Lecture 2
Production

Macroeconomics (Quantitative)

Econ 101B

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A Model of Production

- Economists use mathematical models to describe and understand the world

- In this lecture, we take our first steps in this direction

Simple Canonical Model

- One Market, Supply and Demand:

$$
\begin{aligned}
q_t &= ap_t + w_t \\
q_t &= -bp_t + p_t^{xyz}
\end{aligned}
$$

- Two equations look very similar
- Which one is the supply curve?
  - If a and b are positive, first equation is supply curve because supply curves slope upward

Components of a Model

1. Endogenous variables
2. Exogenous variables and parameters
3. Equations that relate these variable to each other

Components of a Model

$$
\begin{aligned}
q_t &= ap_t + w_t \\
q_t &= -bp_t + p_t^{xyz}
\end{aligned}
$$

- Endogenous variables: p_t and q_t
- Exogenous variables: w_t and p_t^{xyz}
- Parameters: a and b

- If you don't know what the endogenous variables are, you don't know what you are solving for.

General Equilibrium Model

- In macro, we usually study general equilibrium models
- A model is a general equilibrium model if price **and** quantity in **all** markets are endogenous variables
- Today we will study our first general equilibrium model with three markets:
  - Labor market, capital market, goods market

Roadmap

- Simple general equilibrium models will consist of:
  - Demand and supply curves for each market
- Important caveat #1: Only relative prices matter
  - This implies that we can set one of the prices equal to 1
  - We call this the numeraire
- Important caveat #2: Walras' Law
  - If all but one market is in equilibrium, then the last one is necessarily also in equilibrium.
  - This implies that we only need to write down equations for all but one of the markets

Roadmap

- To solve a model, you need as many equations as there are endogenous variables
- Example: 3 markets (labor, capital, goods)
  - How many endogenous variables?
  - Five endogenous variables: L, w, K, r, Y.
    - Price of goods is the numeraire
  - We will need 5 equations
    - E.g.: demand and supply in the labor and capital markets and one more equation to give us output.

Production Function

- We represent production process by a production function:
$$Y = F(K, L)$$
- Example: How much ice cream can we make if we have L workers and K ice cream machines?
- Y: amount of ice cream produced
- L: Number of workers (or hours of work)
- K: Number of machines (amount of capital)
- Inputs or factors of production: Labor, Capital

Properties of Production Functions

- What properties should a reasonable production function satisfy?

$$Y = F(K, L)$$

- Suppose we increase L
  – Output increases
  – But at a declining rate
- Suppose we increase K
  – Output increases
  – But at a declining rate

Properties of Production Functions

- What properties should a reasonable production function satisfy?
- Positive marginal products of factors:

$$\frac{\partial F}{\partial L} \geq 0 \text{ and } \frac{\partial F}{\partial K} \geq 0$$

- Diminishing marginal products:

$$\frac{\partial^2 F}{\partial L^2} \leq 0 \text{ and } \frac{\partial^2 F}{\partial K^2} \leq 0$$

(What about $\partial^2 F / \partial L \partial K$?)

Cobb-Douglas Production Function

- Most commonly used production function

$$Y = AK^{a}L^{1-a}$$

$$\frac{\partial Y}{\partial L} = (1-a)AK^{a}L^{-a} \geq 0 \text{ and } \frac{\partial Y}{\partial K} = aAK^{a-1}L^{1-a} \geq 0$$

$$\frac{\partial^2 Y}{\partial L^2} = -a(1-a)AK^{a}L^{-a-1} \leq 0 \text{ and } \frac{\partial^2 Y}{\partial K^2} = -a(1-a)AK^{a-2}L^{1-a} \leq 0$$

$$\frac{\partial^2 Y}{\partial L \partial K} = a(1-a)AK^{a-1}L^{-a} \geq 0$$

Returns to Scale

- Suppose:

$$Y = AK^a L^{1-a}$$

- What happens if we double all inputs to production? (i.e., both K and L)

$$\begin{aligned} F(2K, 2L) &= A(2K)^a (2L)^{1-a} \\ &= A2^a K^a 2^{1-a} L^{1-a} \\ &= 2AK^a L^{1-a} \\ &= 2F(K, L) \end{aligned}$$

- Twice as much of every input yields twice as much output
- Production function is constant returns to scale

Returns to Scale

- What is it about $Y = AK^{a}L^{1-a}$ that implies constant returns to scale?
  - Exponents on K and L sum to one
- If exponents sum to more than one:
  - Increasing returns to scale
- If exponents sum to less than one:
  - Decreasing returns to scale
- Economically, why should we model production as having constant returns to scale at the aggregate level?

Replication Argument

- Replication argument:
  - If you build two identical factories. You should get twice as much output as from one.
- Counter-arguments?
  - Fixed factors: Some inputs cannot be doubled. E.g., land.
    - Not strictly speaking about the production function itself.
    - But has similar implications (diminishing returns to remaining factors)
  - Externalities: There may be either positive or negative spillovers between factories.

Behavior of Firms

- How should we model the behavior of firms?
- Usual assumptions in economics:
  - Individuals maximize utility
  - Firms maximize profits
- Tricky for firms. … What is a firm?
  - Shareholders put up money and “own” the firm
  - Managers hired by shareholders
  - Employees hired by managers
  - Money borrowed from creditors
- Conflicts of interest left and right!!

Firm’s Problem

- Choose how many workers to hire and machines to rent to maximize profits

$$\max_{K,L} F(K,L) - rK - wL$$

- Profits are production minus cost of inputs
- We set the price of output to 1 (the numeraire)
- r: rental price of machines, w: wage of workers
- Firm takes r and w as given. Why?
  – Competitive markets for labor and capital (Simplifying assumption – left wing critique)

Firm's Problem: Solution

- How do we go about solving the firm's problem?

$$\max_{K,L} F(K,L) - rK - wL$$

1. Choose optimal level of capital
   - Maximize profits with respect to K treating L as fixed
2. Choose optimal level of labor
   - Maximize profits with respect to L treating K as fixed

Firm’s Problem: Solution

- We are making use of a simple but powerful mathematical result:

$$\max_{x,y} f(x, y)$$

is given by the solution to the following two equations

$$\frac{\partial f(x, y)}{\partial x} = 0$$

$$\frac{\partial f(x, y)}{\partial y} = 0$$

Firm's Problem: Solution

$$\max_{K,L} F(K,L) - rK - wL$$

- We start by plugging in for $F(K,L)$:

$$\max_{K,L} AK^{a}L^{1-a} - rK - wL$$

Optimal Choice of Capital by Firm

- Differentiate profit function with respect to K (holding L constant)

$$\frac{\partial}{\partial K} AK^{a}L^{1-a} - rK - wL = aAK^{a-1}L^{1-a} - r$$

- Set this equal to zero to maximize:

$$aAK^{a-1}L^{1-a} - r = 0$$

$$aAK^{a-1}L^{1-a} = r$$

Optimal Choice of Labor by Firm

- Differentiate profit function with respect to L (holding K constant)

$$\frac{\partial}{\partial L} AK^{a}L^{1-a} - rK - wL = (1-a)AK^{a}L^{-a} - w$$

- Set this equal to zero to maximize:

$$aAK^{a}L^{-a} - w = 0$$

$$(1-a)AK^{a}L^{-a} = w$$

Firm's Problem: Solution

- Firm maximizes profits if:

$$aAK^{a-1}L^{1-a} = r$$

$$(1-a)AK^{a}L^{-a} = w$$

- Seems a bit inscrutable?
- How can we interpret these conditions?

Firm’s Optimal Choice of Capital

$$aAK^{a-1}L^{1-a} = r$$

- RHS: Price of capital
- LHS:

$$\frac{\partial}{\partial K}Y = \frac{\partial}{\partial K}AK^{a}L^{1-a} = aAK^{a-1}L^{1-a}$$

- LHS: Marginal product of capital
- Optimal for firms to rent capital to the point where the marginal product of capital is equal to price of capital

Firm's Optimal Choice of Labor

$$(1-a)AK^{a}L^{-a} = w$$

- RHS: price of labor (wage)
- LHS

$$\frac{\partial}{\partial L}Y = \frac{\partial}{\partial L}AK^{a}L^{1-a} = (1-a)AK^{a}L^{-a}$$

- LHS: Marginal product of labor
- Optimal for firms to hire labor to the point where the marginal product of labor is equal to price of labor (the wage)

Firm’s Problem

- Maximize profits:

$$\max_{K,L} AK^{a}L^{1-a} - rK - wL$$

- Optimal choice of capital implies:

$$aAK^{a-1}L^{1-a} = r$$

- Optimal choice of labor implies:

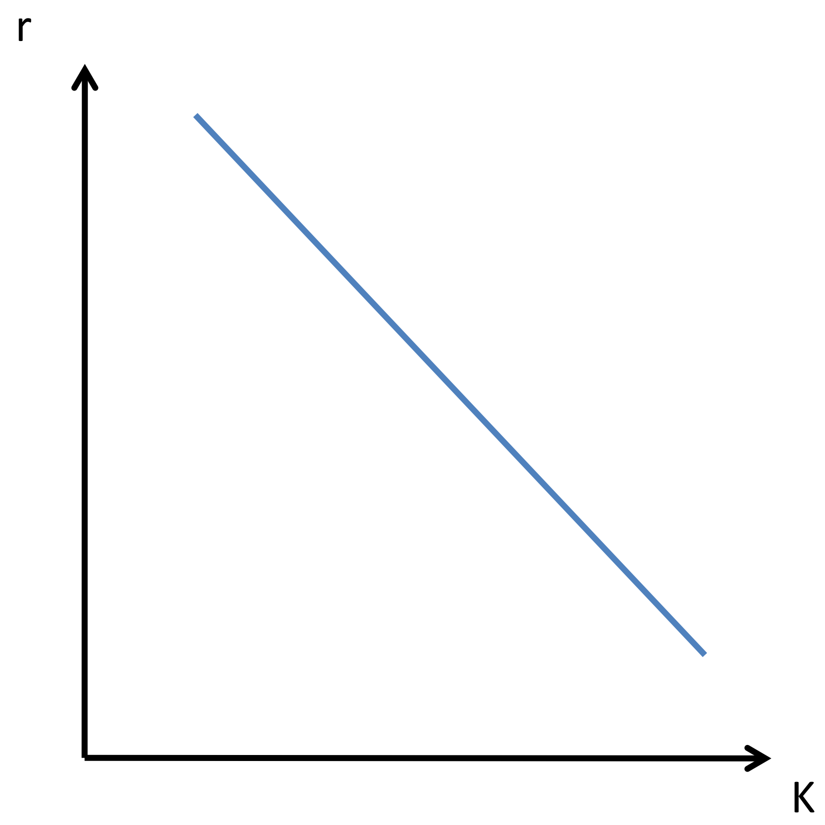
$$(1-a)AK^{a}L^{-a} = w$$

Math vs. Economics

- We can derive optimal behavior by firms in two ways:
  – Mathematically
    - Differentiate profits with respect to capital holding labor fixed
    - Differentiate profits with respect to labor holding capital fixed
  – By economic reasoning:
    - Firm sets MPK equal to price of capital
    - Firm sets MPL equal to price of labor

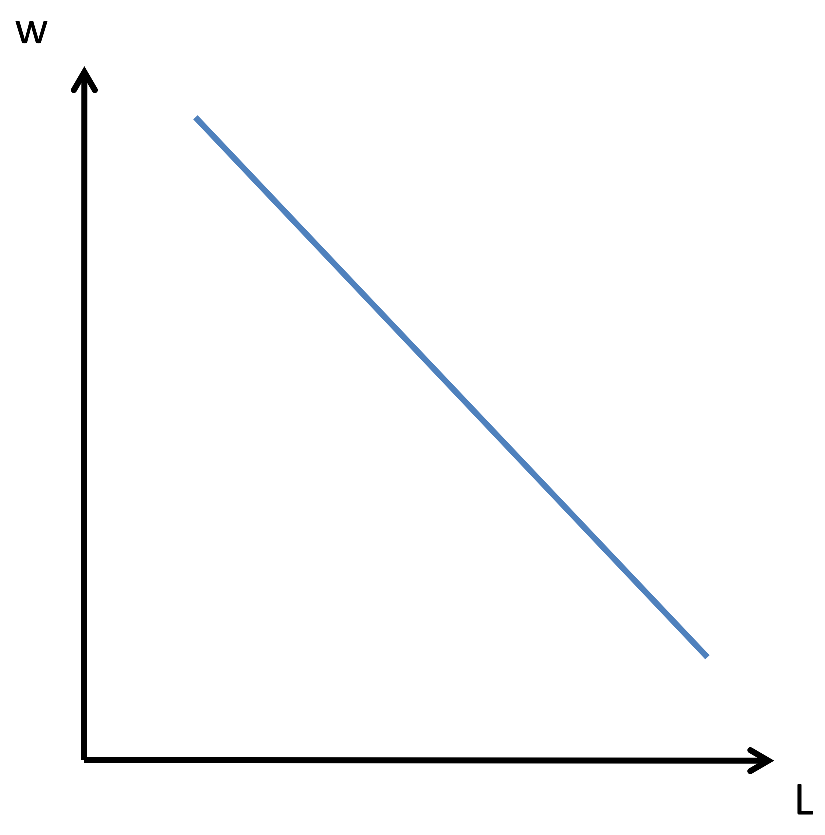
Firm Behavior

- Optimal choice of capital yields:
$$aAK^{a-1}L^{1-a} = r$$
- This equation can be represented graphically by a “curve” in (K,r) space
- The curve is all the points in (K,r) space that satisfy the equation (for some set values of other variables, i.e., L, A, a)
- Why is this curve downward sloping?
- Does this “curve” have a name?
  - Demand curve in capital market



Firm Behavior

- Optimal choice of labor yields:
$$(1-a)AK^{a}L^{-a} = w$$
- This equation can also be represented graphically by a "curve" in (L,w) space
- Curve is all the points in (L,w) space that satisfy the equation (for some set values of other variables)
- Does this "curve" have a name?
  – Demand curve in labor market



Power of Competition

- We assume labor market is perfectly competitive
- This implies that workers are paid their marginal product
  - Workers paid value of what they produce at the margin (value of what last worker produces)
- Why aren’t they paid less? (Why don’t firms exploit the workers?)
  - Each firm is held in check by competition with other firms in the labor market (if one pays less, another firm will find it profitable to hire away that firm’s workers)

Power of Competition

- But perhaps perfect competition is not a realistic assumption
- Perhaps we only make this assumption because it simplifies the math
- And then we forget that we make it
- And we start to think markets are efficient in the real world
- And we might even forget about the very possibility that markets might not be competitive
- Daniel Kahneman called this “theory-induced blindness”

Completing the Model

- We need demand and supply curves for the capital and labor markets + one more equation
- We have labor and capital demand
- What about labor and capital supply?
- Simplifying assumption: labor and capital supplied inelastically

$$K = \overline{K}$$

$$L = \overline{L}$$

Completing the Model

- Last equation: Something that gives output as a function of other variables
- We can use production function:

$$Y = AK^{a}L^{1-a}$$

Model

- Five equations:

$$aAK^{a-1}L^{1-a} = r$$

$$(1-a)AK^{a}L^{-a} = w$$

$$K = \overline{K}$$

$$L = \overline{L}$$

$$Y = AK^{a}L^{1-a}$$

- Five endogenous variables: K, L, r, w, Y

An Equilibrium

- In economics, the solution to a model is called an equilibrium
  - What happens when things play out in the markets under consideration (“when markets clear”)
- Model: System of five equations with five unknowns
- “Solving” the model:
  - Solving for endogenous variables as functions only of exogenous variables and parameters
  - Rewriting the system with the endogenous variables on one side of the equations and only exogenous variables and parameters on the other side.

Equilibrium

- "Trivial" solution in our case:

$$K = \bar{K}$$

$$L = \bar{L}$$

$$r = aA\bar{K}^{a-1}\bar{L}^{1-a}$$

$$w = (1-a)A\bar{K}^{a}\bar{L}^{-a}$$

$$Y = A\bar{K}^{a}\bar{L}^{1-a}$$

Factor Shares

- Revenue firms receive is paid out as:
  1. Wages to workers
  2. Returns to capital
  3. Pure profits

- Accounting measures divide revenue up somewhat differently

- Firms raise funds through:
  – Debt
  – Equity
- Accounting measures view:
  – interest payments as a cost
  – Returns to equity as a part of profits
- Economics:
  – Interest payments on debt a part of returns to capital
  – Competitive return on equity part of return to capital

Labor Share in Our Model

- Labor compensation is wL. Labor share is wL/Y.
- From labor demand curve we have

$$w = (1-a)AK^aL^{-a} = (1-a)\frac{Y}{L}$$

- Multiplying through by L yield

$$wL = (1-a)Y$$

- Labor share is $(1-a)$

Capital Share in Our Model

- Capital compensation is rK. Capital share is rK/Y.
- From the capital demand curve we have

$$r = aAK^{a-1}L^{1-a} = a\frac{Y}{K}$$

- Multiplying through by K yields

$$rK = aY$$

- Capital share is a

Profit Share

- How much of firm revenue is left as profits?
- None!
- Why?
  - Factor markets are perfectly competitive
  - Production function is constant returns to scale
- Euler's theorem:

$$F(L,K) = \frac{\partial F(L,K)}{\partial L} L + \frac{\partial F(L,K)}{\partial K} K$$

$$Y = wL + rK$$

Why Cobb-Douglas

- Cobb-Douglas is tractable
- Arguably more important is that Cobb-Douglas implies constant factor shares
- Factor share have in fact been remarkably close to constant since WWII

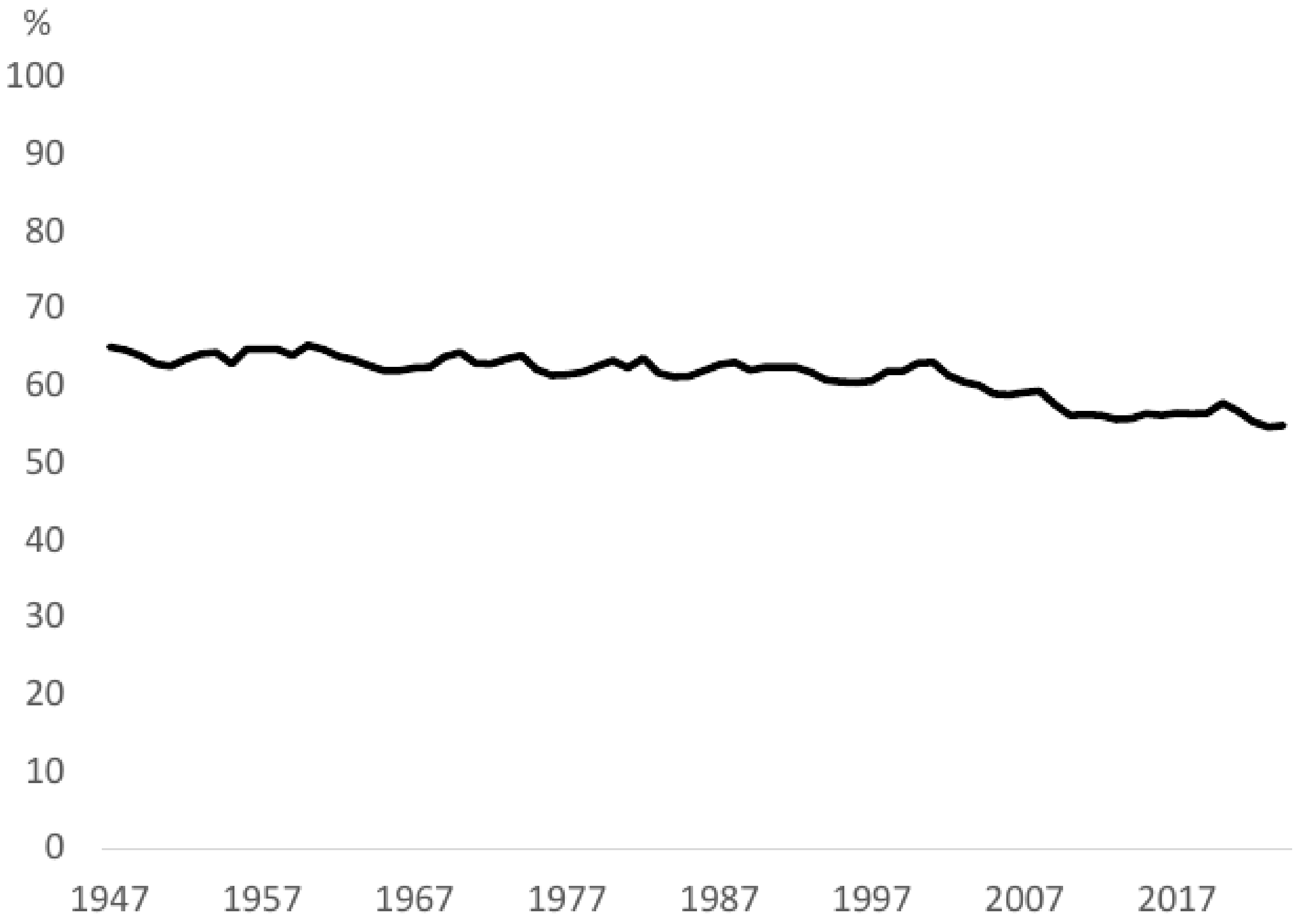


Figure 2: Labor Share in U.S. Nonfarm Business Sector

Worries about Labor-Saving Technology

- Worries ever since Industrial Revolution that machines will take our jobs
- Labor share constant despite huge amounts of labor-saving technology
- Stark contrast with horses
- But is our luck about to run out?

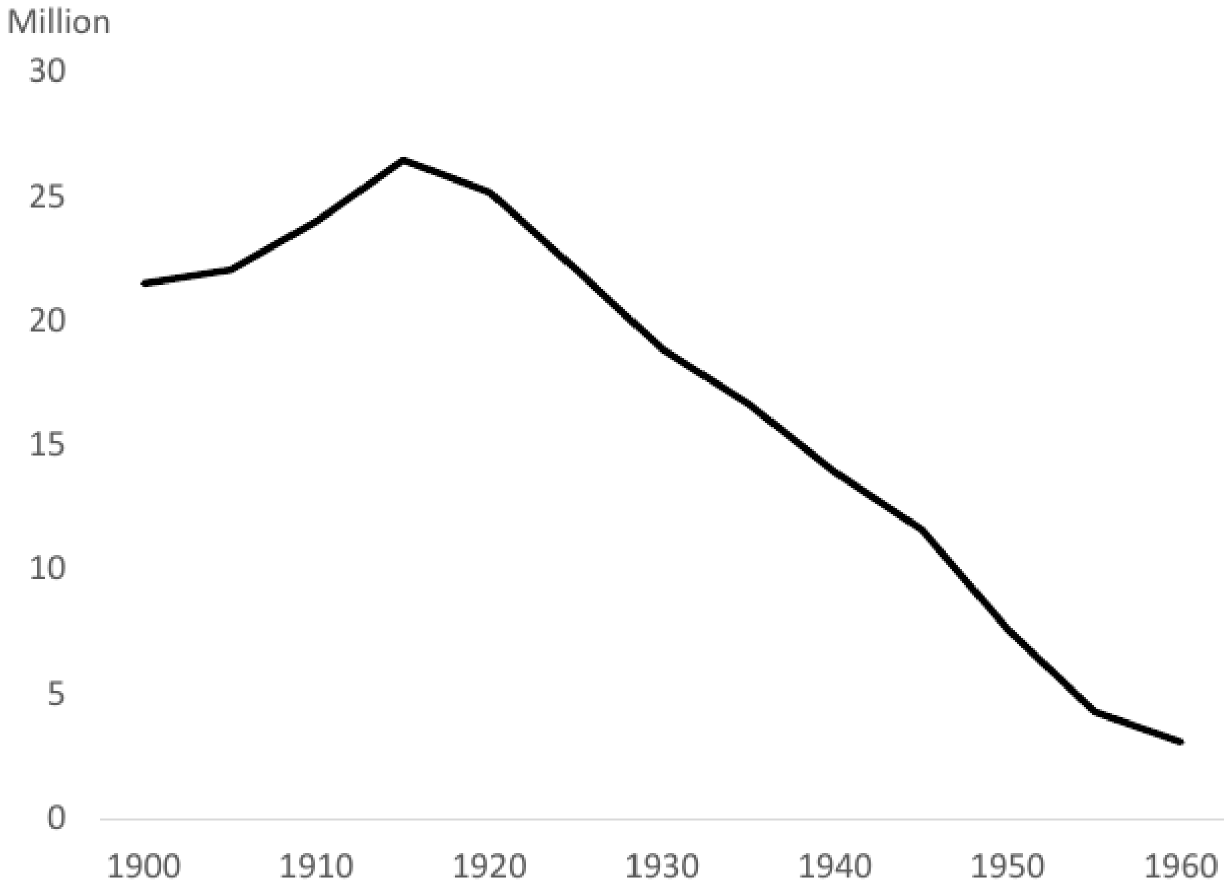


Figure 3: Number of Horses and Mules in the United States, 1900-1960

Is the Labor Share Falling?

- Closer looks reveals downward trend in labor share
- Are machines finally doing us in?

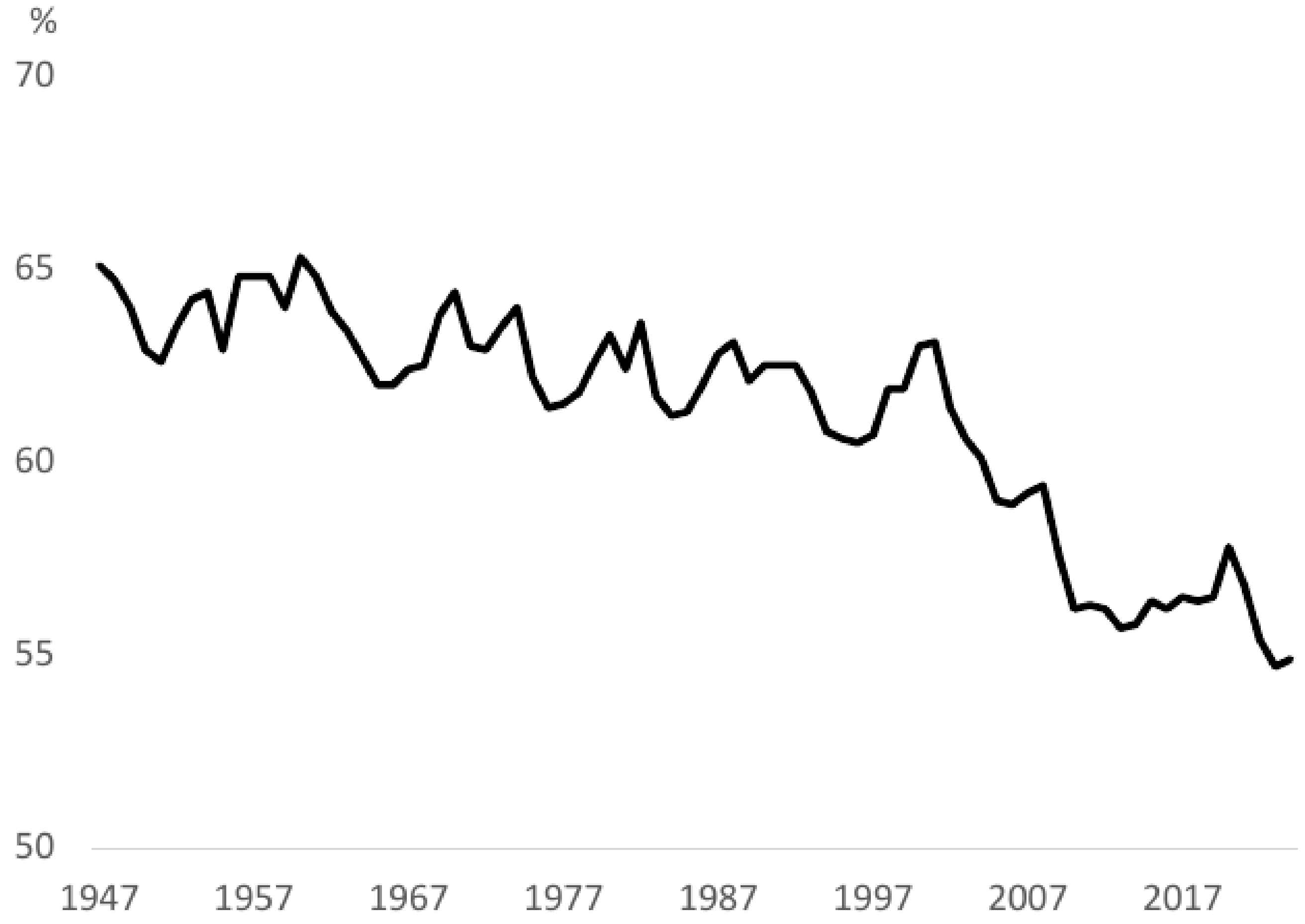


Figure 4: Labor Share in U.S. Nonfarm Business Sector, A Closer Look

Is the Labor Share Falling?

- Recent research has argued that apparent fall in labor share is due to mismeasurement
- Labor share using pre-1999 methods has not fallen
- Difference: Reclassification of intellectual property as capital
- But aren't the new methods better?

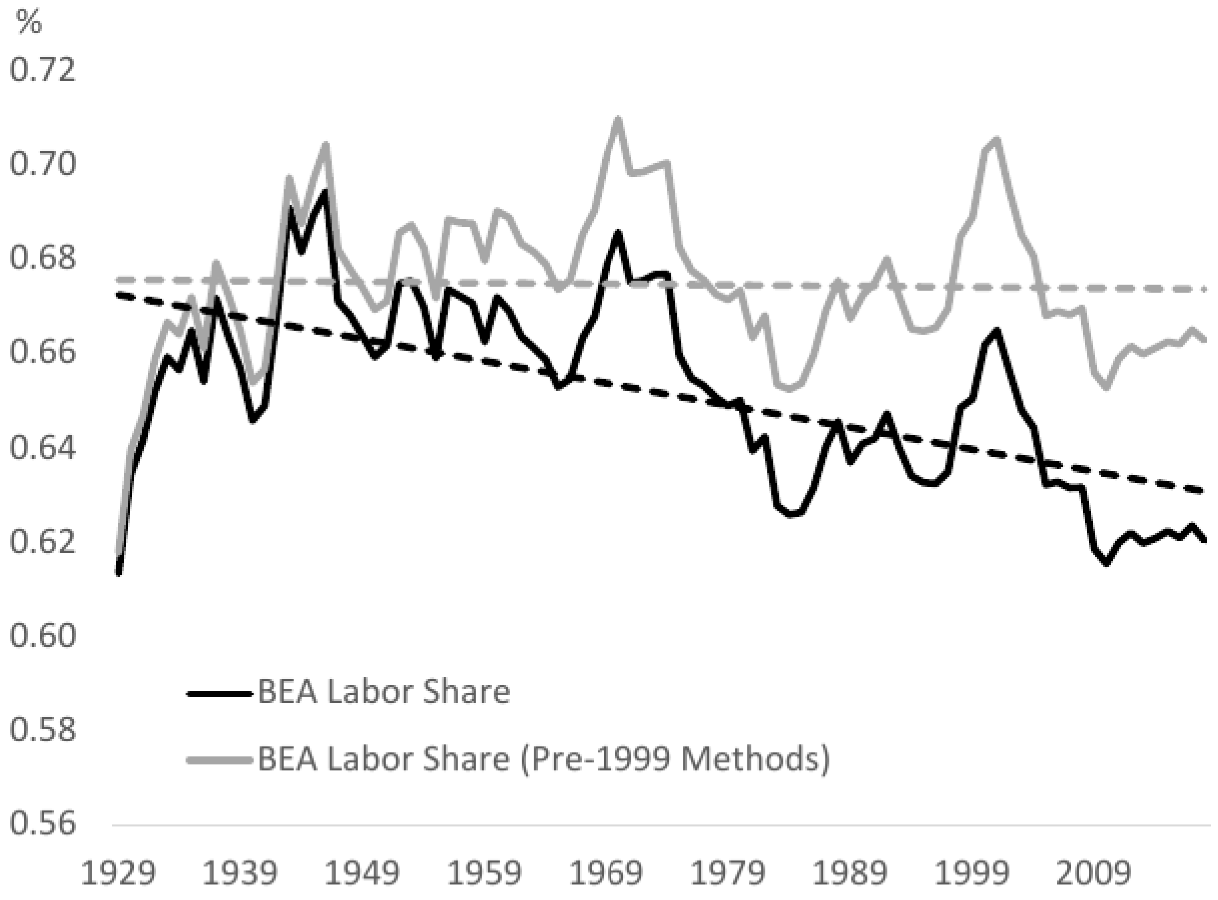


Figure 5: Economy-Wide U.S. Labor Share

Is Labor Share Falling?

- Barro (2021): GDP double counts investment
  - Once when built, again when it yields a service flow
- GDP used for many purposes, including business cycle measurement. Perhaps appropriate for that purpose
- We are interested in welfare of workers and owners of capital
  - How much they can afford to consume
- GDP not appropriate for this because of double-counting
  - Owners of capital must maintain capital (can’t consumer that part)
  - Must also build new capital (can’t consume that part)

Is Labor Share Falling?

- Net domestic product (NDP) more appropriate for labor share calculations
- If depreciation rises, difference between GDP and NDP increases
- Role of high depreciation capital has been increasing (e.g., software)
- Gross labor share falling, but not clear share of output available for consumption is falling

Better and Cheaper Machines?

- One reason why labor share may be falling: better or cheaper machines
- Doesn't work with Cobb-Douglas production function

$$Y = A(zK)^a L^{1-a}$$

- In this case:

$$r = az^a AK^{a-1} L^{1-a} = a\frac{Y}{K}$$

$$rK = aY$$

- Labor share independent of z

More General Production Function

- Turns out to be very knife-edge
- Constant Elasticity of Substitution (CES) Production function:

$$Y_t = \left[a\left(A_{K,t}K_t\right)^{\frac{\sigma-1}{\sigma}} + (a-1)\left(A_{L,t}L_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

- In this case, labor share is

$$s_{L,t} = 1 - a^{\sigma}\left(\frac{A_{K,t}}{r_t}\right)^{\sigma-1}$$

Labor Share in CES Case

$$s_{L,t} = 1 - a^{\sigma} \left(\frac{A_{K,t}}{r_t} \right)^{\sigma - 1}$$

- σ key parameter. If $\sigma = 1$ back in Cobb-Douglas case
- σ is elasticity of substitution between capital and labor
  - If price of capital falls by 1% relative to price of labor, quantity of capital used in production increases by σ% relative to quantity of labor
  - This number is 1 for Cobb-Douglas. But that is a special case.

Labor Share in CES Case

$$s_{L,t} = 1 - a^{\sigma} \left(\frac{A_{K,t}}{r_t} \right)^{\sigma - 1}$$

- $\sigma > 1$:
  - Increase in $A_{K,t}$ and decrease in r_t reduce the labor share (increase the capital share)
  - In this case, improvements in capital imply that firms use a lot more capital, increasing the capital share

Better and Cheaper Machines

- Machines have been getting cheaper (r_t has been falling)
- Perhaps this is why labor share has been falling?
- Only works if $\sigma > 1$
- But whether this is the case is not clear.

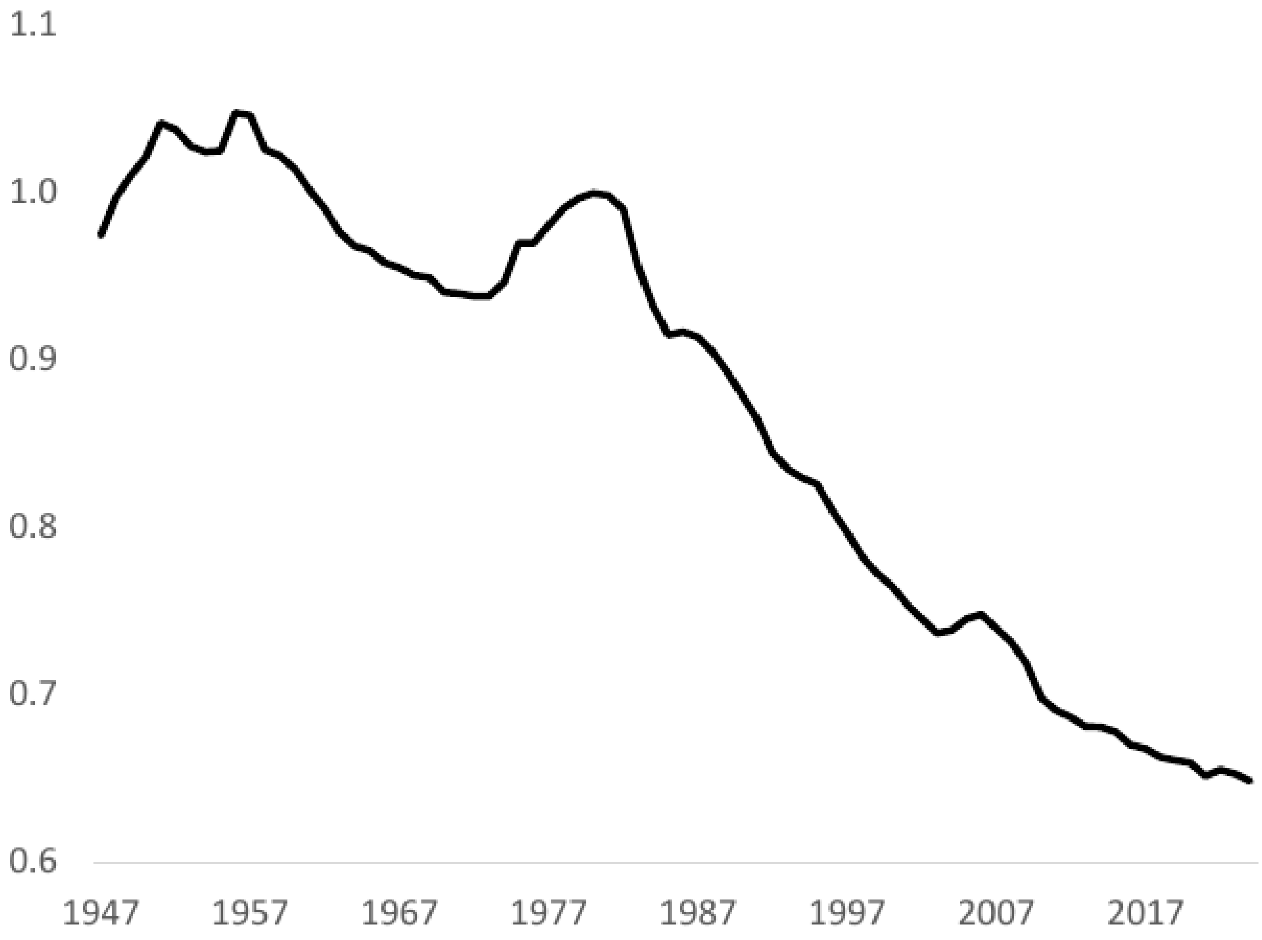


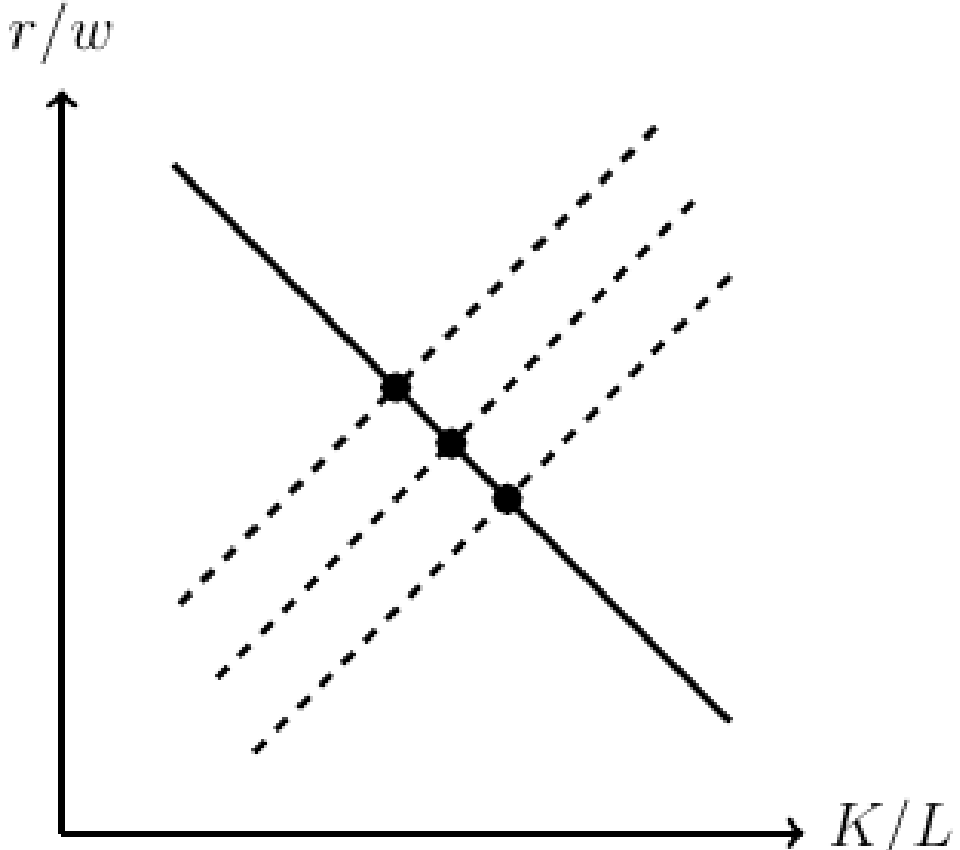
Figure 6: The Price of Investment Goods Relative to the Price of Output

Estimating σ is Hard

Two problems:

- Aggregate σ can be arbitrarily larger than firm-level σ (Houthakker, 1955)

- Simultaneous equations problem: Need pure variation in supply of L_t/K_t to estimate slow of relative demand curve



"Good" Variation