University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
Fall 2025

Properties of preferences
(Kreps Ch. 2 and Rubinstein Ch. 4 w/o differentiability)

Sep 9 and 11

The roadmap

$\succsim$		u
monotone	$\Longrightarrow$	nondecreasing
strongly monotone	$\Longrightarrow$	strictly increasing
continuous	$\Longrightarrow$	continuous (Debreu's Theorem)
convex	$\Longrightarrow$	quasi-concave (but not concave)
strictly convex	$\Longrightarrow$	strictly concave (and strictly quasi-concave)
homothetic (and continuous)	$\Longrightarrow$	continuous and homogeneous
(so-called) quasi-linear	$\Longrightarrow$	quasi-linear
(so-called) differentiable	$\Longrightarrow$	differentiable
separable	$\Longrightarrow$	separable (form)
strongly separable	$\Longrightarrow$	additively separable (form)

$\Longrightarrow$ If $\succsim$ are monotone then all u-representations are nondecreasing, but $\succsim$ are monotone is implied if only some u-representations are nondecreasing.

Next we discuss a “special case” of a $\mathcal{DM}$ – a consumer who makes choices between combinations of commodities (bundles).

> Rubinstein: “... I have a certain image in mind: my late mother going to the marketplace with money in hand and coming back with a shopping bag full of fruit and vegetables...”

A less abstract set of choices $X = \mathbb{R}_+^K$ — a bundle $x \in X$ is a combination of K commodities where $x_k \geq 0$ is the quantity of commodity k.

Classical (well-behaved) preferences

We impose some restrictions on $\succsim$ in addition to completeness, transitivity and reflexivity.

An additional three "classical" restrictions/conditions based on the mathematical structure of $X = \mathbb{R}_+^K$ are:

monotonicity + continuity + convexity

We refer to the map of indifference curves $\{y | \, y \sim x\}$ for some x demonstrating such $\succsim$ as well-behaved.

Monotonicity

(more is better...)

Increasing the amount of some x_k is preferred and increasing the amount of all x_k is strictly preferred:

- $\succsim$ satisfies *monotonicity* if for all $x, y \in X$ and for all k

$$\text{if } x_k \geq y_k \implies x \succsim y \text{ and if } x_k > y_k \implies x \succ y.$$

- $\succsim$ satisfies *strong monotonicity* if for all $x, y \in X$ and for all k

$$\text{if } x_k \geq y_k \text{ and } x \neq y \implies x \succ y.$$

Leontief preferences $\min\{x_1, ..., x_k\}$ satisfy monotonicity but not strong monotonicity.

– $\succsim$ satisfies *local nonsatiation* if for all $y \in X$ and every $\varepsilon > 0$, there is $x \in X$ such that

$$\|x - y\| \leq \varepsilon \text{ and } x \succ y.$$

A thick indifference set violates local nonsatiation. Show the following:

strong monotonicity $\Longrightarrow$ monotonicity $\Longrightarrow$ local nonsatiation.

Continuity

We will use the topological structure of $\mathbb{R}^K_+$ (with a standard distance function) in order to apply the definition of continuity:

For any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n \; \forall \; n$,

$$x = \lim_{n\to\infty} x_n \text{ and } y = \lim_{n\to\infty} y_n$$

we have $x \succsim y$. That is, $\succsim$ on X is *continuos* if it preserved under limits.

Debreu's Theorem: Any continuous $\succsim$ is represented by some continuous u. If we also assume monotonicity, then have a simple/elegant proof.

Proof:

- We show that for every bundle x, there is a bundle on the diagonal $(t, .., t)$ for $t \geq 0$ such that the $\mathcal{DM}$ is indifferent between that bundle and the x:

$$(\max_k\{x_k\}, ..., \max_k\{x_k\}) \succsim x \succsim (0, ..., 0)$$

so (by continuity) there is a bundle on the main diagonal that is indifferent to x and (by monotonicity) this bundle is unique.

Denote this bundle by $(t(x), ..., t(x))$ and let $u(x) = t(x)$ and note that

$$\begin{array}{rcl} x & \succsim & y \\ & \Updownarrow & \\ (t(x), ..., t(x)) & \succsim & (t(y), ..., t(y)) \\ & \Updownarrow & \\ t(x) & \geq & t(y). \end{array}$$

where the 2nd $\Updownarrow$ is by monotonicity.

To show that u is continuous, let (x^n) be a sequence such that $x = \lim_{n\to\infty} x_n$ and assume (towards contradiction) that $t(x) \neq \lim_{n\to\infty} t(x_n)$ but there is nothing 'elegant' in this part...

Convexity

$\succsim$ on X is *convex* if for every $x \in X$ the upper counter set

$$\{y \in X : y \succsim x\}$$

is convex – if $y \succsim x$ and $z \succsim x$ then $\alpha y + (1 - \alpha) z \succsim x$ for any $\alpha \in [0, 1]$.

(1) $\succsim$ is convex if

$$x \succsim y \Longrightarrow \alpha x + (1 - \alpha) y \succsim y \text{ for any } \alpha \in (0, 1).$$

(2) $\succsim$ is convex if for any $x, y, z \in X$ such that $z = \alpha x + (1 - \alpha) y$ for some $\alpha \in (0, 1)$

$$z \succsim x \text{ or } z \succsim y.$$

In words,

(1) If $x \succsim y$, then "going only part of the way" from y to x is also an improvement over y.

(2) If z is "between" x and y, then it is impossible that both $x \succ z$ and $y \succ z$.

$\succsim$ on X is *strictly convex* if for every $x, y, z \in X$ and $y \neq z$ we have that

$$y \succsim x \text{ and } z \succsim x \Longrightarrow \alpha y + (1-\alpha)z \succ x \text{ for any } \alpha \in (0,1).$$

Concavity and quasi-concavity:

u is concave if for all x, y and $\lambda \in [0, 1]$ we have

$$u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$$

and it is quasi-concave if for all $y \in X$

$$\{x \in X : u(x) \geq u(y)\}$$

is convex. Any function that is concave is also quasi-concave.

If $x \succsim y \Leftrightarrow u(x) \geq u(y)$ then

$$\succsim \text{ is convex}$$
$$\Updownarrow$$
$$u \text{ is quasi-concave}$$

but $\succsim$ is convex does not imply that u is concave, for example if $X = \mathbb{R}$

$$x \succsim y \text{ if } x \geq y \text{ or } y < 0.$$

Back to the roadmap...

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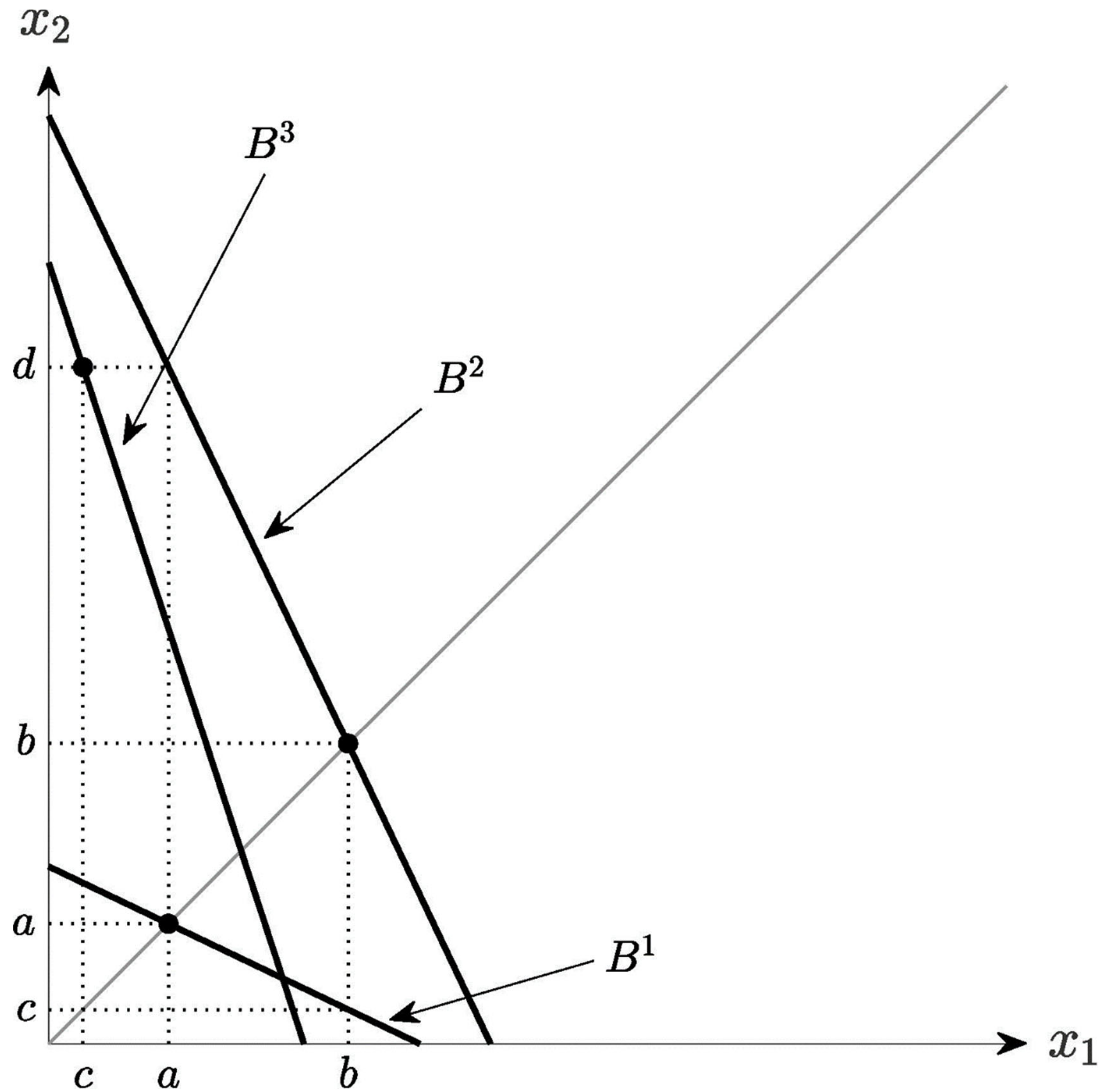
Should we go beyond the basic properties?!

"I can tell you of an important new result I got recently. I have what I suppose to be a completely general treatment of the revealed preference problem..." – A letter from Sydney Afriat to Oskar Morgenstern, 1964.

Afriat's Theorem The following conditions are equivalent: (i) The data satisfy GARP. (ii) There exists u that rationalizes the data. (iii) There exists a continuous, increasing, concave u that rationalizes the data.

$\Longrightarrow$ We <u>should</u> assume that $\succsim$ satisfy (some versions of) monotonicity, continuity, and convexity and will refer to a $\mathcal{DM}$ with such well-behaved $\succsim$ as a "classical consumer."

A simple violation of EU-rationalizability



EU requires that

$$
\begin{aligned}
U(a,a) &= 2u(a) \geq u(b) + u(c) \\
U(b,b) &= 2u(b) \geq u(a) + u(d)
\end{aligned}
$$

b/c $(a,a)R^D(b,c)$ and $(b,b)R^D(a,d)$.

But rearranging yields

$$u(a) + u(b) \geq u(c) + u(d)$$

which contradicts that $(c,d)R^D(a,b)$.

Rubinstein's view:

– "... the reason for abandoning the "generality" of the classical consumer is because empirically we observe only certain kinds of consumers who are described by special classes of preferences..."

– "... stronger assumptions are needed in economic models in order to make them interesting models, just as an engaging story of fiction cannot be based on a hero about which the reader knows very little..."

I beg to disagree...

Economics and consumer behavior

ANGUS DEATON and
JOHN MUELLBAUER

Homotheticity

$\succsim$ are homothetic if $x \succsim y \Longrightarrow$ that $\alpha x \succsim \alpha y$ for all $\alpha \geq 0$. A $\succsim$ on X that admits a u-representation that is homogeneous of degree λ

$$u(\alpha x) = \alpha^{\lambda} u(x) \text{ for all } x > 0.$$

is homothetic.

For any degree λ

$$\begin{aligned} x \succsim y &\iff u(x) \geq u(y) \\ &\iff \alpha^{\lambda} u(x) \geq \alpha^{\lambda} u(y) \\ &\iff u(\alpha x) \geq u(\alpha y) \qquad \iff \alpha x \succsim \alpha y \end{aligned}$$

Any homothetic, continuous, and <u>monotonic</u> $\succsim$ on X can be represented by a continuous utility u that is homogeneous of degree one.

We have already proved that for any $x \in X$

$$x \sim (t(x), ..., t(x))$$

and that the function $u(x) = t(x)$ is a continuous u-representation of $\succsim$. Because $\succsim$ are homothetic

$$\alpha x \sim (\alpha t(x), ..., \alpha t(x))$$

and therefore

$$u(\alpha x) = \alpha t(x) = \alpha u(x).$$

Quasi-linearity

$\succsim$ on X is quasi-linear in x_1 (the "numeraire" good) if

$$x \succsim y \Longrightarrow (x + \varepsilon e_1) \succsim (y + \varepsilon e_1)$$

where $e_1 = (1, 0, ..., 0)$ and $\varepsilon > 0$. The indifference curves of $\succsim$ that are quasi-linear in x_1 are parallel to each other (relative to the x_1-axis).

A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}_+^{K-1}$ is quasi-linear in x_1 if and only if it admits a u-representation of the form

$$u(x) = x_1 + v(x_{-1}).$$

Proof: Assume that $\succsim$ is also strongly monotonic and the following lemma (which you should prove):

- If $\succsim$ is strongly monotonic, continuous, quasi-linear in x_1 then for any (x_{-1}) there is a number $v(x_{-1})$ such that

$$(v(x_{-1}), 0, ..., 0) \sim (0, x_{-1}).$$

- By quasi-linearity in x_1

$$(x_1 + v(x_{-1}), 0, ..., 0) \sim (x_1, x_{-1}).$$

  and by strong monotonicity (in x_1), $u(x) = x_1 + v(x_{-1})$ represents $\succsim$.

If $\succsim$ is strongly monotonic, continuous, quasi-linear in $x_1, ..., x_K$ then it admits a linear u-representation

$$u(x) = \alpha_1 x_1 + \cdots + \alpha_K x_K.$$

Proof (for $K = 2$): We need to show that $v(a+b) = v(a) + v(b)$ for all a and b:

– By the definition of v

$$v(0, a) \sim (v(a), 0) \text{ and } v(0, b) \sim (v(b), 0)$$

and by quasi-linearity in x_1 and x_2

$$(v(b), a) \sim (v(a) + v(b), 0) \text{ and } (v(b), a) \sim (0, a + b).$$

– Thus,

$$(v(a)+v(b),0) \sim (0,a+b) \Longrightarrow v(a+b)=v(a)+v(b).$$

– Let $v(1)=c$. Then, for any natural numbers m and n we have

$$v(\frac{m}{n}) = c\frac{m}{n}.$$

Since $v(0)=0$ and v is an increasing function, $v(x)=cx$.

! Note: w/out monotonicity, Cauchy's functional equation—$v(a+b) = v(a)+v(b)$—can be satisfied also by nonlinear functions.

Just separability...
(not weak vs. additive)

$\succsim$ satisfies *separability* if for any x_i

$$\begin{array}{ccc}(x_i, x_{-i}) & \succsim & (x'_i, x_{-i}) \\ & \Updownarrow & \\ (x_i, x'_{-i}) & \succsim & (x'_i, x'_{-i}).\end{array}$$

Such $\succsim$ admits an additive u-representation

$$u(x) = v_1(x_1) + \cdots + v_K(x_K).$$

A common assumption used in demand analysis that allows for a clear demarcation (see R4 problem 6) $\Longrightarrow$ two-stage bundling in demand analysis...

What about differentiability?

It is often (always?) assumed in empirical work that u is differentiable...

... but what are 'differentiable' preferences?!