|_{\mu(N')}$ is the restriction of agent i's preference to the remaining houses. The ***reduced allocation w.r.t.*** N', $\mu_{N'} \colon N' \to \mu(N')$ is the bijection defined by $\mu_{N'}(i) = \mu(i)$, for each $i \in N'$.

A correspondence φ is ***consistent*** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $\emptyset \neq N' \subset N$ and any $\mu \in \varphi(\mathcal{E})$, one has $\mu_{N'} \in \varphi(r^{\mu}_{N'}(\mathcal{E}))$. Note that the union of *consistent* correspondences is *consistent*. A correspondence φ is ***pairwise consistent*** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $N' \subset N$ with $|N'| = 2$ and any $\mu \in \varphi(\mathcal{E})$, one has $\mu_{N'} \in \varphi(r^{\mu}_{N'}(\mathcal{E}))$. It is ***conversely consistent*** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ with $|N| \geq 2$ and any allocation μ for $\mathcal{E}$ such that for any $N' \subset N$ with $|N'| = 2$ we have $\mu_{N'} \in \varphi(r^{\mu}_{N'}(\mathcal{E}))$, we have $\mu \in \varphi(\mathcal{E})$.[6] By changing set memberships to equalities, one obtains the definitions of *consistency*, *pairwise consistency* and *converse consistency* for rules.

[5] Since agents have strict preferences, given a linear order $\succeq$, a subset $\mathcal{M} \subset \mathcal{N}$ and a problem $\mathcal{E}$, there *exists a unique* sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$ for $\mathcal{E}$. Therefore, $\varphi^{\succeq,\mathcal{M}}$ is well defined as a rule.

[6] An alternative definition of *converse consistency* would require that if for any proper subset N' of N with $|N'| \geq 2$, one has $\mu_{N'} \in \varphi(r^{\mu}_{N'}(\mathcal{E}))$ then $\mu \in \varphi(\mathcal{E})$. These two definitions turn out to be equivalent in

Anonymity requires that a correspondence should be independent of the names of the agents. More precisely, a correspondence φ is ***anonymous*** if for any $\emptyset \neq H \subset \mathcal{H}$, any two problems $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N', H, (R'_i)_{i \in N'})$ and any $\mu \in \varphi(\mathcal{E})$, *if* $\pi : N \to N'$ is a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad aR_i b \iff aR'_{\pi(i)} b,$$

then $\mu \circ \pi^{-1} \in \varphi(\mathcal{E}')$.

Neutrality requires that a correspondence should be independent of the particular labeling of the houses. More precisely, a correspondence φ is ***neutral*** if for any $\emptyset \neq N \subset \mathcal{N}$, any two problems $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N, H', (R'_i)_{i \in N})$ and any $\mu \in \varphi(\mathcal{E})$, *if* $\pi : H \to H'$ is a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad aR_i b \iff \pi(a) R'_i \pi(b),$$

then $\pi \circ \mu \in \varphi(\mathcal{E}')$. By changing the quantifier "for any $\emptyset \neq N \subset \mathcal{N}$" to "for any $N \subset \mathcal{N}$ with $|N| = 2$", we obtain the definition of ***pairwise neutrality***.

3 Results

Thomson (1998) points out that the Pareto correspondence is *consistent* in allocation problems where goods are privately appropriable. We start by noting this in our special context of house allocation problems.

Proposition 1 *The Pareto correspondence is consistent.*

The following proposition asserts that the Pareto correspondence is not *conversely consistent*. It is analogous to Tadenuma and Thomson's (1991) result about the lack of *converse consistency* of the Pareto correspondence in economies with indivisible goods and money.

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the context of house allocation problems. It is straightforward to check the equivalence of these definitions via induction on the number of players, by using the following two ***transitivity*** properties of reduction: if $\emptyset \neq N'' \subset N' \subset N$, $\mathcal{E} = (N, H, (R_i)_{i \in N})$ is a problem and μ is an allocation for $\mathcal{E}$ then $r^{\mu_{N'}}_{N''}\left(r^{\mu}_{N'}(\mathcal{E})\right) = r^{\mu}_{N''}(\mathcal{E})$ and $(\mu_{N'})_{N''} = \mu_{N''}$. By using the same properties, one can also show that *pairwise consistency* and *converse consistency* imply *consistency*. The latter statement is a direct consequence of Lemma 2. Thomson (1996) points out that in any class of allocation problems where admissible problems involve finitely many agents and reduction is *transitive*, the two forms of *converse consistency* are equivalent.

Proposition 2 *The Pareto correspondence is not conversely consistent. The weak Pareto correspondence is neither pairwise consistent nor conversely consistent.*

Proof Let 1, 2, 3 be three distinct potential agents and a, b, c three distinct potential houses. To see that the Pareto correspondence is not *conversely consistent*, consider the following problem $\mathcal{E}$:

P_1	P_2	P_3
a	c	b
$\underline{b}$	$\underline{a}$	$\underline{c}$
c	b	a

Let μ be the allocation corresponding to the underlined selection. Note that for any $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$, the allocation $\mu_{\{i,j\}}$ is chosen by the Pareto correspondence in the reduced problem $r^{\mu}_{\{i,j\}}(\mathcal{E})$. Thus, if the Pareto correspondence were *conversely consistent*, then μ should be in the set of *Pareto optimal* allocations for $\mathcal{E}$. However, μ is strongly and therefore weakly Pareto dominated in $\mathcal{E}$, so it is not in the Pareto correspondence for $\mathcal{E}$, showing that the Pareto correspondence is not *conversely consistent*. The same example shows that the weak Pareto correspondence is not *conversely consistent*.

To see that the weak Pareto correspondence is not *pairwise consistent*, consider the following problem $\mathcal{E}$:

P_1	P_2	P_3
$\underline{a}$	c	b
b	$\underline{b}$	$\underline{c}$
c	a	a

Let μ be the allocation corresponding to the underlined selection. The allocation μ is not strongly Pareto dominated in $\mathcal{E}$, therefore it is in the weak Pareto correspondence for $\mathcal{E}$. However, the reduced allocation $\mu_{\{2,3\}}$ is strongly Pareto dominated in the reduced problem $r^{\mu}_{\{2,3\}}(\mathcal{E})$, thus $\mu_{\{2,3\}}$ is not in the weak Pareto correspondence for $r^{\mu}_{\{2,3\}}(\mathcal{E})$, showing that the weak Pareto correspondence is not *pairwise consistent*. □

For any $i \in \mathcal{N}$, any linear order $\succeq$ on $\mathcal{N}$ and any $\emptyset \neq N \subset \mathcal{N}$, let $L(i, \succeq, N) = \{j \in N | i \succeq j\}$. A property that *characterizes* serial dictatorships in the context of assignment problems is that an agent never envies those who are ranked below him in the serial dictatorship order. This idea is generalized to sequential solutions in the following lemma.

Lemma 1 *Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $M \subset N$ and let $\succeq$ be a linear order on N. An allocation μ for $\mathcal{E}$ is the* ***sequential allocation induced by $\succeq$ and M for $\mathcal{E}$*** *if and only if the following are true:*

1. *$\mu(i)\ R_i\ \mu(j)$ for any $i \in N \cap M$ and any $j \in L(i, \succeq, N)$,*
2. *$\mu(j)\ R_i\ \mu(i)$ for any $i \in N \setminus M$ and any $j \in L(i, \succeq, N)$.*

Proof Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $M \subset N$ and let $\succeq$ be a linear order on N.

First, assume that μ is the sequential allocation induced by $\succeq$ and M for $\mathcal{E}$. Let $i \in N$ and $j \in L(i, \succeq, N)$. Since $i \succeq j$, j does not come before i in the sequential solution order. Therefore, $\mu(j)$ is not previously allocated at the step when i receives his house. If $i \in M$, then $\mu(i)$ is the top-ranked house among the remaining ones w.r.t. R_i, at the step when i's assignment is made. In particular, $\mu(i)\ R_i\ \mu(j)$. Similarly, if $i \notin M$, then $\mu(i)$ is the bottom-ranked house among the remaining ones w.r.t. R_i, at the step when i's assignment is made. In particular, $\mu(j)\ R_i\ \mu(i)$.

For the converse, assume that the allocation μ for $\mathcal{E}$ is such that Conditions 1 and 2 are satisfied. For each $k \in \{1, \ldots, |N|\}$, let $i^k \in N$ be the k^{th} person from the top in N w.r.t. $\succeq$. Let $k \in \{1, \ldots, |N|\}$ and $a \in \{\mu(i^k), \mu(i^{k+1}), \ldots, \mu(i^{|N|})\}$. Then $a = \mu(j)$ for some $j \in L(i^k, \succeq, N)$. Therefore, if $i^k \in M$, we have $\mu(i^k)\ R_{i^k}\ a$ by Condition 1, and $\mu(i^k)$ is the top-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \ldots, \mu(i^{|N|})\}$ w.r.t. R_{i^k}. Similarly, if $i^k \notin M$, by Condition 2, we have that $a\ R_{i^k}\ \mu(i^k)$, and $\mu(i^k)$ is the bottom-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \ldots, \mu(i^{|N|})\}$ w.r.t. R_{i^k}. So, initially, if $i^1 \in M$, then i^1 receives his top-ranked house in H. Otherwise, he receives his bottom-ranked house in H. At the k^{th} step, if $i^k \in M$, then i^k receives his top-ranked house among those that are not already allocated in earlier steps, otherwise i^k receives his bottom-ranked house among the remaining ones. Therefore, μ is the sequential allocation induced by $\succeq$ and M for $\mathcal{E}$. $\square$

Proposition 3 *Simple sequential solutions are consistent, conversely consistent, and neutral.*

We omit the proof of Proposition 3 since it is straightforward using Lemma 1.

Proposition 4 *Simple serial dictatorships are Pareto optimal, consistent, conversely consistent, and neutral.*

Proof By Abdulkadiroğlu and Sönmez (1998), serial dictatorships lead to *Pareto optimal* allocations. Therefore, simple serial dictatorships are *Pareto optimal.* The other claims directly follow from Proposition 3. □

Theorem 1 *If a rule is pairwise consistent and pairwise neutral, then it is a simple sequential solution.*

Proof Let φ be any *pairwise consistent* and *pairwise neutral* rule. Let $a, b \in \mathcal{H}$ be two distinct houses and $i, j \in \mathcal{N}$ two distinct agents. Let the problems $\mathcal{E}^1$ and $\mathcal{E}^2$ be as follows:

$\mathcal{E}^1$

P_i^1	P_j^1
a	a
b	b

$\mathcal{E}^2$

P_i^2	P_j^2
a	b
b	a

Depending on the values that φ takes (the underlined selections below) in the problems $\mathcal{E}^1$ and $\mathcal{E}^2$, exactly one of the following four cases prevails:

•*Case 1:* $\bar{i} \succeq j$

P_i^1	P_j^1	P_i^2	P_j^2
$\underline{a}$	a	$\underline{a}$	$\underline{b}$
b	$\underline{b}$	b	a

•*Case 2:* $\underline{i} \succeq j$

P_i^1	P_j^1	P_i^2	P_j^2
a	$\underline{a}$	a	b
$\underline{b}$	b	$\underline{b}$	$\underline{a}$

•*Case 3:* $\bar{j} \succeq i$

P_i^1	P_j^1	P_i^2	P_j^2
a	$\underline{a}$	$\underline{a}$	$\underline{b}$
$\underline{b}$	b	b	a

•*Case 4:* $\underline{j} \succeq i$

P_i^1	P_j^1	P_i^2	P_j^2
$\underline{a}$	a	a	b
b	$\underline{b}$	$\underline{b}$	$\underline{a}$

By the *pairwise neutrality* assumption, the four cases above are independent of the choice of houses a and b. Therefore, we have:

•*Case 1:* $\bar{i} \succeq j$ *In any problem* $\mathcal{E}$ *involving* i *and* j, i *does not envy* j *under* $\varphi(\mathcal{E})$.

Indeed, suppose that there exists a problem $\mathcal{E} = (N, H, (R_i)_{i\in N})$ involving i and j, such that i envies j under $\mu = \varphi(\mathcal{E})$. Then, $\mu(j)\ P_i\ \mu(i)$, i.e. $\mu_{\{i,j\}}(j)\ P_i|_{\mu(\{i,j\})}\ \mu_{\{i,j\}}(i)$. In conjunction with the *pairwise consistency* of φ, this implies that one of the following cases prevails in the reduced problem $r^{\mu}_{\{i,j\}}(\mathcal{E})$:

$P_i\|_{\mu(\{i,j\})}$	$P_j\|_{\mu(\{i,j\})}$
$\mu_{\{i,j\}}(j)$	$\underline{\mu_{\{i,j\}}(j)}$
$\underline{\mu_{\{i,j\}}(i)}$	$\mu_{\{i,j\}}(i)$

$P_i\|_{\mu(\{i,j\})}$	$P_j\|_{\mu(\{i,j\})}$
$\mu_{\{i,j\}}(j)$	$\mu_{\{i,j\}}(i)$
$\underline{\mu_{\{i,j\}}(i)}$	$\underline{\mu_{\{i,j\}}(j)}$

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where the underlined allocations represent φ's selection for the reduced problem. In either case, we obtain a contradiction to $\bar{i} \succeq j$ by setting $a = \mu_{\{i,j\}}(j)$ and $b = \mu_{\{i,j\}}(i)$.

• *Case 2: $\underline{i} \succeq j$ In any problem $\mathcal{E}$ involving i and j, i envies j under $\varphi(\mathcal{E})$.*

Suppose that there exists a problem $\mathcal{E} = (N, H, (R_i)_{i\in N})$ involving i and j, such that i does not envy j under $\mu = \varphi(\mathcal{E})$. Then, $\mu(i)\ P_i\ \mu(j)$, i.e. $\mu_{\{i,j\}}(i)\ P_i|_{\mu(\{i,j\})}\ \mu_{\{i,j\}}(j)$. By *pairwise consistency* of φ, one of the following cases prevails in the reduced problem $r^{\mu}_{\{i,j\}}(\mathcal{E})$:

$P_i\|_{\mu(\{i,j\})}$	$P_j\|_{\mu(\{i,j\})}$
$\underline{\mu_{\{i,j\}}(i)}$	$\mu_{\{i,j\}}(i)$
$\mu_{\{i,j\}}(j)$	$\underline{\mu_{\{i,j\}}(j)}$

$P_i\|_{\mu(\{i,j\})}$	$P_j\|_{\mu(\{i,j\})}$
$\underline{\mu_{\{i,j\}}(i)}$	$\underline{\mu_{\{i,j\}}(j)}$
$\mu_{\{i,j\}}(j)$	$\mu_{\{i,j\}}(i)$

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where the underlined allocations represent φ's selection for the reduced problem. In either case, we obtain a contradiction to $\underline{i} \succeq j$ by setting $a = \mu_{\{i,j\}}(i)$ and $b = \mu_{\{i,j\}}(j)$.

The two other cases are exactly symmetric. Since the four cases considered are independent of the choice of houses a and b, we can define a *reflexive* relation $\succeq$ on $\mathcal{N}$ by letting $i \succeq j$ if and only if $\bar{i} \succeq j$ or $\underline{i} \succeq j$, for any two distinct $i, j \in \mathcal{N}$. The relation $\succeq$ is *complete* since one of the four cases prevails and it is *antisymmetric* since the four cases are mutually exclusive. Moreover, it is shown in the appendix that for any three distinct agents $i, j, k \in \mathcal{N}$, the following implications hold:

$$\left(\bar{i} \succeq j \textit{ and } j \succeq k \Longrightarrow \bar{i} \succeq k\right) \text{ and } \left(\underline{i} \succeq j \textit{ and } j \succeq k \Longrightarrow \underline{i} \succeq k\right).$$

In particular, the relation $\succeq$ is *transitive*. Therefore, $\succeq$ is a linear order on $\mathcal{N}$.

Let $i \in \mathcal{N}$ not be the minimal element[7] of $\mathcal{N}$ w.r.t. $\succeq$. Then, there exists $k \in L(i, \succeq, \mathcal{N})$ with $i \succ k$.[8] Let $j \in L(i, \succeq, \mathcal{N})$ such that $j \neq k$ and $i \succ j$, then:

• $\underline{\bar{i} \succeq k \Longrightarrow \bar{i} \succeq j}$*:* Suppose that $\bar{i} \succeq k$. If $k \succeq j$, we immediately have that $\bar{i} \succeq j$. Otherwise if $j \succeq k$, suppose that it is not true that $\bar{i} \succeq j$. But then since $i \succeq j$, we must have $\underline{i} \succeq j$. Along with $j \succeq k$, this implies that $\underline{i} \succeq k$, a contradiction. Therefore, $\bar{i} \succeq j$.

• $\underline{\underline{i} \succeq k \Longrightarrow \underline{i} \succeq j}$*:* Suppose that $\underline{i} \succeq k$. Similarly, if $k \succeq j$, we immediately have that $\underline{i} \succeq j$. Otherwise if $j \succeq k$, suppose that it is not true that $\underline{i} \succeq j$. But then since $i \succeq j$, we must have $\bar{i} \succeq j$. Along with $j \succeq k$, this implies that $\bar{i} \succeq k$, a contradiction. Therefore, $\underline{i} \succeq j$.

Therefore, we may validly define the set $\mathcal{M} \subset \mathcal{N}$ as follows. If there exists a minimal element of $\mathcal{N}$ w.r.t. $\succeq$, let it belong to $\mathcal{M}$. For any other $i \in \mathcal{N}$, let $i \in \mathcal{M}$ if and only if $\bar{i} \succeq k$ for some—or for any $k \in L(i, \succeq, \mathcal{N})$ with $i \succ k$.

Finally, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $i \in N$, $j \in L(i, \succeq, N)$ and $\mu = \varphi(\mathcal{E})$. If $i = j$ then $\mu(i) = \mu(j)$ and therefore $\mu(i)\ R_i\ \mu(j)$ and $\mu(j)\ R_i\ \mu(i)$. Otherwise $i \neq j$, so we have $i \succ j$. In this case, if $i \in N \cap \mathcal{M}$ then by construction $\bar{i} \succeq j$, i.e. i never envies j, i.e. $\mu(i)\ R_i\ \mu(j)$. Otherwise if $i \in N \setminus \mathcal{M}$ then by construction $\underline{i} \succeq j$, i.e. i always envies j, i.e. $\mu(j)\ R_i\ \mu(i)$. By Lemma 1, μ is the sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$ for $\mathcal{E}$. Therefore, φ is the simple sequential solution induced by $\succeq$ and $\mathcal{M}$. $\square$

Corollary 1 *If a rule is weakly Pareto optimal, pairwise consistent, and pairwise neutral, then it is a simple serial dictatorship.*

Proof Let φ be a *weakly Pareto optimal*, *pairwise consistent*, and *pairwise neutral* rule. By Theorem 1, φ is a simple sequential solution, i.e. there exists $\succeq$ and $\mathcal{M} \subset \mathcal{N}$ such that $\varphi = \varphi^{\succeq, \mathcal{M}}$. Suppose that $\varphi \neq \varphi^{\succeq}$. Then there exists $i \in \mathcal{N} \setminus \mathcal{M}$ such that i is not the minimal element in $\mathcal{N}$ w.r.t. $\succeq$. Let $j \in \mathcal{N}$ be such that $i \succ j$ and let the problem $\mathcal{E}$ be as follows:

[7] For any $i \in \mathcal{N}$, i is the ***minimal element*** of $\mathcal{N}$ w.r.t. $\succeq$, if for any $j \in \mathcal{N}$, we have $j \succeq i$. For the case when $\mathcal{N}$ is finite, this is equivalent to saying that i is the bottom-ranked agent in $\mathcal{N}$ w.r.t. $\succeq$. By the *antisymmetry* of $\succeq$, there exists at most one minimal element of $\mathcal{N}$ w.r.t. $\succeq$.

[8] $\succ$ is used to denote the asymmetric part of $\succeq$.

P_i	P_j
a	b
b	a

Since $i \notin \mathcal{M}$, under the allocation $\varphi(\mathcal{E}) = \varphi^{\succeq,\mathcal{M}}(\mathcal{E})$, agents i and j receive b and a, respectively. However, both are made strictly better off by exchanging their assigned houses, a contradiction to φ being *weakly Pareto optimal*. Therefore, $\varphi = \varphi^{\succeq}$. □

The Pareto correspondence is *anonymous* but not *conversely consistent*. The next proposition shows that there does not exist a *nonempty valued* correspondence that is *weakly Pareto optimal*, *anonymous*, and *conversely consistent*.

Proposition 5 *Any nonempty valued, anonymous, and conversely consistent correspondence is not weakly Pareto optimal.*

Proof Let φ be a *nonempty valued*, *anonymous*, and *conversely consistent* correspondence. Then, for any distinct $i, j \in \mathcal{N}$ and $a, b \in \mathcal{H}$, the correspondence φ will choose both allocations from the problem:

P_i	P_j
a	a
b	b

since it is *nonempty valued* and *anonymous*. By *converse consistency* of φ, the underlined allocation μ will be chosen by φ from the following problem $\mathcal{E}$:

P_1	P_2	P_3
a	c	b
$\underline{b}$	$\underline{a}$	$\underline{c}$
c	b	a

Note that the allocation μ is strongly Pareto dominated in $\mathcal{E}$, showing that φ is not *weakly Pareto optimal.* □

For any correspondence or rule φ, let $\varphi|_2$ be its restriction to two-person problems. A correspondence φ is an ***extension*** of a correspondence $\overline{\varphi}$ defined for two-person problems if

$\varphi|_2 = \overline{\varphi}$. The following lemma states that *consistent* and *conversely consistent* correspondences are characterized by their restrictions to two-person problems.

Lemma 2 *For any correspondence $\overline{\varphi}$ defined for two-person problems, there exists a consistent and conversely consistent extension φ that is unique up to one-person problems.*[9] *The extension φ is defined as follows. For any problem $\mathcal{E} = (N, H, (R_i)_{i\in N})$ with $|N| \geq 2$ and for any allocation μ for $\mathcal{E}$:*

$$\mu \in \varphi(\mathcal{E}) \iff \forall N' \subset N \text{ with } |N'| = 2 : \quad \mu_{N'} \in \overline{\varphi}\left(r^{\mu}_{N'}(\mathcal{E})\right).$$

In particular, any consistent and conversely consistent correspondence φ is expressed as in above where $\overline{\varphi} = \varphi|_2$.

Proof Let $\overline{\varphi}$ be any correspondence defined for two-person problems. Let the correspondence φ be defined as in above for problems involving more than one person and W.L.O.G. let φ select the unique allocation in one-person problems. Since $\overline{\varphi} = \varphi|_2$, φ is *conversely consistent* by definition. To see that φ is *consistent*, let $\mathcal{E} = (N, H, (R_i)_{i\in N})$, $\emptyset \neq N' \subset N$ and $\mu \in \varphi(\mathcal{E})$. Assume W.L.O.G. that $|N'| \geq 2$. Consider the reduced problem $\mathcal{E}' = r^{\mu}_{N'}(\mathcal{E})$ and the allocation $\mu_{N'}$ for $\mathcal{E}'$. For any $N'' \subset N'$ with $|N''| = 2$, we have $(\mu_{N'})_{N''} = \mu_{N''} \in \overline{\varphi}\left(r^{\mu}_{N''}(\mathcal{E})\right) = \varphi\left(r^{\mu}_{N''}(\mathcal{E})\right) = \varphi\left(r^{\mu_{N'}}_{N''}\left(r^{\mu}_{N'}(\mathcal{E})\right)\right) = \overline{\varphi}\left(r^{\mu_{N'}}_{N''}(\mathcal{E}')\right)$, since $N'' \subset N$ with $|N''| = 2$ and $\mu \in \varphi(\mathcal{E})$. Therefore, by definition of φ, we have $\mu_{N'} \in \varphi(\mathcal{E}') = \varphi\left(r^{\mu}_{N'}(\mathcal{E})\right)$, showing that φ is *consistent*.

To show uniqueness of φ up to one-person problems, let φ' be any other *consistent* and *conversely consistent* extension of $\overline{\varphi}$. *Consistency* of φ' requires that $\varphi' \subset \varphi$. Similarly, *converse consistency* of φ' requires that $\varphi \subset \varphi'$ in problems involving more than one person, showing the uniqueness of the extension up to one-person problems.[10]

Any *consistent* and *conversely consistent* correspondence φ is in particular a *consistent* and *conversely consistent* extension of $\varphi|_2$. By uniqueness of the *consistent* and *conversely*

[9] In other words, if there exist two different *consistent* and *conversely consistent* extensions of $\overline{\varphi}$, they would only differ in their selections from one-person problems.

[10] The proof of the uniqueness part makes implicit use of the "Elevator Lemma" in Thomson (1998). The Elevator Lemma states that if $\varphi|_2 \subset \varphi'|_2$, φ is *consistent* and φ' is *conversely consistent*, then $\varphi \subset \varphi'$ up to one-person problems.

consistent extension of $\varphi|_2$ up to one-person problems, φ is expressed as in above where $\overline{\varphi} = \varphi|_2$. □

For any correspondence $\overline{\varphi}$ defined for two-person problems, let $Ext\left(\overline{\varphi}\right)$ be the *consistent* and *conversely consistent* extension of $\overline{\varphi}$ selecting the unique allocation in one-person problems. The correspondence $Ext\left(\overline{\varphi}\right)$ is uniquely defined by Lemma 2. For any two correspondences $\overline{\varphi}$ and $\overline{\varphi}'$ defined for two-person problems such that $\overline{\varphi} \subset \overline{\varphi}'$, we have $Ext\left(\overline{\varphi}\right) \subset Ext\left(\overline{\varphi}'\right)$. In particular, if $\{\overline{\varphi}_\alpha\}_{\alpha \in I}$ is any collection of correspondences defined for two-person problems, we have:

$$\bigcup_{\alpha \in I} Ext\left(\overline{\varphi}_\alpha\right) \subset Ext\left(\bigcup_{\alpha \in I} \overline{\varphi}_\alpha\right).$$

Moreover, for any linear order $\succeq$ on $\mathcal{N}$ and any $\mathcal{M} \subset \mathcal{N}$, we have:

$$Ext\left(\varphi^{\succeq,\mathcal{M}}|_2\right) = \varphi^{\succeq,\mathcal{M}},$$

by Proposition 3 and Lemma 2.

Let $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ be an indexed family of relations on $\mathcal{N}$. The family $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ is $\mathbf{3^+}$**–*acyclic*** if there do not exist distinct elements $i_1, i_2, \ldots, i_n \in \mathcal{N}$ and $a_1, a_2, \ldots, a_n \in \mathcal{H}$ with $n \geq 3$ such that $i_1 \succeq_{a_1,a_2} i_2 \succeq_{a_2,a_3} \cdots \succeq_{a_{n-1},a_n} i_n \succeq_{a_n,a_1} i_1$.

For any correspondence $\overline{\varphi}$ defined for two-person problems, we can naturally induce a family of relations $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ on $\mathcal{N}$ as follows. For any $a, b \in \mathcal{H}$, if $a = b$, let $\succeq_{a,b} = \emptyset$, otherwise if $a \neq b$, let $\succeq_{a,b}$ be the *reflexive* relation such that for any distinct $i, j \in \mathcal{N}$, we have $i \succeq_{a,b} j$ if and only if $\overline{\varphi}$ chooses the underlined allocation from the following problem:

P_i	P_j
$\underline{a}$	a
b	$\underline{b}$

Lemma 3 *If a Pareto optimal and conversely consistent correspondence φ is such that $\varphi|_2$ is nonempty valued, then $\varphi|_2$ induces a 3^+–acyclic family of relations on $\mathcal{N}$. Conversely, for any consistent correspondence φ such that $\varphi|_2$ induces a 3^+–acyclic family of relations on $\mathcal{N}$ and is Pareto optimal, the correspondence φ is Pareto optimal.*

We defer the proof of Lemma 3 to the appendix. A restatement of Lemma 3 gives us the following characterization of *Pareto optimal*, *consistent*, and *conversely consistent* correspondences.

Corollary 2 *A correspondence φ is nonempty valued in one and two-person problems, Pareto optimal, consistent, and conversely consistent if and only if $\varphi = Ext\,(\overline{\varphi})$ for some nonempty valued and Pareto optimal correspondence $\overline{\varphi}$ defined for two-person problems that induces a 3^+-acyclic family of relations on $\mathcal{N}$.*

A relation $\succeq$ on $\mathcal{N}$ is $\mathbf{3^+}$**–*acyclic*** if there do not exist distinct elements $i_1, i_2, \ldots, i_n \in \mathcal{N}$ with $n \geq 3$ such that $i_1 \succeq i_2 \succeq \ldots \succeq i_n \succeq i_1$. Any *complete* and 3^+–*acyclic* relation is *transitive*. A *transitive* relation is 3^+–*acyclic* if and only if its indifference classes are of size smaller than 3. For any *complete* and 3^+–*acyclic* relation $\succeq$ on $\mathcal{N}$, let $\overline{\varphi}^{\succeq}$ be the *nonempty valued*, *Pareto optimal* and *neutral* correspondence defined for two-person problems, such that for any two distinct agents $i, j \in \mathcal{N}$ and any problem $\mathcal{E}$ of type:

P_i	P_j
$\underline{a}$	a
b	$\underline{b}$

the set $\overline{\varphi}^{\succeq}(\mathcal{E})$ contains the underlined allocation if and only if $i \succeq j$. In this case, the family of relations $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ induced by $\overline{\varphi}^{\succeq}$ are such that for any distinct $a, b \in \mathcal{H}$, we have $\succeq_{a,b}=\succeq$.

By the axiom of choice, for any *reflexive*, *complete*, and 3^+–*acyclic* relation $\succeq$ on $\mathcal{N}$, there exists a linear order $\succeq'\subset\succeq$. If $\succeq$ has n indifference classes of size 2, then there exist exactly 2^n such linear orders. For example, when $\mathcal{N} = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ with $|\mathcal{N}| = 6$, a typical *reflexive*, *complete* and 3^+–*acyclic* relation $\succeq$ on $\mathcal{N}$ and the 4 linear orders $\succeq_1, \ \succeq_2, \ \succeq_3, \ \succeq_4 \ \subset \ \succeq$ obtained by arbitrarily breaking indifferences in $\succeq$ are depicted as follows:

$\succeq$
i_3 i_6
i_2
i_1 i_5
i_4

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$
i_3	i_3	i_6	i_6
i_6	i_6	i_3	i_3
i_2	i_2	i_2	i_2
i_1	i_5	i_1	i_5
i_5	i_1	i_5	i_1
i_4	i_4	i_4	i_4

.

We also have that $\overline{\varphi}^{\succeq} = \varphi^{\succeq_1}|_2 \cup \varphi^{\succeq_2}|_2 \cup \varphi^{\succeq_3}|_2 \cup \varphi^{\succeq_4}|_2$.

Lemma 4 *A correspondence φ is nonempty valued, Pareto optimal, consistent, conversely consistent, and neutral if and only if there exists a reflexive, complete, and 3^+–acyclic relation $\succeq$ on $\mathcal{N}$ such that $\varphi = Ext\left(\overline{\varphi}^{\succeq}\right)$.*

We defer the proof of Lemma 4 to the appendix. The following theorem states that a correspondence is *nonempty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* if and only if it can be expressed as a union of simple serial dictatorships in a particular manner.

Theorem 2 *A correspondence φ is nonempty valued, Pareto optimal, consistent, conversely consistent, and neutral if and only if there exists a reflexive, complete, and 3^+–acyclic relation $\succeq$ on $\mathcal{N}$ such that:*

$$\varphi = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha},$$

where $\{\succeq_\alpha\}_{\alpha \in I}$ is the set of linear orders contained in $\succeq$.

Proof Let $\succeq$ be any *reflexive*, *complete*, and 3^+*–acyclic* relation on $\mathcal{N}$ and let $\{\succeq_\alpha\}_{\alpha \in I}$ be the set of linear orders contained in $\succeq$. We will show that

$$Ext\left(\overline{\varphi}^{\succeq}\right) = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}.$$

This will prove the theorem, by Lemma 4. We already know that

$$\overline{\varphi}^{\succeq} = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}|_2 = \left(\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}\right)|_2.$$

By showing that $\bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}$ is *consistent* and *conversely consistent*, we will have that $\bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}$ is a *consistent* and *conversely consistent* extension of $\overline{\varphi}^{\succeq}$, which will imply the desired equality, by Lemma 2. By Proposition 3, simple serial dictatorships are *consistent*. Therefore, $\bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}$ is *consistent* as a union of *consistent* correspondences. To see that $\bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}$ is *conversely consistent*, let $\mathcal{E} = (N, H, (R_i)_{i\in N})$ with $|N| \geq 2$ and let μ be an allocation for $\mathcal{E}$, such that for any $N' \subset N$ with $|N'| = 2$, we have $\mu_{N'} \in \bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}(r^\mu_{N'}(\mathcal{E}))$. So, for any $\{i,j\} \subset N$ with $i \neq j$, we can choose $\alpha_{\{i,j\}} \in I$ such that $\mu_{\{i,j\}} \in \varphi^{\succeq_{\alpha_{\{i,j\}}}}(r^\mu_{\{i,j\}}(\mathcal{E}))$. Hence, we can define a *reflexive* and *complete* relation $\succeq'$ on N, such that for any distinct $i, j \in N$, we have $i \succeq' j$ if and only if $i \succeq_{\alpha_{\{i,j\}}} j$. Then, $\succeq'$ is 3^+*–acyclic*, since $\succeq' \subset \bigcup_{\alpha\in I}(\succeq_\alpha|_N) = \succeq|_N$. Also note that $\succeq'$ is *transitive*, since it is *complete* and 3^+*–acyclic*. Moreover, since each $\succeq_{\alpha_{\{i,j\}}}$ is *antisymmetric*, we have that $\succeq'$ is *antisymmetric*, showing that $\succeq'$ is a linear order on N. Since $\succeq' \subset \succeq|_N$, there exists $\beta \in I$ such that $\succeq' = \succeq_\beta|_N$. Moreover, since μ is the serial dictatorship allocation induced by $\succeq'$ for $\mathcal{E}$, for any $N' \subset N$ with $|N'| = 2$, we have $\mu_{N'} = \varphi^{\succeq_\beta}(r^\mu_{N'}(\mathcal{E}))$. But then, since $\varphi^{\succeq_\beta}$ is a simple serial dictatorship, it is *conversely consistent*, therefore we have $\{\mu\} = \varphi^{\succeq_\beta}(\mathcal{E}) \subset \bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}(\mathcal{E})$, showing that $\bigcup_{\alpha\in I}\varphi^{\succeq_\alpha}$ is *conversely consistent.* □

4 Concluding remarks

This paper investigates the role of the *consistency* principle in *house allocation problems.* Classes of allocation rules and correspondences satisfying *consistency* and its *converse* are identified. The class of rules satisfying weak forms of *efficiency*, *consistency*, and *neutrality* are characterized by serial dictatorships where each agent is assigned his best house, following a sequence determined by an exogenous priority ordering. The more general class of *consistent* and *neutral* rules turn out to be characterized by sequential solutions generalizing serial dictatorships, where certain agents receive their least preferred house when their turn comes. The latter result is negative in the sense that one can not recover other properties of interest by dropping the *efficiency* axiom. The impossibilities concerning *anonymity* and equal treatment of equals remain present even in the case of *multivalued* correspondences.

5 Appendix

5.1 Independence of Axioms

Let $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{H} = \{a, b, c\}$ with $|\mathcal{N}| = |\mathcal{H}| = 3$ in the following examples which establish the independence of axioms in Theorem 1 and Corollary 1.

(i) Let φ select the serial dictatorship allocation induced by the order $\succeq$: $1 \succ 2 \succ 3$ in three-person problems, and the serial dictatorship allocation induced by the order $\succeq'$: $2 \succ' 1 \succ' 3$ in all other problems. The rule φ is *Pareto optimal* and *neutral*. To see that φ is not *pairwise consistent*, consider the problem $\mathcal{E}$ depicted below:

P_1	P_2	P_3
$\underline{a}$	a	a
b	$\underline{b}$	b
c	c	$\underline{c}$

Note that φ chooses the underlined allocation μ for $\mathcal{E}$. Let $\mu' = \varphi\left(r^{\mu}_{\{1,2\}}(\mathcal{E})\right)$, then $\mu'(1) = b \neq a = \mu_{\{1,2\}}(1)$, i.e., $\varphi\left(r^{\mu}_{\{1,2\}}(\mathcal{E})\right) = \mu' \neq \mu_{\{1,2\}}$. Therefore, φ is not *pairwise consistent*.

(ii) Let $\succeq$: $1 \succ 2 \succ 3$ and consider the simple sequential solution $\varphi^{\succeq,\emptyset}$. By Proposition 3, $\varphi^{\succeq,\emptyset}$ is *neutral* and *consistent*. To see that $\varphi^{\succeq,\emptyset}$ is not *weakly Pareto optimal* consider the following problem $\mathcal{E}$:

P_1	P_2
a	b
$\underline{b}$	$\underline{a}$

where the underlined allocation $\varphi^{\succeq,\emptyset}(\mathcal{E})$ is strongly Pareto dominated in $\mathcal{E}$.

(iii) Let $\succeq$: $1 \succ 2 \succ 3$ and $\succeq'$: $2 \succ' 1 \succ' 3$. At each $\mathcal{E} = (N, H, (R_i)_{i \in N})$, define the allocation rule φ by $\varphi(\mathcal{E}) = \varphi^{\succeq}(\mathcal{E})$ if $1, 2 \in N$, $a \in H$ and both 1 and 2 rank a in the top, and $\varphi(\mathcal{E}) = \varphi^{\succeq'}(\mathcal{E})$ otherwise. The rule φ is *Pareto optimal*, *consistent*, and *conversely consistent*.[11] Consider the underlined selections of φ from the following problems:

[11]Gibbard (1973) and Satterthwaite (1975) show that for a large class of social choice functions, *strategyproofness* is equivalent to *dictatorship*. However, note that φ above is *strategyproof* but not *dictatorial*, showing that *strategyproofness* does not imply *dictatorship* in the context of house allocation problems. The rule φ is the

$\mathcal{E}$

P_1	P_2
$\underline{a}$	a
b	$\underline{b}$

$\mathcal{E}'$

P'_1	P'_2
b	$\underline{b}$
$\underline{a}$	a

Let $\mu = \varphi(\mathcal{E})$ and $\mu' = \varphi(\mathcal{E}')$. Agent 1 is assigned a under μ. Therefore, if φ is *pairwise neutral*, agent 1 should be assigned b under μ', but this is not the case. Thus, φ is not *pairwise neutral.*

5.2 Proofs

Proof (Part of Theorem 1) Let $a, b, c \in \mathcal{H}$ be three distinct houses and let $i, j, k \in \mathcal{N}$ be any three distinct agents. Then,

$\underline{\bar{i} \succeq j \textit{ and } \bar{j} \succeq k \Longrightarrow \bar{i} \succeq k}$*:*

Suppose that $\bar{i} \succeq j$ and $\bar{j} \succeq k$. Then, in any problem involving i, j and k, we have that i does not envy j and j does not envy k. Consider the following problem $\mathcal{E}$ and the underlined allocation μ:

P_i	P_j	P_k
$\underline{a}$	a	a
b	$\underline{b}$	b
c	c	$\underline{c}$

Note that μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j does not envy k. Therefore, $\mu = \varphi(\mathcal{E})$. Consider the reduced problem $r^{\mu}_{\{i,k\}}(\mathcal{E})$:

P_i	P_k
$\underline{a}$	a
c	$\underline{c}$

By *pairwise consistency* of φ, the underlined selection $\mu_{\{i,k\}} \in \varphi\left(r^{\mu}_{\{i,k\}}(\mathcal{E})\right)$. Therefore, either $\bar{i} \succeq k$ or $\underline{k} \succeq i$.

Consider the following problem $\mathcal{E}$ and the underlined allocation μ:

variable population extension of a "hierarchical exchange function", introduced in Papai (1997) and of a "top trading cycles mechanism", introduced in Abdulkadiroğlu and Sönmez (1999).

P_i	P_j	P_k
$\underline{a}$	a	$\underline{c}$
b	$\underline{b}$	b
c	c	a

Note that μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j does not envy k. Therefore, $\mu = \varphi(\mathcal{E})$. Consider the reduced problem $r^{\mu}_{\{i,k\}}(\mathcal{E})$:

P_i	P_k
$\underline{a}$	$\underline{c}$
c	a

By *pairwise consistency* of φ, the underlined selection $\mu_{\{i,k\}} \in \varphi\left(r^{\mu}_{\{i,k\}}(\mathcal{E})\right)$. Therefore, either $\bar{i} \succeq k$ or $\bar{k} \succeq i$. By the above paragraph, $\bar{i} \succeq k$.

Similarly, one can show that ($\bar{i} \succeq j$ *and* $\underline{j} \succeq k \Longrightarrow \bar{i} \succeq k$), ($\underline{i} \succeq j$ *and* $\bar{j} \succeq k \Longrightarrow \underline{i} \succeq k$) and ($\underline{i} \succeq j$ *and* $\underline{j} \succeq k \Longrightarrow \underline{i} \succeq k$) which altogether imply that:

$$\left(\bar{i} \succeq j \textit{ and } j \succeq k \Longrightarrow \bar{i} \succeq k\right) \quad \text{and} \quad \left(\underline{i} \succeq j \textit{ and } j \succeq k \Longrightarrow \underline{i} \succeq k\right).$$

Proof (Lemma 3) Let φ be any *Pareto optimal* and *conversely consistent* correspondence such that $\varphi|_2$ is *nonempty valued.* Let $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ be the family of relations induced by $\varphi|_2$ on $\mathcal{N}$. Suppose that there exist distinct elements $i_1, i_2, \ldots, i_n \in \mathcal{N}$ and $a_1, a_2, \ldots, a_n \in \mathcal{H}$ with $n \geq 3$ such that $i_1 \succeq_{a_1,a_2} i_2 \succeq_{a_2,a_3} \cdots \succeq_{a_{n-1},a_n} i_n \succeq_{a_n,a_1} i_1$. Consider the following problem $\mathcal{E}$ and the underlined allocation μ:

P_{i_1}	P_{i_2}	P_{i_3}	$\ldots$	P_{i_n}
a_n	a_1	a_2		a_{n-1}
$\underline{a_1}$	$\underline{a_2}$	$\underline{a_3}$		$\underline{a_n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$

Note that for any two distinct integers $l, k \in \{1, 2, \ldots, n\}$, we have that $\mu_{\{i_l,i_k\}} \in \varphi|_2\left(r^{\mu}_{\{i_l,i_k\}}(\mathcal{E})\right) = \varphi\left(r^{\mu}_{\{i_l,i_k\}}(\mathcal{E})\right)$. But then, by *converse consistency* of φ, we have $\mu \in \varphi(\mathcal{E})$, a contradiction to φ being *Pareto optimal* and μ being Pareto dominated in $\mathcal{E}$. Therefore, the family $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ is 3^+*–acyclic.*

For the converse, let φ be any *consistent* correspondence such that $\varphi|_2$ induces a 3^+–*acyclic* family of relations $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ on $\mathcal{N}$ and is *Pareto optimal.* Suppose that φ is not *Pareto optimal.* Then there exists $\mathcal{E} = (N, H, (R_i)_{i\in N})$ and $\mu \in \varphi(\mathcal{E})$ such that μ is weakly Pareto dominated by another allocation μ' for $\mathcal{E}$. We will show that there exists a cycle of agents $i_0, i_1, \ldots, i_n = i_0 \in N$ such that each one envies the next, under μ. Let $\emptyset \neq N' \subset N$ be the set of agents who are strictly better off under μ'. Since preferences are *antisymmetric* and the agents in $N \setminus N'$ are indifferent between μ and μ', their assignments are the same in either case, i.e. $\mu'|_{N\setminus N'} = \mu|_{N\setminus N'}$. Therefore, we have that $\mu'(N') = \mu(N')$. Now, consider the bijection $\pi = \mu^{-1} \circ \mu' \colon N \to N$. Since $\mu'|_{N\setminus N'} = \mu|_{N\setminus N'}$ and $\mu'(N') = \mu(N')$, we have that $\pi|_{N\setminus N'}$ is the identity map on $N \setminus N'$ and $\pi|_{N'}$ is a permutation of N'. Choose $i \in N'$ and let $N_i = \left\{ \pi^k(i) \mid k \in \{0, 1, 2, \ldots\} \right\} \subset N'$.[12] Let $j \in N_i$. Since $\mu\left(\pi(j)\right) = \mu\circ(\mu^{-1} \circ \mu')\,(j) = \mu'(j)$ and $j \in N'$, we have that $\mu\left(\pi(j)\right) = \mu'(j)\ P_j\ \mu(j)$, i.e. j envies $\pi(j)$ under μ. In particular, $j \neq \pi(j)$ and $j, \pi(j) \in N_i$, therefore, $n = |N_i| \geq 2$. Note that for any $j = \pi^l(i) \in N_i$ and any positive integer k such that $\pi^k(j) = j$, we have $\pi^k(i) = \pi^{-l}\circ\pi^k\circ\pi^l(i) = \pi^{-l}\circ\pi^k(j) = \pi^{-l}(j) = \pi^{-l}\circ\pi^l(i) = i$. Then, $N_i = \left\{i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i)\right\}$, so $k \geq |N_i| = n$. Therefore, the agents $i, \pi(i), \pi^2(i), \ldots, \pi^{n-1}(i)$ are all distinct, for otherwise there exists $j \in N_i$ and a positive integer $k < n$ such that $\pi^k(j) = j$, a contradiction. Then, we have $N_i = \{i, \pi(i), \pi^2(i), \ldots, \pi^{n-1}(i)\}$. Moreover, since $\pi^n(i) \in N_i$, by a similar argument, we can only have that $\pi^n(i) = i$. Letting $i_k = \pi^k(i)$ and $a_k = \mu(i_k)$, we have that $a_0 = a_n P_{i_{n-1}} a_{n-1} \ldots a_2 P_{i_1} a_1 P_{i_0} a_0$. Moreover, for any $k \in \{0, 1, \ldots, n-1\}$, we have $a_{k+1} P_{i_{k+1}} a_k$, for otherwise there exists $k \in \{0, 1, \ldots, n-1\}$ such that $a_k P_{i_{k+1}} a_{k+1} P_{i_k} a_k$. But then, by *consistency* of φ, the following underlined allocation is chosen by φ and hence by $\varphi|_2$, from the reduced problem $r^{\mu}_{\{i_k, i_{k+1}\}}(\mathcal{E})$:

$P_{i_k}\vert_{\{a_k,a_{k+1}\}}$	$P_{i_{k+1}}\vert_{\{a_k,a_{k+1}\}}$
a_{k+1}	a_k
$\underline{a_k}$	$\underline{a_{k+1}}$

a contradiction to $\varphi|_2$ being *Pareto optimal.* In particular, we have $n \geq 3$, for otherwise, if $n = 2$, then $a_0 = a_2 P_{i_1} a_1$, a contradiction to $a_{k+1} P_{i_{k+1}} a_k$ for $k = 0$. Moreover, for every

[12]Here, π^k denotes the map π composed k times with itself and $\pi^{-k} = \left(\pi^{-1}\right)^k$. The map π^0 denotes the identity.

$k \in \{0, 1, \ldots, n-1\}$, the reduced problem $r^{\mu}_{\{i_k, i_{k+1}\}}(\mathcal{E})$ is of the form:

$P_{i_k}\|_{\{a_k, a_{k+1}\}}$	$P_{i_{k+1}}\|_{\{a_k, a_{k+1}\}}$
a_{k+1}	$\underline{a_{k+1}}$
$\underline{a_k}$	a_k

where the underlined allocation is chosen by $\varphi|_2$, by *consistency* of φ. But then, $i_0 \succeq_{a_0, a_{n-1}} i_{n-1} \succeq_{a_{n-1}, a_{n-2}} \ldots \succeq_{a_2, a_1} i_1 \succeq_{a_1, a_0} i_0$ and $n \geq 3$, a contradiction to $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ being 3^+*–acyclic*. Therefore, φ is *Pareto optimal*, completing the proof of the lemma. □

Proof (Lemma 4) Let $\succeq$ be a *reflexive*, *complete*, and 3^+*–acyclic* relation on $\mathcal{N}$. Let $\varphi = Ext\left(\overline{\varphi}^{\succeq}\right)$. By the axiom of choice, there exists a linear order $\succeq' \subset \succeq$. Then, $\varphi^{\succeq'}|_2 \subset \overline{\varphi}^{\succeq}$, i.e. $\varphi^{\succeq'} = Ext\left(\varphi^{\succeq'}|_2\right) \subset Ext\left(\overline{\varphi}^{\succeq}\right) = \varphi$. Since φ contains the simple serial dictatorship $\varphi^{\succeq'}$, it is *nonempty valued*. The correspondence φ is *consistent* and *conversely consistent*. By Lemma 3, it is also *Pareto optimal*. To see that φ is *neutral*, let $\emptyset \neq N \subset \mathcal{N}$, $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and $\mathcal{E}' = (N, H', (R_i')_{i \in N})$. W.L.O.G., assume that $|N| \geq 2$. Suppose that there exists a bijection $\pi \colon H \to H'$ satisfying:

$$\forall i \in N, \forall a, b \in H: \quad aR_ib \iff \pi(a)R_i'\pi(b),$$

Let $\mu \in \varphi(\mathcal{E})$. By definition of φ, we have:

$$\forall N' \subset N \text{ with } |N'| = 2: \quad \mu_{N'} \in \overline{\varphi}^{\succeq}\left(r^{\mu}_{N'}(\mathcal{E})\right).$$

Take any $N' \subset N$ with $|N'| = 2$. From above, $\mu_{N'} \in \overline{\varphi}^{\succeq}\left(r^{\mu}_{N'}(\mathcal{E})\right)$. Note that $\pi|_{\mu(N')} \colon \mu(N') \to \pi \circ \mu(N')$ is a bijection between the house sets of the reduced problems $r^{\mu}_{N'}(\mathcal{E})$ and $r^{\pi \circ \mu}_{N'}(\mathcal{E}')$ satisfying:

$$\forall i \in N', \forall a, b \in \mu(N'): \quad aR_i|_{\mu(N')}b \iff \pi|_{\mu(N')}(a)\ R_i'|_{\pi \circ \mu(N')}\ \pi|_{\mu(N')}(b),$$

So, by *neutrality* of $\overline{\varphi}^{\succeq}$ for two-person problems, $(\pi \circ \mu)|_{N'} = \pi|_{\mu(N')} \circ \mu_{N'} \in \overline{\varphi}^{\succeq}\left(r^{\pi \circ \mu}_{N'}(\mathcal{E}')\right)$. Since this is true for any such N', by definition of φ, we have that $\pi \circ \mu \in \varphi(\mathcal{E}')$. Therefore, φ is *neutral*.

For the converse, let φ be a *nonempty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* correspondence. By Lemma 2, $\varphi = Ext\,(\varphi|_2)$ up to one player problems, and by Lemma 3, $\varphi|_2$ induces a 3^+–*acyclic* family of relations $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ on $\mathcal{N}$. Note that in this case, since φ is *nonempty valued*, the equality $\varphi = Ext\,(\varphi|_2)$ also holds for one-person problems and for any pair of distinct $a, b \in \mathcal{H}$, the relation $\succeq_{a,b}$ is *complete*. Since φ is *neutral*, in particular, $\varphi|_2$ is *neutral*. *Neutrality* of $\varphi|_2$ requires in turn that the *reflexive* and *complete* relation $\succeq_{a,b}$ is identical for any pair of distinct $a, b \in \mathcal{H}$. Let $\succeq=\succeq_{a,b}$ for some—or for any pair of distinct $a, b \in \mathcal{H}$. Then, $\varphi|_2 = \overline{\varphi}^{\succeq}$. Moreover, since the family $\{\succeq_{a,b}\}_{(a,b)\in\mathcal{H}\times\mathcal{H}}$ is 3^+–*acyclic*, the relation $\succeq$ is 3^+–*acyclic*. Therefore, $\varphi = Ext\,(\overline{\varphi}^{\succeq})$, where $\succeq$ is a *reflexive*, *complete*, and 3^+–*acyclic* relation on $\mathcal{N}$, completing the proof of the lemma. □

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