Advanced Microeconomics
(Economics 104)
Fall 2011
Strategic games II

Topics

- Existence of Nash equilibrium.
- Randomization.
- Mixed strategy.
- vNM preferences.
- Mixed strategy Nash equilibrium.
- Examples.
- Existence of mixed strategy Nash equilibrium.
- Dominance.

Recall that:

A strategic game is a triple $\langle N, (A_i), (\succsim_i)\rangle$ where N is a finite set of players, and

for each player $i \in N$

- a non-empty set A_i of actions
- a preference relation $\succsim_i$on the set $A = \times_{j\in N} A_j$ of possible outcomes.

When $\succsim_i$can be represented by a utility function $u_i : A \to \mathbb{R}$ a strategic game is a triple $\langle N, (A_i), (u_i)\rangle$.

Kakutani's fixed point theorem

Let

$X \subseteq \mathbb{R}^n$ non-empty compact (closed and bounded) and convex

$f : X \to X$ set-valued function for which

- the set $f(x)$ is non-empty and convex $\forall x \in X$.
- the graph of f is closed.
  $y \in f(x)$ for any $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ $\forall n$, $x_n \longrightarrow x$ and $y_n \longrightarrow y$.

Than, $\exists x^* \in X$ such that $x^* \in f(x^*)$.

Note that the concept of a closed graph is simply the usual notion of closedness relative to $X \times X$ applied to the set

$$\{(x, y) \in X \times X : y \in f(x)$$

Also, f is *upper hemicontinuous* if it has a closed graph and the images of compact sets are bounded.

Necessity of conditions in Kakutani's theorem

X is compact

- $X = \mathbb{R}^1$ and $f(x) = x + 1$

X is convex

- $X = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ and f is $90°$ clock-wise rotation.

$f(x)$ is convex for any $x \in X$

- $X = [0, 1]$ and

$$f(x) = \begin{cases} \{1\} & \text{if} \quad x < \frac{1}{2} \\ \{0, 1\} & \text{if} \quad x = \frac{1}{2} \\ \{0\} & \text{if} \quad x > \frac{1}{2} \end{cases}$$

f has a closed graph

- $X = [0, 1]$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$$

Existence of Nash equilibrium (OR 2.4)

The strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a NE if for all $i \in N$

- A_i - non-empty, compact, convex subset of the Euclidian space.
- $\succsim_i$- continuous and quasi-concave on A_i.

Proof.

- A_i is compact and $\succsim_i$is continuous $\implies$ $B_i(a_{-i}) \neq \emptyset$.
- $\succsim_i$is quasi-concave on A_i $\implies$ $B_i(a_{-i})$ is convex.
- $\succsim_i$is continuous $\implies$ B has a closed graph.

Then, B has a fixed point by Kakutani.

Mixed strategies

Suppose that, each player i can randomize among all her strategies so choices are not deterministic.

Then, we need to add theses specifications to the primitives of the model of strategic game.

A mixed strategy of player i is given by

$$\alpha_i \in \Delta(A_i)$$

where $\Delta(A_i)$ is the set of all probability distributions over A_i.

- A profile $(\alpha_i)_{i \in N}$ of mixed strategies induces a probability distribution over the set A so a mixed strategy profile is given by

$$\times_{i \in N} \Delta(A_i)$$

- Assuming independence, the probability of a pure action profile $a \in A$ is

$$\textstyle\prod_{i \in N} \alpha_i(a_i)$$

Randomization (O 4.1-4.2, OR 3.1)

Player i expected payoffs are given by a vNM utility function

$$U_i : \times_{j\in N}\Delta(A_j) \to \mathbb{R}$$

which represents player i's preferences over the set of lotteries over A.

– For any mixed strategy profile $\alpha = (\alpha_j)_{j\in N} \in \times_{j\in N}\Delta(A_j)$

$$U_i(\alpha) = \sum_{a\in A}(\prod_{j\in N}\alpha_j(a_j))u_i(a)$$

which is linear in α.

The mixed extension of a the strategic game

$$\langle N, (A_i), (u_i)\rangle$$

is the strategic game

$$\langle N, (\Delta(A_i)), (U_i)\rangle$$

Mixed strategy Nash equilibrium (O 4.3, OR 3.1-3.2)

A mixed strategy NE of a strategic game $\langle N, (A_i), (U_i)\rangle$ is a profile $\alpha^* \in \times_{i\in N}\Delta(A_i)$ of actions such that

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i})$$

$\forall i \in N$ and $\forall \alpha_i \in \Delta(A_i)$ where $U_i(\alpha)$ is player i's expect payoff to a mixed strategy profile α.

Examples

- Battle of the Sexes (BoS)

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

- Coordination Game

	B	S
B	$2, 2$	$0, 0$
S	$0, 0$	$1, 1$

- Hawk-Dove

	D	H
D	$3, 3$	$1, 4$
H	$4, 1$	$0, 0$

- Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Consider a 2×2 game and let the probability that player 1 assigns to her strategy T be p and hence she assigns probability $1-p$ to her strategy B.

Similarly, player 2 assigns probability q and $1-q$ to her strategies L and R respectively. Note that the probabilities of the optional outcomes are as follows:

$$\begin{aligned}
prob(T, L) &= pq, \\
prob(T, R) &= p(1-q), \\
prob(B, L) &= (1-p)q, \\
prob(B, R) &= (1-p)(1-q).
\end{aligned}$$

The notion of mixed strategies gives the following existence result.

If we admit mixed strategy as well as pure. Every finite player, finite strategy game has

- at least one Nash equilibrium
- an odd number of Nash equilibria.

Thus, for the class of games with a finite number of players and a finite number of strategies to each player, a Nash equilibrium always exists. This is given below.

Existence of mixed strategy Nash equilibrium

This result is an extension to the existence of pure strategy Nash equilibrium and given here just for completeness.

Every finite (the set of actions of each player is finite!) strategic game has a mixed strategy NE.

Proof.

- Take a mixed extension of a strategic game $\langle N, (\Delta(A_i)), (U_i)\rangle$.
- Let m_i be the number of $a_i \in A_i$ (pure strategies).
- Then, the set of player i's mixed strategies $\Delta(A_i)$ is given by

  $$\{(p_k)_{k=1}^{m_i} : \textstyle\sum_{k=1}^{m_i} p_k = 1 \text{ and } p_k \geq 0 \ \forall k\}$$

  which is non empty, convex and compact.
- vNM expected utility is linear probabilities so U_i is quasi-concave and continuous.

Therefore, the mixed extension has a NE by Kakutani's.

Calculating a mixed strategy Nash equilibrium in the *BoS*

Let p and q be the probabilities that player 1 and 2 respectively assign to the strategy *Game*. Player 2 will be indifferent between using her strategy B and S when player 1 assigns a probability p such that her expected payoffs from playing B and S are the same. That is,

$$\begin{gathered} 1p + 0(1-p) = 0p + 2(1-p) \\ p = 2 - 2p \\ p^* = 2/3 \end{gathered}$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy B and probability $1 - p^* = 1/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

Similarly, player 1 will be indifferent between using her strategy B and S when player 2 assigns a probability q such that her expected payoffs from playing B and S are the same. That is,

$$\begin{gathered} 2q + 0(1-q) = 0q + 1(1-q) \\ 2q = 1 - q \\ q^* = 1/3 \end{gathered}$$

Hence, when player 2 assigns probability $q^* = 1/3$ to her strategy B and probability $1 - q^* = 2/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

So, the *BoS* has two Nash equilibria in pure strategies $\{(B, B), (S, S)\}$ and one in mixed strategies $\{(2/3, 1/3)\}$.

- In terms of best responses

$$B_2(p) = \begin{cases} q = 1 & if \quad p > 2/3 \\ q \in [0, 1] & if \quad p = 2/3 \\ q = 0 & if \quad p < 2/3 \end{cases}$$

$$B_1(q) = \begin{cases} p = 1 & if \quad q > 1/3 \\ p \in [0, 1] & if \quad p = 1/3 \\ p = 0 & if \quad p < 1/3 \end{cases}$$

Two results on mixed strategy Nash equilibrium

Result 1

- A pure strategy NE of a strategic game is a NE of its mixed extension.
- The set of pure strategy NE of a strategic game is a subset of its set of mixed strategy NE.

Result 2

- A profile of mixed strategies is a NE iff for each player every pure strategy in the support of is a best response.
- Every action in the support of any player's NE mixed strategy yields the same payoff.

Dominance (O 4.4 OR 4.2)

An action $a_i \in A_i$ of player i is strictly dominated if there exists a mixed strategy α_i such that

$$U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$.

An action $a_i \in A_i$ of player i is weakly dominated if there exists a mixed strategy α_i such that

$$U_i(a_{-i}, \alpha_i) \geq U_i(a_{-i}, a_i)$$

for all $a_{-i} \in A_{-i}$ and

$$U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$$

for some $a_{-i} \in A_{-i}$.

Two results on dominated strategies

Result 1

- An action of a player in a finite strategic game is never-best response if and only if it is strictly dominated.

Result 2

- Consider a game G and a game G' obtained by iterated removal of all (weakly and strictly) dominated strategies. Than, any a which is a NE of G' is also a NE of G and the converse holds for the iterated removal of strictly dominated strategies.