Strategic games I

Sep 19, 2016
Side note 1: individual preferences

Consider some (finite) set of alternatives \((x, y, z, \ldots)\).

- Formally, we represent the decision-maker’s preferences by a binary relation \(\succsim\) defined on the set of consumption bundles.

- For any pair of bundles \(x\) and \(y\), if the decision-maker says that \(x\) is at least as good as \(y\), we write

\[
x \succeq y
\]

and say that \(x\) is weakly preferred to \(y\).

Bear in mind: economic theory often seeks to convince you with simple examples and then gets you to extrapolate. This simple construction works in wider (and wilder circumstances).
From the weak preference relation $\succeq$ we derive two other relations on the set of alternatives:

- Strict performance relation

  $x \succ y$ if and only if $x \succeq y$ and not $y \succeq x$.

  The phrase $x \succ y$ is read $x$ is \textit{strictly preferred} to $y$.

- Indifference relation

  $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$.

  The phrase $x \sim y$ is read $x$ is \textit{indifferent} to $y$. 
Side note II: individual rationality

Economic theory begins with two assumptions about preferences. These assumptions are so fundamental that we can refer to them as “axioms” of decision theory.

[1] Completeness

\[ x \succeq y \text{ or } y \succeq x \]

for any pair of bundles \( x \) and \( y \).

[2] Transitivity

\[ \text{if } x \succeq y \text{ and } y \succeq z \text{ then } x \succeq z \]

for any three bundles \( x, y \) and \( z \).
Together, completeness and transitivity constitute the formal definition of *rationality* as the term is used in economics. Rational economic agents are ones who

have the ability to make choices [1], and whose choices display a logical consistency [2].

(Only) the preferences of a rational agent can be represented, or summarized, by a *utility function*. 
Strategic games

A strategic game consists of

- a set of players (decision makers)
- for each player, a set of possible actions
- for each player, preferences over the set of action profiles (outcomes).

In strategic games, players move simultaneously. A wide range of situations may be modeled as strategic games.
A two-player (finite) strategic game can be described conveniently in a so-called bi-matrix.

For example, a generic $2 \times 2$ (two players and two possible actions for each player) game

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & A_1, A_2 & B_1, B_2 \\
B & C_1, C_2 & D_1, D_2 \\
\end{array}
\]

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2).
Applying the definition of a strategic game to the $2 \times 2$ game above yields:

- Players: $\{1, 2\}$

- Action sets: $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$

- Action profiles (outcomes):
  \[
  A = A_1 \times A_2 = \{(T, L), (T, R), (B, L), (B, R)\}
  \]

- Preferences: $\preceq_1$ and $\preceq_2$ are given by the bi-matrix.
Rock-Paper-Scissors
(over a dollar)

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<thead>
<tr>
<th></th>
<th>$R$</th>
<th>$P$</th>
<th>$S$</th>
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<tbody>
<tr>
<td>$R$</td>
<td>0,0</td>
<td>−1,1</td>
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<tr>
<td>$P$</td>
<td>1,−1</td>
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<td>$S$</td>
<td>−1,1</td>
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Each player’s set of actions is \{Rock, Paper, Scissors\} and the set of action profiles is 

\{RR, RP, RS, PR, PP, PS, SR, SP, SS\}.
In rock-paper-scissors

\[ PR \sim_1 SP \sim_1 RS \succ_1 PP \sim_1 RR \sim_1 SS \succ_1 PS \sim_1 SR \sim_1 PS \]

and

\[ PR \sim_2 SP \sim_2 RS \prec_2 PP \sim_2 RR \sim_2 SS \prec_2 PS \sim_2 SR \sim_2 PS \]

This is a zero-sum or a strictly competitive game.
Classical $2 \times 2$ games

- The following simple $2 \times 2$ games represent a variety of strategic situations.

- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.

- These classical games “span” the set of almost all games (strategic equivalence).
Game I: Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Work</th>
<th>Goof</th>
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<tbody>
<tr>
<td><strong>Work</strong></td>
<td>3, 3</td>
<td>0, 4</td>
</tr>
<tr>
<td><strong>Goof</strong></td>
<td>4, 0</td>
<td>1, 1</td>
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A situation where there are gains from cooperation but each player has an incentive to “free ride.”

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.
Game II: Battle of the Sexes (BoS)

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<tr>
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<th>Ball</th>
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<tbody>
<tr>
<td>Ball</td>
<td>2,1</td>
<td>0,0</td>
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<tr>
<td>Show</td>
<td>0,0</td>
<td>1,2</td>
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Like the Prisoner’s Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.
Game III-V: Coordination, Hawk-Dove, and Matching Pennies

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<tr>
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<td><strong>Show</strong></td>
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<table>
<thead>
<tr>
<th></th>
<th>Dove</th>
<th>Hawk</th>
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<tr>
<td><strong>Dove</strong></td>
<td>3,3</td>
<td>1,4</td>
</tr>
<tr>
<td><strong>Hawk</strong></td>
<td>4,1</td>
<td>0,0</td>
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<th>Head</th>
<th>Tail</th>
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<tr>
<td><strong>Head</strong></td>
<td>1,−1</td>
<td>−1,1</td>
</tr>
<tr>
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<td>−1,1</td>
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Best response and dominated actions

Action $T$ is player 1’s *best response* to action $L$ player 2 if $T$ is the optimal choice when 1 *conjectures* that 2 will play $L$.

Player 1’s action $T$ is *strictly* dominated if it is never a best response (inferior to $B$ no matter what the other players do).

In the Prisoner’s Dilemma, for example, action *Work* is strictly dominated by action *Goof*. As we will see, a strictly dominated action is not used in any Nash equilibrium.
An action \( a_i \in A_i \) of player \( i \) is *strictly dominated* if there exists an action \( a_i' \neq a_i \) such that
\[
u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})
\]
or equivalently,
\[
(a_i, a_{-i}) \prec_i (a_i', a_{-i})
\]
for all \( a_{-i} \in A_{-i} \).

One interesting result on dominated strategies is that an action of a player (in a finite strategic game) is never a best response if and only if it is strictly dominated.
An action \( a_i \in A_i \) of player \( i \) is weakly dominated if there exists an action \( a_i' \neq a_i \) such that

\[
u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i}) \quad \text{or equivalently} \quad (a_i, a_{-i}) \succeq_i (a_i', a_{-i})
\]

for all \( a_{-i} \in A_{-i} \) and

\[
u_i(a_i, a_{-i}) < u_i(a_i', a_{-i}) \quad \text{or equivalently} \quad (a_i, a_{-i}) \prec_i (a_i', a_{-i})
\]

for some \( a_{-i} \in A_{-i} \).
Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a NE is a set of actions such that all players are doing their best given the actions of the other players.
Nash equilibrium ($NE$) is a steady state of the play of a strategic game. Formally, a $NE$ of a strategic game is a profile $a^* \in A$ of actions such that

$$u_i(a^*_i, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$$

or equivalently,

$$(a^*_i, a_{-i}^*) \succeq_i (a_i, a_{-i}^*)$$

$\forall i \in N$ and $\forall a_i \in A_i$.

In words, no player has a profitable deviation given the actions of the other players.
Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

\[
\begin{array}{ccc}
 & q & 1-q \\
T & pq & p(1-q) \\
L & & \\
B & (1-p)q & (1-p)(1-q) \\
\end{array}
\]

Let \( p \) and \( q \) be the probabilities that player 1 and 2 respectively assign to the strategy \( \text{Ball} \).
Player 2 will be indifferent between using her strategy $B$ and $S$ when player 1 assigns a probability $p$ such that her expected payoffs from playing $B$ and $S$ are the same. That is,

$$1p + 0(1 - p) = 0p + 2(1 - p)$$

$$p = 2 - 2p$$

$$p^* = 2/3$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy $B$ and probability $1 - p^* = 1/3$ to her strategy $S$, player 2 is indifferent between playing $B$ or $S$ any mixture of them.
Similarly, player 1 will be indifferent between using her strategy \( B \) and \( S \) when player 2 assigns a probability \( q \) such that her expected payoffs from playing \( B \) and \( S \) are the same. That is,

\[
2q + 0(1 - q) = 0q + 1(1 - q) \\
2q = 1 - q \\
q^* = 1/3
\]

Hence, when player 2 assigns probability \( q^* = 1/3 \) to her strategy \( B \) and probability \( 1 - q^* = 2/3 \) to her strategy \( S \), player 2 is indifferent between playing \( B \) or \( S \) any mixture of them.
In terms of best responses:

\[ B_1(q) = \begin{cases} 
  p = 1 & \text{if } q > 1/3 \\
  p \in [0, 1] & \text{if } q = 1/3 \\
  p = 0 & \text{if } q < 1/3 
\end{cases} \]

\[ B_2(p) = \begin{cases} 
  q = 1 & \text{if } p > 2/3 \\
  q \in [0, 1] & \text{if } p = 2/3 \\
  q = 0 & \text{if } p < 2/3 
\end{cases} \]

The BoS has two Nash equilibria in pure strategies \{ (B, B), (S, S) \} and one in mixed strategies \{ (2/3, 1/3) \}. In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).