

**Advanced Microeconomics
(Economics 104)
Spring 2001
Repeated games**

The main idea (O 14.1, OR 8.1-2)

In the *Prisoner's Dilemma*

	<i>C</i>	<i>D</i>
<i>C</i>	3, 3	0, 4
<i>D</i>	4, 0	1, 1

no cooperation (D, D) is the unique *NE* since D strictly dominates C , but both players are better off when the outcome is (C, C) .

When played repeatedly, cooperation (C, C) in every period is stable if

- each player believes that choosing D will end cooperation, and
- subsequent losses outweigh the immediate gain.

Hence, the socially desirable outcome (C, C) can be sustained if players have long-term objectives.

So the question is what strategies support cooperation (C, C) ? Only (credible) punishments for choosing D .

In general, we can think that strategies are social norms, cooperation, threats and punishments where threats are carried out as punishments when the social norms require it.

Folk theorems give the set of payoffs (and not outcomes!) that can be sustained by equilibria.

The problem is that the set of equilibrium outcome is huge, so it lacks predictive power.

Strategies

Grim trigger strategy

$$\boxed{C : C} \xrightarrow{(\cdot, D)} \boxed{D : D}$$

Limited punishment

$$\begin{array}{ccccccc} \dashrightarrow & \boxed{\mathcal{P}_0 : C} & \longrightarrow & \boxed{\mathcal{P}_1 : D} & \longrightarrow & \boxed{\mathcal{P}_2 : D} & \longrightarrow & \boxed{\mathcal{P}_3 : D} & \dashrightarrow \\ & & & (\cdot, D) & & (\cdot, \cdot) & & (\cdot, \cdot) & & (\cdot, \cdot) \end{array}$$

Tit-for-tat

$$\dashrightarrow \boxed{C : D} \xrightarrow{(\cdot, D)} \boxed{D : D} \dashrightarrow (\cdot, C)$$

Payoffs

Suppose that each player's preferences over streams $(\omega^1, \omega^2, \dots)$ of payoffs are represented by the discounted sum

$$V = \sum_{t=1}^{\infty} \delta^{t-1} \omega^t,$$

where $0 < \delta < 1$.

The discounted sum of stream (c, c, \dots) is $c/(1-\delta)$, so a player is indifferent between the two streams if

$$c = (1 - \delta)V.$$

Hence, we call $(1-\delta)V$ the discounted average of stream $(\omega^1, \omega^2, \dots)$, which represent the same preferences.

Nash equilibria

Grim trigger strategy

$$(1 - \delta)(3 + \delta + \delta^2 + \dots) = (1 - \delta) \left[3 + \frac{\delta}{(1 - \delta)} \right] = 3(1 - \delta) + \delta$$

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1 - \delta) + \delta \leq 2,$$

or $\delta \geq 1/2$.

If $\delta \geq 1/2$, then the strategy pair in which each player's strategy is grim strategy is a Nash equilibrium which generates the outcome (C, C) in every period.

Limited punishment (k periods)

$$(1 - \delta)(3 + \delta + \delta^2 + \dots + \delta^k) = (1 - \delta) \left[3 + \frac{(1 - \delta^k)}{(1 - \delta)} \right] = 3(1 - \delta) + \delta(1 - \delta^k)$$

Note that after deviating at period t a player should choose D from period $t + 1$ through $t + k$.

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1 - \delta) + \delta(1 - \delta^k) \leq 2(1 - \delta^{k+1}).$$

Note that for $k = 1$, then no $\delta < 1$ satisfies the inequality.

Tit-for-tat

A deviator's best-reply to tit-for-tat is to alternate between D and C or to always choose D , so tit-for-tat is a best-reply to tit-for-tat if and only if

$$(1 - \delta)(3 + 0 + 3\delta^2 + 0 + \dots) = (1 - \delta) \frac{3}{1 - \delta^2} = \frac{3}{1 + \delta} \leq 2$$

and

$$(1 - \delta)(3 + \delta + \delta^2 + \dots) = (1 - \delta) \left[3 + \frac{\delta}{(1 - \delta)} \right] = 3 - 2\delta \leq 2.$$

Both conditions yield $\delta \geq 1/2$.

Subgame perfect equilibria

Grim trigger strategy

For the Nash equilibria to be subgame perfect, "threats" must be credible: punishing the other player if she deviates must be optimal.

Consider the subgame following the outcome (C, D) in period 1 and suppose player 1 adheres to the grim strategy.

Claim: It is not optimal for player 2 to adhere to his grim strategy in period 2.

If player 2 adheres to the grim strategy, then the outcome in period 2 is (D, C) and (D, D) in every subsequent period, so her discounted average payoff in the subgame is

$$(1 - \delta)(0 + \delta + \delta^2 + \dots) = \delta,$$

where as her discounted average payoff is 1 if she choose D already in period 2.

But, the "modified" grim trigger strategy for an infinitely repeated prisoner's dilemma

$$\boxed{C : C} \rightarrow \boxed{D : D} \\ (\cdot, \cdot) / (C, C)$$

is a subgame perfect equilibrium strategy if $\delta \geq 1/2$.

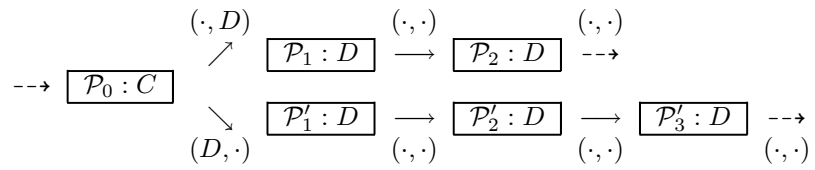
Limited punishment

The game does not have such subgame perfect equilibria from the same reason that a pair of grim strategies is never subgame perfect.

But, we can modify the limited punishment strategy in the same way that we modified the grim strategy to obtain subgame perfect equilibrium for δ sufficiently high.

The number of periods for which a player chooses D after a history in which not all the outcomes were (C, C) must depend on the identity of the deviator.

Consider the strategy of player 2, where the top part entails her reaction to her own deviation



Tit-for-tat

The optimality of tit-for-tat after histories ending in (C, C) is covered by our analysis of Nash equilibrium.

If both players adhere to tit-for-tat after histories ending in (C, D) : then the outcome alternates between (D, C) and (C, D) .

(The analysis is the same for histories ending in (D, C) , except that the roles of the players are reversed.)

Then, player 1's discounted average payoff in the subgame is

$$(1 - \delta)(3 + 3\delta^2 + 3\delta^4 + \dots) = \frac{3}{1 + \delta},$$

and player 2's discounted average payoff in the subgame is

$$(1 - \delta)(3\delta + 3\delta^3 + 3\delta^5 + \dots) = \frac{3\delta}{1 + \delta}.$$

Next, we check if tit-for-tat satisfies the one-deviation property of subgame perfection.

If player 1 (2) chooses C (D) in the first period of the subgame, and subsequently adheres to tit-for-tat, then the outcome is (C, C) ((D, D)) in every subsequent period. Such a deviation is profitable for player 1 (2) if and only if

$$\frac{3}{(1 + \delta)} \geq 2, \text{ or } \delta \leq 1/2$$

and

$$\frac{3\delta}{(1 + \delta)} \geq 1, \text{ or } \delta \geq 1/2,$$

respectively.

Finally, after histories ending in (D, D) , if both players adhere to tit-for-tat, then the outcome is (D, D) in every subsequent period.

On the other hand, if either player deviates to C , then the outcome alternates between (D, C) and (C, D) (see above).

Thus, a pair of tit-for-tat strategies is a subgame perfect equilibrium if and only if $\delta = 1/2$.