

# CONSUMPTION: BASIC PERMANENT INCOME MODEL

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*Consumption is the sole end and purpose of all production.*

Adam Smith

# CONSUMPTION: MACRO VS. MICRO

- Micro:
  - Apples versus Oranges (vs. TVs vs. trips to Hawaii, etc)
  - Normal goods, luxury goods, inferior goods, Giffen goods, etc.
  - Slutsky equation, Shephard's lemma, Roy's identity, etc.
  - Demand systems: CES, AIDS, Translog, etc.
- Macro:
  - Behavior of overall individual consumption
  - Response of consumption to changes in income and interest rates
  - Consumption and saving over the life-cycle
  - Response of economy-wide aggregate consumption to income and interest rates

- Household consumption problem is to maximize:

$$E_0 \sum_{t=0}^T \beta^t U(C_t)$$

$$C_t = \left[ \int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

subject to a budget constraint

- Problem can be divided into two parts:
  - Solve for  $c_t(i)$  subject to  $C_t$
  - Solve for  $C_t$

# CONSUMPTION: MACRO VS. MICRO

- Expenditure on goods in budget constraint:

$$\int_0^1 p(i)c(i)di$$

- Define  $P_t$  as the minimum expenditure needed to purchase one unit of the composite consumption good  $C_t$
- Then it turns out that

$$\int_0^1 p(i)c(i)di = P_t C_t$$

- So, we can write budget constraint without reference to  $c(i)$ s and  $p(i)$ s
- $P_t$  is the ideal price index

# MOTIVATING QUESTIONS

- Social Programs / Taxes / Inequality :
  - Do people save “enough” for retirement?
  - How does consumption respond to an unemployment spell?
  - How much do the super-rich save?
- Business Cycles:
  - How does consumption respond to monetary policy?
  - How does consumption respond to stimulus checks / UI extensions?
- Long-Run Growth:
  - What are the determinants of aggregate savings?

# History of Thought

# KEYNES' CONSUMPTION FUNCTION

Keynes (1936, p. 96):

*The fundamental psychological law, upon which we are entitled to depend with great confidence both a priori and from our knowledge of human nature and from detailed facts of experience, is that men are disposed, as a rule and on average, to increase their consumption as their income increases, but not by as much as the increase in their income.*



# KEYNES' CONSUMPTION FUNCTION

Keynes (1936, p. 93-94):

*The usual type of short-period fluctuation in the rate of interest is not likely, however, to have much direct influence on spending either way. There are not many people who will alter their way of living because the rate of interest has fallen from 5 to 4 per cent, if their aggregate income is the same as before.*

# KEYNES' CONSUMPTION FUNCTION

$$C_t = \alpha + \gamma(Y_t - T_t)$$

- Consumption a function of after-tax income
- Marginal propensity to consume ( $\gamma$ ) between zero and one
- Interest rates not important
- Future income not important

# CONSUMPTION: INTROSPECTION

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- How about a one-time \$100,000 scholarship?
  
- How about a person like me?

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- How much less would you consume over a year as a fraction of your annual consumption?

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- Suppose interest rates rose by 1 percentage point.
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- Suppose you received news that short term interest rates were going to be 1 percentage point higher (than you thought before) 5 years from now for one year.
- How much less would you consume over this coming year as a fraction of your annual consumption?



# KEYNESIAN CROSS

- Suppose  $I$ ,  $G$ ,  $NX$  are exogenous  
(i.e., not functions of output directly or indirectly)
- Planned expenditure (aggregate demand):

$$PE_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

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$$PE_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

- Suppose the output is completely **demand determined**
- Output must equal  $PE_t$ :

$$Y_t = \alpha + \gamma(Y_t - T_t) + I_t + G_t + NX_t$$

- A little algebra then yields:

$$Y_t = \frac{1}{1 - \gamma} [\alpha - \gamma T_t + I_t + G_t + NX_t]$$

$$\text{Government purchases multiplier} = \frac{1}{1 - \gamma}$$

$$\text{Tax cut multiplier} = \frac{\gamma}{1 - \gamma}$$

- With  $\text{MPC} = 2/3$ :  $G$  multiplier is 3 and  $T$  multiplier is 2
- Logic:
  - Government spends:  $\Delta G$  (which raises income by  $\Delta G$ )
  - First round change in consumption:  $\gamma \Delta G$
  - Second round change in consumption:  $\gamma^2 \Delta G$
  - Etc.
- Multiplier for change to “autonomous spending” (i.e.,  $\alpha$ ) same as for  $G$

# KEYNESIAN CROSS

- Keynesian Economics in its simplest form
- VERY strong assumptions!!
  1. Simplistic consumption function
  2. Investment exogenous
  3. No prices change as output changes  
(i.e., economy completely demand determined)
  4. No monetary policy response  
(but wouldn't matter since nothing responds to interest rate)
- IS-LM:  $I(r)$  + monetary policy response
- New Keynesian model: "Modern" consumption function  
+ Phillips curve + monetary policy

- Classical Economics:
  - Saving is good
  - Foundation for capital accumulation
- (Old) Keynesian Economics:
  - Increased saving / fall in “autonomous” spending (i.e.,  $\alpha$ ) thought to have contributed to causing the Great Depression
  - Widespread worry during WWII about “secular stagnation”
    - As people get richer, they will save larger share of income ( $MPC < 1$ )
    - Eventually too much saving
    - Not enough demand, not enough investment opportunities

# EMPIRICS OF CONSUMPTION FUNCTION

- Early work looked at budget studies  
(i.e., cross section at a point in time)
- $\Delta C / \Delta Y \approx 2/3$
- Also analyzed aggregate saving over course of Great Depression
  - Savings rose as economy recovered

# THREE LANDMARK EMPIRICAL STUDIES

“dealt a fatal blow to this extraordinarily simple view of the savings process”  
(Modigliani 86)

- Simon Kuznetz (1946):
  - National Income and Product Accounts back to 1899
  - No rise in aggregate savings over time
- Dorothy Brady and Rose D. Friedman (1947):
  - Re-analyze budget study data
  - Consumption function shifts up over time as average income increases
- Margaret Reid (unpublished):
  - Re-analyzes budget study data
  - Introduces concept of “permanent component of income”

(See Burns (2022) for history of “Hidden Figures.”)

Permanent Income Hypothesis  
and  
Life-Cycle Hypothesis



- Originally developed independently by:
  - Modigliani and Brumberg (1954) (Life-Cycle Hypothesis)
  - Friedman (1957) (Permanent Income Hypothesis)
- Basic idea:
  - Utility maximization and perfect markets imply that current consumption is determined by net present value of life-time income

# HOUSEHOLD CONSUMPTION-SAVING PROBLEM

Simplifying assumptions:

- Known finite horizon  $T$
- No uncertainty
- Constant interest rate
- No durable goods (houses/cars/etc)
- Exogenous income process
- Costless enforcement of contracts
- No bankruptcy (i.e., full commitment to repay debt)
- Natural borrowing limit

# HOUSEHOLD'S PROBLEM: SETUP

- Preferences:

$$\sum_{t=0}^T \beta^t U(C_t)$$

- Savings/Borrowing technology:
  - Household can save at rate  $r$
  - Household can borrow at rate  $r$  up to some limit
  - Household assets denoted  $A_t$
- Initial assets:  $A_{-1}$
- Income stream:  $Y_t$

# HOUSEHOLD'S PROBLEM

- Maximize

$$\sum_{t=0}^T \beta^t U(C_t)$$

- Subject to “budget constraint”:

$$\frac{A_t}{1+r} + C_t = Y_t + A_{t-1}$$

- But, mathematically, this is not really a constraint (doesn't constrain the problem)
- Mathematically, this is just a definition of  $A_t$

Real constraint is constraint on  $A_t$  sequence

- Simplest: “Natural” borrowing limit:  $A_T \geq 0$   
(i.e., household cannot die with debt)
- Alternative: No (unsecured) borrowing:  $A_t > 0$   
(much tighter / much more realistic)

# INTERTEMPORAL BUDGET CONSTRAINT

- With natural borrowing limit, sequence of one-period budget constraints can be consolidated into a single intertemporal budget constraint:

$$\sum_{t=0}^T \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^T \frac{Y_t}{(1+r)^t}$$

- Present value of consumption cannot be larger than present value of income and assets
- This embeds the  $A_T \geq 0$  constraint

# HOUSEHOLD'S PROBLEM

$$\max \sum_{t=0}^T \beta^t U(C_t)$$

subject to:

$$\sum_{t=0}^T \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^T \frac{Y_t}{(1+r)^t}$$

- Important to differentiate between:
  - Choice variables:  $C_t$  (and  $A_t$ , for  $t \geq 0$ )
  - Exogenous variables:  $A_{-1}$  and  $Y_t$  (and  $r$  and  $\beta$ )

- Krusell (2015, ch. 4) is preferred reading on dynamic optimization
  - Hard to strike right balance on technical details (this is not a math class)
  - Sims lecture notes, Stokey, Lucas, with Prescott (1989), Ljungqvist and Sargent (2018), Acemoglu (2009) are more techy
  - Romer (2019) is less techy
- Various readings present slightly different versions of the problem
  - E.g., Krusell (2015) presents planner problem with production
  - Good for you to see slight variations in notation and setup



# HOUSEHOLD'S PROBLEM: SOLUTION

- One way to solve household's problem is to set up a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^T \beta^t U(C_t) - \lambda \left( \sum_{t=0}^T \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^T \frac{Y_t}{(1+r)^t} \right)$$

and derive Kuhn-Tucker conditions

- Differentiating yield first order conditions:

$$\beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}$$

- The full set of optimality conditions additionally includes a complementary slackness condition:

$$\lambda \left( \sum_{t=0}^T \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^T \frac{Y_t}{(1+r)^t} \right) = 0$$

# COMPLEMENTARY SLACKNESS CONDITION

$$\lambda \left( \sum_{t=0}^T \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^T \frac{Y_t}{(1+r)^t} \right) = 0$$

- Notice from first order conditions that:

$$\lambda = \beta^t (1+r)^t U'(C_t)$$

- If  $U'(C_t) > 0$ , then  $\lambda > 0$
- Implies that:

$$\sum_{t=0}^T \frac{C_t}{(1+r)^t} - A_{-1} - \sum_{t=0}^T \frac{Y_t}{(1+r)^t} = 0$$

- Since  $U'(C_t) > 0$ , intertemporal budget constraint holds with equality.
- Often we just impose this from the beginning

# CONSUMPTION EULER EQUATION

$$\beta^t U'(C_t) = \frac{\lambda}{(1+r)^t}$$

$$\beta^{t+1} U'(C_{t+1}) = \frac{\lambda}{(1+r)^{t+1}}$$

Divide one by the other:

$$\frac{U'(C_t)}{\beta U'(C_{t+1})} = (1+r)$$

Rearrange:

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

This equation is usually referred to as the consumption Euler equation

# CALCULUS OF VARIATIONS

- Lagrangian math does not yield much intuition
- Alternative: Calculus of variations
- We seek to maximize

$$V(C) = \sum_{t=0}^T \beta^t U(C_t)$$

subject to

$$\sum_{t=0}^T \frac{C_t}{(1+r)^t} \leq A_{-1} + \sum_{t=0}^T \frac{Y_t}{(1+r)^t}$$

- Here  $C$  denotes the sequence  $\{C_0, C_1, \dots, C_{T-1}, C_T\}$

- Suppose we have a candidate optimal path  $C_t^*$
- Let's consider a variation on this path:
  - Save  $\epsilon$  more at time  $t$
  - Consume proceeds at time  $t + 1$
- Utility from new path:

$$V(C) = \dots + \beta^t U(C_t^* - \epsilon) + \beta^{t+1} U(C_{t+1}^* + \epsilon(1 + r)) + \dots$$

- If  $C_t^*$  is the optimum, then

$$\left. \frac{dV}{d\epsilon} \right|_{\epsilon=0} = 0$$

- At the optimum, benefit of small variation must be zero

$$\frac{dV}{d\epsilon} = -\beta^t U'(C_t^* - \epsilon) + (1+r)\beta^{t+1} U'(C_{t+1}^* + \epsilon(1+r))$$

$$\left. \frac{dV}{d\epsilon} \right|_{\epsilon=0} = -\beta^t U'(C_t^*) + (1+r)\beta^{t+1} U'(C_{t+1}^*) = 0$$

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

- The generic first order condition in calculus of variations is called the Euler equation (or Euler-Lagrange equation)
- This is where the consumption Euler equation gets its name

# PERMANENT INCOME HYPOTHESIS

- Suppose  $U(C_t) = \log C_t$
- Then

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

becomes:

$$\frac{C_{t+1}}{C_t} = \beta(1+r)$$

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- How does consumption growth  $C_{t+1}/C_t$  depend on income growth  $Y_{t+1}/Y_t$ ?



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- How does consumption growth  $C_{t+1}/C_t$  depend on income growth  $Y_{t+1}/Y_t$ ?
- It doesn't!!

# PERMANENT INCOME HYPOTHESIS

- Suppose for simplicity that  $\beta(1 + r) = 1$
- Consumption Euler equation becomes

$$U'(C_t) = U'(C_{t+1})$$

which implies

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- Consumers optimally smooth their consumption

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- Consumers optimally smooth their consumption
- Variation in consumption only due to:
  - Variation in interest rates
  - Variation in marginal utility  $U'_t(C_t)$  (e.g., children, health)

# PERMANENT INCOME HYPOTHESIS

- Let's plug  $C_t = C_{t+1}$  into intertemporal budget constraint:

$$C_0 = \Phi(r) \left( A_{-1} + \sum_{t=0}^T \frac{Y_t}{(1+r)^t} \right)$$

$$\Phi(r) = \frac{1 - \frac{1}{1+r}}{1 - \left(\frac{1}{1+r}\right)^{T+1}}$$

- Consumption a function of present value of life-time income
- Current income is not special

# PERMANENT INCOME HYPOTHESIS

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- Marginal propensity to consume (MPC): How much of a windfall extra dollar a household spends over some period of time
- What does Permanent Income Hypothesis imply about MPC?
- MPC out of windfall gain is equal to  $\Phi(r)$
- Suppose  $T = 40$  and  $r = 0.02$ :

$$\Phi(r) = 0.035$$

# INFINITE HORIZON AND UNCERTAINTY

Let's consider a version of the household problem with:

- Infinite horizon
- Uncertainty
- Heterogeneous preferences

We still maintain:

- No durable goods (houses/cars/etc)
- Exogenous income process
- Costless enforcement of contracts
- No bankruptcy (i.e., full commitment to repay debt)
- Natural borrowing limit

# WHY INFINITE HORIZON?

## 1. Altruism: We love our children

- If we value our children's consumption like our own, intergenerational discounting is the same as intragenerational discounting
- If we however value giving (not children's consumption) things are different (warm glow bequests)

## 2. Simplicity:

- Infinite horizon makes problem more stationary
- In finite horizon problem, horizon is a state variable (i.e., affects optimal choice)
- Solution to problem with long horizon similar to one with infinite horizon



- Household  $i$  faces uncertainty about future income  $Y_{it+j}$   
(include  $i$  to emphasize that risk is partly idiosyncratic)
- Heterogeneity in income across households potentially yields heterogeneity in consumption:  $C_{it}$

# WHAT ASSETS ARE TRADED?

- If households are risk averse, they will want to “buy insurance” against income risk
- Whether they can depends on what assets are traded
- Two polar cases:
  - Complete markets: Complete set of state contingent assets available
  - Bonds only: Only non-state contingent asset available
- We will start by considering the complete markets case

# NATURAL BORROWING LIMIT

- What is the “natural” borrowing limit in the infinite horizon case?

# NATURAL BORROWING LIMIT

- What is the “natural” borrowing limit in the infinite horizon case?
- Household can “borrow” (sell assets) up to the point where it can repay for sure in all states of the world
- Rules out “Ponzi schemes”:
  - Sell asset at time  $t$
  - Sell more assets at time  $t + 1$  to pay off interest/principle coming due
  - Keep doing this ad infinitum
- Natural borrowing limit can be quite “tight”:
  - If non-zero probability of zero future income, natural borrowing limit is zero

# HOUSEHOLD'S PROBLEM

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U_i(C_{it}(s^t))$$

subject to:

$$C_{it}(s^t) + \sum_{s^{t+1}|s^t} Q_t(s^{t+1})A_{it}(s^{t+1}) = Y_{it}(s^t) + A_{i,t-1}(s^t),$$

a No Ponzi scheme constraint, and given  $A_{i,-1}$

- $s^t = [s_1, s_2, \dots, s_t]$  denotes history of states up to date  $t$
- $Q_t(s^{t+1})$  denotes the time  $t$  price of Arrow security that pays off one unit of consumption in state  $s^{t+1}$
- $A_{it}(s^{t+1})$  denotes quantity of Arrow security that pays off in state  $s^{t+1}$  that is purchased at time  $t$  by household  $i$

# HOUSEHOLD'S PROBLEM: SOLUTION

- One solution method (again) is to set up a Lagrangian:

$$\mathcal{L}_t = E_t \sum_{j=0}^{\infty} \beta^j \left( U_i(C_{i,t+j}(s^{t+j})) - \lambda_{i,t+j}(s^{t+j}) \left( C_{i,t+j}(s^{t+j}) \right. \right. \\ \left. \left. + \sum_{s^{t+j+1}|s^{t+j}} Q_{t+j}(s^{t+j+1}) A_{i,t+j}(s^{t+j+1}) - Y_{i,t+j}(s^{t+j}) - A_{i,t+j-1}(s^{t+j}) \right) \right)$$

- The choice variables at time  $t$  are  $C_{it}(s^t)$  and  $A_{it}(s^{t+1})$

- Differentiation of time  $t$  Lagrangian yields:

$$\frac{\partial \mathcal{L}_t}{\partial C_{it}(s^t)} : U'_i(C_{it}(s^t)) = \lambda_{it}(s^t)$$

$$\frac{\partial \mathcal{L}_t}{\partial A_{it}(s^{t+1})} : \lambda_{it}(s^t) Q_t(s^{t+1}) = E_t[\beta \lambda_{it+1}(s^{t+1}) I(s^{t+1})]$$

where  $I(s^{t+1})$  is an indicator for whether state  $s^{t+1}$  occurs

- The latter of these can be rewritten:

$$\lambda_{it}(s^t) Q_t(s^{t+1}) = \beta p_t(s^{t+1}) \lambda_{it+1}(s^{t+1})$$

where  $p_t(s^{t+1})$  is the time  $t$  probability of state  $s^{t+1}$  occurring

- See Sims Lecture Notes for more general “cookbook”

- Combining equations from last slide to eliminate  $\lambda_{it}$  we get:

$$Q_t(\mathbf{s}^{t+1})U'_i(C_{it}(\mathbf{s}^t)) = \beta p_t(\mathbf{s}^{t+1})U'_i(C_{it+1}(\mathbf{s}^{t+1}))$$

- This is a version of the consumption Euler equation
  - Trades off consumption today and consumption in one particular state tomorrow
  - Cost today:  $Q_t(\mathbf{s}^{t+1})U'_i(C_{it}(\mathbf{s}^t))$
  - Expected benefit tomorrow:  $\beta p_t(\mathbf{s}^{t+1})U'_i(C_{it+1}(\mathbf{s}^{t+1}))$



- Since Euler equation holds for each state  $s^{t+1}$ , it also holds on average

$$\begin{aligned} \sum_{s^{t+1}|s^t} [Q_t(s^{t+1})U'_i(C_{it}(s^t))] &= \sum_{s^{t+1}|s^t} [\beta p_t(s^{t+1})U'_i(C_{it+1}(s^{t+1}))] \\ \Rightarrow U'_i(C_{it}(s^t)) \sum_{s^{t+1}|s^t} Q_t(s^{t+1}) &= \beta E_t[U'_i(C_{it+1}(s^{t+1}))] \\ \Rightarrow U'_i(C_{it}(s^t))(1 + R_{ft}(s^t))^{-1} &= \beta E_t[U'_i(C_{it+1}(s^{t+1}))] \\ \Rightarrow U'_i(C_{it}(s^t)) &= \beta(1 + R_{ft})E_t[U'_i(C_{it+1}(s^{t+1}))] \end{aligned}$$

where  $R_{ft}(s^t)$  is the riskless interest rate in state  $s^t$

- Notice that buying one unit of each Arrow security is the same as buying a riskless bond

# TRANSVERSALITY CONDITION

- In finite horizon case, there was a complementary slackness condition that said that household should not die with positive wealth
- What is the counterpart in infinite horizon case?

# TRANSVERSALITY CONDITION

- In finite horizon case, there was a complementary slackness condition that said that household should not die with positive wealth
- What is the counterpart in infinite horizon case?
- Transversality condition:

$$\lim_{j \rightarrow \infty} \beta^j E_t[U'_i(C_{it+j}(s^{t+j}))A_{it+j}(s^{t+j})] \leq 0$$

- Intuitively:
  - Cannot be optimal to choose a plan that leaves resources with positive net present value today unspent in the infinite future
  - Cannot be optimal to allow your wealth to explode at a rate faster than discounted marginal utility is falling
- Are Harvard/Princeton/Stanford etc. optimizing?

# TRANSVERSALITY VS. NO PONZI

- Transversality and No Ponzi are sometimes confused
- Very different in nature!!!
- No Ponzi:
  - Debt cannot explode
  - Constraint imposed by lenders
- Transversality:
  - Wealth cannot explode (too fast)
  - Necessary condition for optimality

- Complete markets and common beliefs imply perfect risk sharing
- The consumption Euler equation

$$Q_t(s^{t+1})U'_i(C_{it}(s^t)) = \beta p_t(s^{t+1})U'_i(C_{it+1}(s^{t+1}))$$

holds for all households

- This implies

$$\frac{Q_t(s^{t+1})}{\beta p_t(s^{t+1})} = \frac{U'_i(C_{it+1}(s^{t+1}))}{U'_i(C_{it}(s^t))} = \frac{U'_k(C_{kt+1}(s^{t+1}))}{U'_k(C_{kt}(s^t))}$$

- Taking the ratio of this equation for states  $s^{t+1}$  and  $s^{*t+1}$  yields

$$\frac{U'_i(C_{it+1}(s^{t+1}))}{U'_i(C_{it+1}(s^{*t+1}))} = \frac{U'_k(C_{kt+1}(s^{t+1}))}{U'_k(C_{kt+1}(s^{*t+1}))}$$

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- Ratio of marginal utility of all households perfectly correlated
- This is called perfect risk sharing
- See Campbell (2018, ch. 4.1.6)
- Empirical literature: Cochrane (1991), Mace (1991), Schulhofer-Wohl (2010) Townsend (1994), Ogaki and Zhang (2001), Ligon, Thomas, Worrall (2002), Fafchamps and Lund (2003), Mazzucco and Saini (2012)

# REPRESENTATIVE HOUSEHOLD

- Perfect risk sharing condition implies all households have the same ordering of marginal utility and consumption across states
- We can number the states  $s^{t+1}$  such that

$$C_{it+1}(1) \leq C_{it+1}(2) \leq \dots \leq C_{it+1}(S)$$

- Define  $\bar{C}_{t+1}(s^{t+1}) = \sum_i C_{it+1}(s^{t+1})$  and we get

$$\bar{C}_{t+1}(1) \leq \bar{C}_{t+1}(2) \leq \dots \leq \bar{C}_{t+1}(S)$$

- We also have

$$\frac{Q_t(1)}{\beta p_t(1)} \geq \frac{Q_t(2)}{\beta p_t(2)} \geq \dots \geq \frac{Q_t(S)}{\beta p_t(S)}$$

i.e., assets that provide insurance are expensive

# REPRESENTATIVE HOUSEHOLD

- We can now define a function  $g(\bar{C}_{t+1}(s^{t+1}))$  such that

$$\frac{g(\bar{C}_t(s^{t+1}))}{g(\bar{C}_t(s^{*t+1}))} = \frac{Q_t(s^{t+1})/\beta p_t(s^{t+1})}{Q_t(s^{*t+1})/\beta p_t(s^{*t+1})}$$

for all states  $s^{t+1}$  and  $s^{*t+1}$

- $g(\bar{C}_{t+1}(s^{t+1}))$  can be interpreted as the marginal utility of a “composite household” or “representative household”
- We can then integrate to get a function  $v(\bar{C}(s^{t+1}))$  such that

$$v'(\bar{C}(s^{t+1})) = g(\bar{C}_{t+1}(s^{t+1}))$$

which is the utility function of the representative household



# REPRESENTATIVE HOUSEHOLD

- Complete market and common beliefs are one way to justify the common representative household assumption
- Important limitations:
  - Utility function of representative household need not be the same as that of individual households
  - Does not generally imply “demand aggregation”: reallocation of wealth alters representative household’s utility function and aggregate demand
  - See Campbell (2018, ch. 4.1.7), Constantinides (1982), Guvenen (2011)
- Demand aggregation requires “Gorman preferences”  
(see MWG ch. 4.D and Acemoglu (2009, ch. 5.2))
- Representative household assumption is a pretty strong assumption

# THE GREAT PARADIGM SHIFT

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  - Backward-looking system

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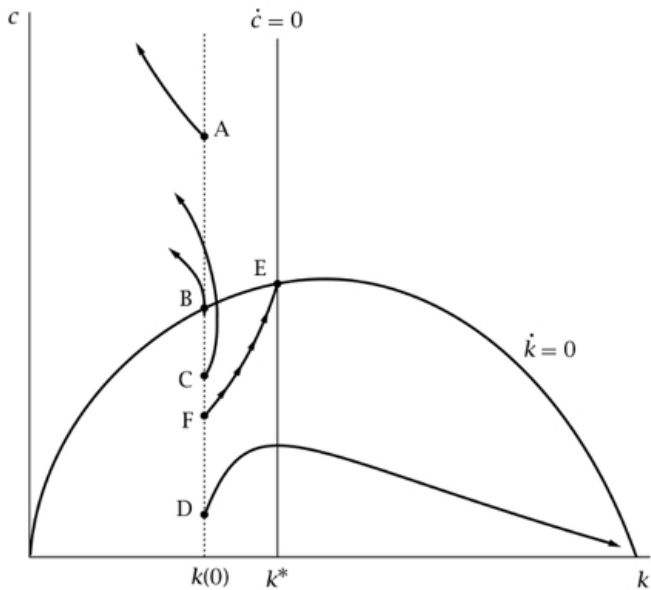
- Milton Friedman, Robert Lucas, etc.
- Pendulum swung really far:

$$c_t = E_t c_{t+1} - \sigma(i_t - E_t \pi_{t+1})$$

- Maybe the world is somewhere in between

# ECONOMICS VERSUS PHYSICS/ENGINEERING

- Biggest difference: People are forward looking!
- How does this show up mathematically?
  - System does not have a full set of initial conditions
  - Rather, some of the boundary conditions are terminal conditions
- Example:
  - Household problem does not come with an initial condition for consumption
  - The boundary condition is the transversality condition
- Water rolling down a hill is not forward looking: Problem comes with a full set of initial conditions (not a transversality condition)
- Presence of expectations not the fundamental difference. Fundamental difference is nature of boundary conditions.



Source: Romer (2019)

One approach:

1. Solve for first order conditions of each agent's problem (household/firm/etc)
  - System of non-linear dynamic equations (difference or differential equations)
  - E.g., consumption Euler equation, capital accumulation equation, labor supply curve, Phillips curve
  - N equations for N unknown variables for each period t plus a set of boundary conditions
2. Solve this system of non-linear dynamic equations
  - If problem doesn't have "kinks": can linearize (perturbation methods), and solve linear dynamic system (e.g., Blanchard-Kahn algorithm)
  - If problem does have "kinks": Need to use "global methods"



- Dynamic Programming is an alternative way to solve dynamic optimization problems
- Has its pros and cons versus Lagrangian methods
- For certain types of problems it is the easiest way to go
  - Problems where continuation value is directly used (e.g., Nash bargaining)
  - Problems where non-linearities are important (sometimes)
- For other problems it is more cumbersome than Lagrangian methods
  - E.g., problems that lend themselves to linearization (e.g., Real Business Cycle models, New Keynesian models)
  - Why? Value function is an extra object. Distinction between state variables and control variables extra headache

# Dynamic Programming

$$V(X_t) = \max_{C_t, A_t} E_t \sum_{t=j}^{\infty} \beta^t U_i(C_{i,t+j}(s^{t+j}))$$

subject to

$$C_{it}(s^t) + \sum_{s^{t+1}|s^t} Q_t(s^{t+1})A_{it}(s^{t+1}) = Y_{it}(s^t) + A_{i,t-1}(s^t),$$

a No Ponzi scheme constraint, and given  $A_{i,t-1}$

- $X_t$  is a vector of “state variables”
- Figuring out what variables are in  $X_t$  is a crucial element of using Dynamic Programming
- The state contains all known variables that affect the household's value function (i.e., the maximized value of the household's objective function).

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- Any variable that helps forecast future income
  - If  $Y_{it}(s^t)$  is i.i.d.: Nothing
  - If  $Y_{it}(s^t)$  is AR(1):  $Y_{it}(s^t)$
  - if  $Y_{it}(s^t)$  is AR(2):  $Y_{it}(s^t)$  and  $Y_{it-1}(s^{t-1})$
  - Etc.

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  - Etc.
- Current asset prices  $Q_t(s^{t+1})$
- Any variables that help forecast future asset prices  
(Sometimes the whole distribution of every agent's wealth)

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(Sometimes the whole distribution of every agent's wealth)

State vector can potentially be quite large and complicated!!



# YOEMAN FARMER MODEL

Consider the following yoeman farmer / planner problem:

$$\max_{\{C_s, K_{s+1}\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

subject to

$$C_t + K_{t+1} \leq f(K_t),$$

$C_t \geq 0$ ,  $K_{t+1} \geq 0$ , and  $K_0$  given.

- No markets. Yoeman farmer doesn't interact with the outside world.
- $f(K_t) = F(K_t, N) + (1 - \delta)K_t$
- Consumption-savings problem where savings technology is productive capital (seeds)

# YOEMAN FARMER MODEL

- If  $U'(C_t) > 0$  for all  $C_t$ , no resources will be wasted and resource constraint will hold with equality
- In this case we can plug resource constraint into objective function and get

$$\max_{\{0 \leq K_{t+1} \leq f(K_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(K_t) - K_{t+1})$$

with  $K_0$  given

- What is the state at time 0?

# YOEMAN FARMER MODEL

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with  $K_0$  given

- What is the state at time 0?
  - Only  $K_0$
  - No exogenous income source, no prices

# VALUE FUNCTION FOR YOEMAN FARMER MODEL

$$V(K_0) = \max_{\{0 \leq K_{t+1} \leq f(K_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(K_t) - K_{t+1})$$

- Let's now make use of the fact that

$$\max_{x,y} f(x,y) = \max_y \{ \max_x f(x,y) \}$$

- I.e., we can maximize in steps:
  - First max over  $x$  given  $y$  (yields a function of  $y$ )
  - Then max resulting function of  $y$  over  $y$
- In our context we divide the problem into  $K_1$  and  $K_t$  for  $t > 1$

# BELLMAN EQUATION

$$V(K_0) = \max_{\{0 \leq K_1 \leq f(K_0)\}} \left\{ U(f(K_0) - K_1) + \max_{\{0 \leq K_{t+1} \leq f(K_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t U(f(K_t) - K_{t+1}) \right\}$$

which simplifies to

$$V(K_0) = \max_{\{0 \leq K_1 \leq f(K_0)\}} \{U(f(K_0) - K_1) + \beta V(K_1)\}$$

and holds for all times  $t$ . So, we can write:

$$V(K) = \max_{\{0 \leq K' \leq f(K)\}} \{U(f(K) - K') + \beta V(K')\}$$

- This equation is called the Bellman equation
- It is a functional equation: the unknown is a function  $V(K)$

- The object of primary interest is actually not  $V(K)$
- It is the decision rule (policy function):

$$K' = g(K)$$

where

$$g(K) = \arg \max_{\{0 \leq K' \leq f(K)\}} \{U(f(K) - K') + \beta V(K')\}$$

$$V(K) = \max_{K' \in \Gamma(K)} \{F(K, K') + \beta V(K')\}$$

Suppose:

- $F$  is continuously differentiable in its two arguments, strictly increasing in  $K$ , and decreasing in  $K'$ , strictly concave, and bounded.
- $\Gamma$  is a nonempty, compact-valued, monotone, and continuous correspondence with a convex graph
- $\beta \in (0, 1)$

Then:

- There exists a unique  $V(K)$  that solves the Bellman equation
- Value function iteration:  $V(K)$  can be found as follows:
  - Pick any initial  $V_0(K)$
  - Find  $V_1(K)$  by evaluating RHS of Bellman equation

Repeat until  $V_{n+1}(K)$  converges to a stable function. This is  $V(K)$ .

- $V$  is strictly concave
- $V$  is strictly increasing
- $V$  is continuously differentiable
- Optimal behavior can be characterized by a function  $g: K' = g(K)$  that is increasing as long as  $F_2$  is increasing in  $K$

For proofs, see Stokey, Lucas, with Prescott (1989)



Those interested in more detail should read:

- Stokey, Lucas, with Prescott (1989) [starting with ch. 2.1, 3, and 4]
- Ljungqvist and Sargent (2018) [starting with ch. 3 and 4]
  - Note: I do not agree with Ljungqvist and Sargent's strong emphasis on the importance of recursive methods. I think they overdo this.

# FINITE HORIZON YOEMAN FARMER MODEL

$$\max_{\{C_s, K_{s+1}\}_{s=0}^T} \sum_{t=0}^T \beta^t U(C_t)$$

subject to

$$C_t + K_{t+1} \leq f(K_t),$$

$C_t \geq 0$ ,  $K_{t+1} \geq 0$ , and  $K_0$  given.

- Non-stationary problem
- Value function will be different in each period  $V_t(K_t)$
- Can be solved by backward induction

# BACKWARD INDUCTION

- Start by solving the problem at time  $T$  as a function of  $K_T$
- Clearly  $K_{T+1} = 0$  is optimal and  $V_{T+1}(K_{T+1}) = 0$
- This implies that

$$\begin{aligned} V_T(K_T) &= \max_{\{0 \leq K_{T+1} \leq f(K_T)\}} \{U(f(K_T) - K_{T+1}) + \beta V_{T+1}(K_{T+1})\} \\ &= U(f(K_T)) \end{aligned}$$

- Then move back one period to  $T - 1$
- The Bellman function for this period is:

$$V_{T-1}(K_{T-1}) = \max_{\{0 \leq K_T \leq f(K_{T-1})\}} \{U(f(K_{T-1}) - K_T) + \beta V_T(K_T)\}$$

- Since  $V_T(K_T)$  is known from prior step, this can be easily solved for  $V_{T-1}(K_{T-1})$

- This process can be iterated backward all the way to  $t = 0$
- Notice that this algorithm is essentially the same as the value function iteration algorithm we discussed for finding  $V(K)$  in the infinite horizon case
- This similarity means that the behavior of a household with a long but finite horizon is similar to the behavior of a household with an infinite horizon

# POLICY FUNCTION ITERATION

- Start with an initial guess for the policy function  $K' = g_0(K)$
- Calculate the value function for this policy function

$$V_0(K) = \sum_{t=0}^{\infty} \beta^t U(f(K) - g_0(K))$$

(In practice a finite sum with a large  $T$ )

- Generate a new policy function

$$g_1(K) = \arg \max_{K'} \{U(f(K) - K') + \beta V_0(K')\}$$

- Iterate on this algorithm until the policy function converges

# SOLVING BELLMAN EQUATIONS IN PRACTICE

- Four methods:
  1. Guess a solution
  2. Iterate on Bellman equation analytically
  3. Iterate on Bellman equation numerically
  4. Iterate on policy function numerically
- First two methods only work for highly special models
- In practice, Dynamic Programming most useful for problems that require numerical solution methods

# FUNCTIONAL EULER EQUATION

- We can derive an Euler equation from the Bellman function

$$V(K) = \max_{K' \in \Gamma(K)} \{F(K, K') + \beta V(K')\}$$

- Using the policy function  $K' = g(K)$  we can rewrite the Bellman equation:

$$V(K) = F(K, g(K)) + \beta V(g(K))$$

- Also, this policy function satisfies the first order condition

$$F_2(K, K') + \beta V'(K') = 0$$

- Evaluating this equation at  $K' = g(K)$  we get

$$F_2(K, g(K)) + \beta V'(g(K)) = 0$$



# FUNCTIONAL EULER EQUATION

- If we differentiate

$$V(K) = F(K, g(K)) + \beta V(g(K))$$

with respect to  $K$  we get

$$V'(K) = F_1(K, g(K)) + g'(K)\{F_2(K, g(K)) + \beta V'(g(K))\}$$

- Second term on RHS is zero (see last slide) and we get:

$$V'(K) = F_1(K, g(K))$$

- The fact that the second term drops out is an application of the envelope theorem

- Combining:

$$F_2(K, g(K)) + \beta V'(g(K)) = 0$$

$$V'(g(K)) = F_1(g(K), g(g(K)))$$

yields

$$F_2(K, g(k)) = -\beta F_1(g(K), g(g(K)))$$

- This is the Euler equation stated as a functional equation

# FUNCTIONAL EULER EQUATION

- In yoeman farmer model:

$$F(K, K') = U(F(K) - K') = U(C(K))$$

$$F_1(K, K') = U'(C(K))F'(K)$$

$$F_2(K, K') = -U'(C(K))$$

- This means that the Euler equation in the yoeman farmer model is

$$U'(C(K)) = \beta F'(K')U'(C(K'))$$

- $F'(K)$  plays the role of the return on investment
- $C(K) = F(K) - g(K)$  is the yoeman farmer's consumption function

$$U'(C(K)) = \beta F'(K') U'(C(K'))$$

- Iterating on the functional Euler equation for  $C(K)$  is an alternative to value function iteration, policy function iteration, and Blanchard-Kahn methods
- Advantages:
  - Level of value function not important (only derivative). Value function iteration waits one polynomial point on getting the level
  - Computationally useful in models with many agents and many frictions (see McKay's notes)
- Disadvantage:
  - May not converge

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