University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
Fall 2024

Choice under uncertainty I
von Neumann-Morgenstern expected utility
(Kreps Ch. 5 and Rubinstein Ch. 7)

Oct 8, 2025

Formulation and representation

We began with $\succsim$ and induce $\succ$ and $\sim$. We will now take $\succ$ as primitive (what the $\mathcal{DM}$ expresses) but it makes (almost) no difference... We say that $\succ$ is a preference relation if it is

- asymmetric: there is no pair $x, y \in X$ such that $x \succ y$ and $x \prec y$.
- negatively transitive: if $x \succ y$ then for any $z \in X$ either $x \succ z$ or $z \succ y$, or both.

$\succ$ is a preference relation on a set X if and only if there is a function $u : X \to \mathbb{R}$ representing $\succ$ in the sense of

$$x \succ y \Leftrightarrow u(x) > u(y).$$

If $\succ$ is a preference relation (asymmetric and negatively transitive) on a set X then it is

- <u>irreflexive</u>: $x \succ x$ for no $x \in X$.
- <u>transitive</u>: $x \succ y$ and $y \succ z \Rightarrow x \succ z$.
- <u>acyclic</u>: $x_1 \succ x_2, x_2 \succ x_3, ..., x_{n-1} \succ x_n \Rightarrow x_1 \neq x_n$.

Uncertain prospects

The goal: X to represent uncertain prospects and to 'specialize' the form of the u-function representing $\succ$ by imposing further conditions on $\succ$ based on the (mathematical) structure of X.

Question How (mathematically) do we model an uncertain prospect and what corresponding forms of functions u should we seek?

The literature contains (basically) three sets of answers to these questions, differing in whether uncertainty is objective or subjective.

(1) and (2) are polar cases and (3) is a middle case:

1. Objective uncertainty: von Neumann-Morgenstern (vNM).
2. Subjective uncertainty: Savage.
3. Horse lottery-roulette wheel theory: Anscombe and Aumann (A-A).

In A-A, the $\mathcal{DM}$ is assumed to have some objective randomizing devices—fair coins, color wheels, roulette wheels, etc.—that s/he can employ to represent her/his subjective uncertainty.

! Understand how these models differ as representations of uncertain prospects and to think why/when one might be a more appropriate model...

Moto Moto learned A-A…

vNM expected utility
(with finite prize spaces)

In the vNM model uncertainty is objective—there is given a quantification of how likely the various outcomes are (given in the form of a probability distribution).

- A given set X of prizes/consequences.
- A set P of probability measures or probability distributions on X.

! P is the choice set—the $\mathcal{DM}$ is choosing/expressing preference among probability distributions.

!! All that matters to the $\mathcal{DM}$ are probabilities and prizes—the randomizing devices and their order are inconsequential.

We take on the easiest case, where the set of possible prizes X is a finite set and P is the set of functions $P : X \to [0, 1]$ such that

$$\sum_{x \in X} p(x) = 1.$$

The $\mathcal{DM}$ is presumed to be making pairwise comparisons between members of P, indicating strict preference by the binary relation $\succ$.

! When X is infinite (countable or not) P can be the set of all simple or discrete (or more complicated) probability measures (mixture-space theorem).

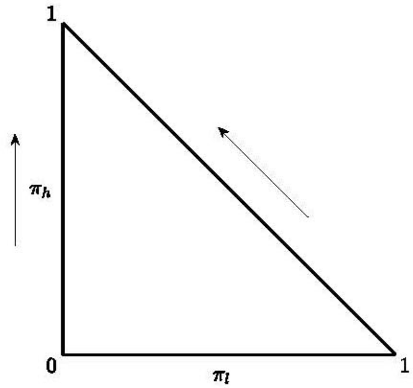
A compound lottery: If $p, q \in P$ and $\alpha \in [0,1]$ then there is an element

$$\alpha p + (1-\alpha)q \in P$$

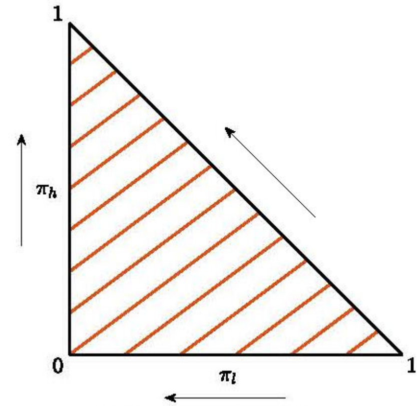
which is defined by taking the convex combinations of the probabilities of each prize separately, or

$$(\alpha p + (1-\alpha)q)(x) = \alpha p(x) + (1-\alpha)q(x)$$

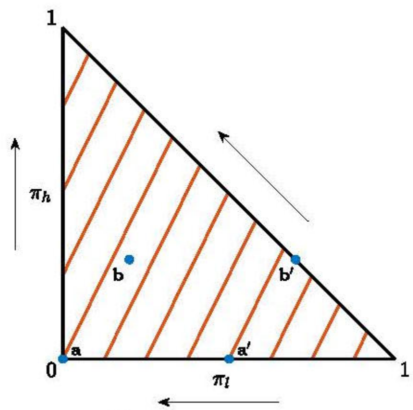
so $\alpha p + (1-\alpha)q$ represents a compound lottery.



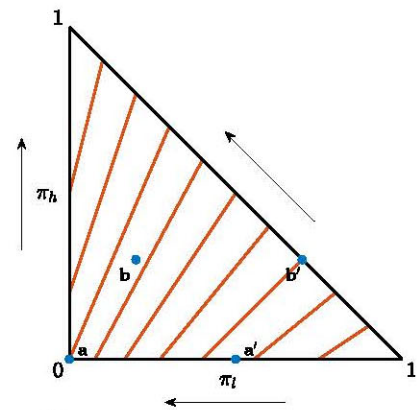
(a) Lottery Space



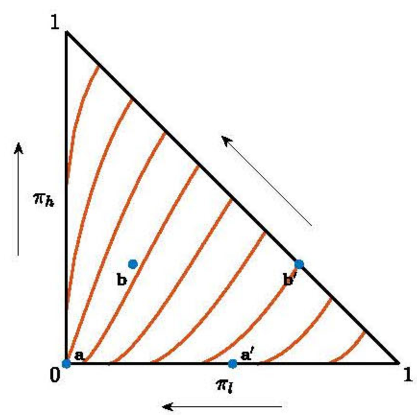
(b) Expected Utility



(c) Allais Paradox



(d) Weighted Expected Utility



(e) Rank-Dependent Utility

Three axioms (about $\succ$ on P)

(A1) $\succ$ is a preference relation (asymmetric and negatively transitive).

(A2) Independence: For all $p, q, r \in P$ and $\alpha \in [0, 1]$

$$p \succ q \Rightarrow \alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r.$$

This is the independence (or substitution) axiom—a normatively compelling principle for choice under uncertainty.

(A3) Continuity: For all $p, q, r \in P$, if $p \succ q \succ r$ then there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha p + (1 - \alpha) r \succ q \succ \beta p + (1 - \beta) r.$$

This is also called the Archimedean axiom—'resemblance' to the Archimedes' principle:

for all $0 < x < y$ there is (an integer) n such that $nx > y$.

? What if p, q and r are (respectively) \$10,000, \$1,000 and death (for sure)?!

Regardless of how you feel about (A1)-(A3), together these axioms yield the following result:

Theorem (vNM): $\succ$ on P satisfies axioms (A1)-(A3) if and only if there exists a function $u : X \longrightarrow \mathbb{R}$ such that

$$p \succ q \Leftrightarrow \sum_x p(x)u(x) > \sum_x q(x)u(x). \qquad (*)$$

And u is unique up to a positive affine transformation: if u represents $\succ$ in the sense of $(*)$, then $u' : X \longrightarrow \mathbb{R}$ also represents $\succ$ if and only if there exist real numbers $c > 0$ and d such that

$$u'(\cdot) = cu(\cdot) + d.$$

Two remarks:

(1) u-representations are unique up to strictly increasing rescalings: if u represents $\succ$ then so will $v(\cdot) = f(u(\cdot))$ for any strictly increasing f.

But if f is an arbitrary increasing function, then the v that results from composing u with f may not have the expected-utility form.

(2) Continuing on this general point, we said that there is no cardinal significance in utility differences.

But in the context of expected utility where u gives an expected-utility representation on P, utility differences have cardinal significance.

To illustrate point (2), suppose

$$u(x) - u(x'') = 2(u(x') - u(x'')) > 0$$

which does not mean that x is twice better than x'' than is x'—it just means that $x \succ x' \succ x''$.

But when u gives an expected-utility representation on P, utility differences have cardinal significance

$$p := \begin{array}{l} \overset{1/2}{\nearrow} \quad x \\ \\ \underset{1/2}{\searrow} \quad x'' \end{array} \sim \quad q := \xrightarrow{1} \quad x'$$

Three lemmas
(and another lemma...)

How is the vNM theorem proven? We first use (A1)-(A3) to obtain three lemmas... If $\succ$ on P satisfies (A1)-(A3) then:

(L1) $p \succ q$ and $0 \leq \alpha < \beta \leq 1 \Rightarrow \beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$.

- If we look at (binary) compounded lotteries, the $\mathcal{DM}$ always (strictly) prefers a higher probability of "winning" the preferred lottery.

(L2) $p \succsim q \succsim r$ and $p \succ r \Rightarrow$ there exists a unique $\alpha^* \in [0,1]$ such that

$$q \sim \alpha^* p + (1-\alpha^*) r.$$

- This result (sometimes simply assumed) is called the calibration property—calibrate the $\mathcal{DM}$'s preference for any lottery in terms of a lottery that involves only the best and worst prizes.

- By (L1), we know there is exactly one value α^* that will do in (L2). This is what causes (A3) to be called a continuity axiom—the preference ordering is continuous in probability.

(L3) $p \sim q$ and $\alpha \in [0,1] \Rightarrow$ for all $r \in P$

$$\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r.$$

– This is just like the independence axiom (A2), except that $\succ$ is replaced by $\sim$. This is sometimes assumed as an axiom (called the substitution axiom).

A sketch of the (L1)-(L3) proofs

(L1) $p \succ q$ and $0 \leq \alpha < \beta \leq 1 \Rightarrow \beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$.

Proof: Let $r = \beta p + (1-\beta)q$. If $\alpha = 0$ then $p \succ q$ and $0 < \beta \leq 1$ with (A2) imply

$$\begin{aligned} r &= \beta p + (1-\beta)q \\ &\succ \beta q + (1-\beta)q \\ &= q \\ &= \alpha p + (1-\alpha)q. \end{aligned}$$

Suppose $\alpha > 0$ so $0 < \dfrac{\alpha}{\beta} < 1$, and $r \succ q$ and (A2) imply

$$\begin{aligned} r &= (1 - \frac{\alpha}{\beta})r + \frac{\alpha}{\beta}r \\ &\succ (1 - \frac{\alpha}{\beta})q + \frac{\alpha}{\beta}r \\ &= (1 - \frac{\alpha}{\beta})q + \frac{\alpha}{\beta}(\beta p + (1-\beta)q) \\ &= \alpha p + (1-\alpha)q. \blacksquare \end{aligned}$$

(L2) $p \succsim q \succsim r$ and $p \succ r \Rightarrow$ there exists a unique $\alpha^* \in [0, 1]$ such that

$$q \sim \alpha^* p + (1 - \alpha^*) r.$$

Proof: Since $p \succ r$, (L1) ensures that if α^* exists it is unique. If $p \sim q$ (resp. $q \sim r$) then $\alpha^* = 1$ (resp. $\alpha^* = 0$) works.

Hence, we only need to consider the case $p \succ q \succ r$. Define

$$\alpha^* = \sup\{\alpha \in [0, 1] : q \succsim \alpha p + (1 - \alpha) r\}$$

(since $\alpha = 0$ is in the set, we are not tacking a **sup** over an empty set...) Assuming,

$$\alpha^* p + (1 - \alpha^*) r \begin{matrix} \succ \\ \\ \prec \end{matrix} q$$

leads to a contradiction by (A3) (verify this!), which leaves us with the third possibility

$$q \sim \alpha^* p + (1 - \alpha^*) r$$

(which is what we want).■

(L3) $p \sim q$ and $\alpha \in [0, 1] \Rightarrow$ for all $r \in P$

$$\alpha p + (1 - \alpha)r \sim \alpha q + (1 - \alpha)r.$$

Proof: Suppose that there is some $s \in P$ with $s \succ p \sim q$ (otherwise, trivial) and that

$$\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$$

toward contradiction.

(A2) implies that for all $\beta \in (0, 1)$

$$\begin{aligned} \beta s + (1-\beta)q &\succ \beta q + (1-\beta)q \\ &= q \\ &\sim p \end{aligned}$$

and (A2) also implies that for all $\alpha \in (0, 1)$

$$\alpha(\beta s + (1-\beta)q) + (1-\alpha)r \succ \alpha p + (1-\alpha)r.$$

Since by assumption

$$\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r \,,$$

(A3) implies that for each β there exists some $\alpha^*(\beta) \in (0, 1)$ such that

$$\begin{gathered} \alpha p + (1 - \alpha)r \\ \succ \\ \alpha^*(\beta)(\alpha(\beta s + (1 - \beta)q) + (1 - \alpha)r) \\ + \\ (1 - \alpha^*(\beta))(\alpha q + (1 - \alpha)r). \end{gathered}$$

Fix, for example, $\beta = 1/2$ and let $\alpha^*(1/2)$ written as α^* then the term on the RHS is

$$[\frac{\alpha^*\alpha}{2}]s + [\frac{\alpha^*\alpha}{2} + (1-\alpha^*)\alpha]q + [1-\alpha]r$$
$$=$$
$$\alpha[\frac{\alpha^*}{2}s + (1-\frac{\alpha^*}{2})q] + (1-\alpha)r.$$

But since $\frac{\alpha^*}{2} > 0$, the last term must be $\succ \alpha p + (1-\alpha)r$, a contradiction.■

Before stating another (final) lemma, we need some notation: For any $x \in X$, let δ_x denote the probability distribution degenerate at x, that is

$$\delta_x(x') = \begin{cases} 1 & \text{if} \quad x' = x \\ 0 & \text{if} \quad x' \neq x \end{cases}$$

(L4) If $\succ$ on P satisfies (A1)-(A3) then for all $p \in P$ there exist $x^\circ, x_\circ \in X$ such that $\delta_{x^\circ} \succsim p \succsim \delta_{x_\circ}$.

The proof (omitted) builds on (A2) and (L3) and uses induction on the size of the support of p.

Proof of the vNM theorem: Showing that if a u-function as in $(*)$ exists then (A1)-(A3) all hold is omitted (straightforward...).

– Suppose that $\succ$ satisfies (A1)-(A3) and use (L4) to produce δ_{x° and $\delta_{x_\circ}$. If $\delta_{x^\circ} \sim \delta_{x_\circ}$ then $u \equiv c$ (constant) satisfies $(*)$ as neither

$$p \succ q \text{ nor } \sum p(x)u(x) > \sum q(x)u(x)$$

is possible.

(u constant is the only possible representation in this case, so u' is any other representation if and only if it is a positive affine transformation of u.)

– From now on, assume $\delta_{x^\circ} \succ \delta_{x_\circ}$ and for any $p \in P$ define

$$f(p) = \alpha \text{ where } \alpha\delta_{x^\circ} + (1-\alpha)\delta_{x_\circ} \sim p.$$

By the lemmas, such an α exists and is unique, so f is well defined.

By (L1) and standard properties of preference relations

$$\begin{array}{rcl} p & \succ & q \\ & \Updownarrow & \\ f(p)\delta_{x^\circ} + (1-f(p))\delta_{x_\circ} & \succ & f(q)\delta_{x^\circ} + (1-f(q))\delta_{x_\circ} \\ & \Updownarrow & \\ f(p) & > & f(q). \end{array}$$

– Hence $f(\cdot)$ is a representation of $\succ$ in the standard sense but we are not done quite yet!

We will have the expected-utility representation $(*)$ as soon as we show that for all $p \in P$

$$f(p) = \sum p(x)u(x). \tag{$\dagger$}$$

The method is to use (††) below and induction on the size of the support of p

$$\{x \in X : p(x) > 0\}.$$

– Note that for all $p, q \in P$ and $\alpha \in [0, 1]$, by repeated application of (L3)

$$\alpha p + (1-\alpha)q$$
$$\sim$$
$$\alpha[f(p)\delta_{x^\circ} + (1-f(p))\delta_{x_\circ}] + (1-\alpha)[f(q)\delta_{x^\circ} + (1-f(q))\delta_{x_\circ}]$$

So by the definition of f

$$f(\alpha p + (1-\alpha)q) = \alpha f(p) + (1-\alpha)f(q). \qquad (\dagger\dagger)$$

– If the support of p has one element, say x', then $p = \delta_{x'}$ and (†) follows trivial. Suppose inductively that (†) is true for p with support of size $n - 1 \geq 1$.

Take any p with support of size $n > 1$, let x' be in the support of p, and defined q as follows:

$$q(x) = \begin{cases} 0 & \text{if } \; x = x' \\ p(x)/(1 - p(x')) & \text{if } \; x \neq x' \end{cases}$$

so q has support of size $n - 1$ and $p = p(x')\delta_{x'} + (1 - p(x'))q$.

By (††) and the induction hypothesis applied to q

$$f(p) = p(x')f(\delta_{x'}) + (1 - p(x'))f(q)$$

$$\shortparallel$$

$$p(x')u(x') + (1 - p(x')) \sum_{x \neq x'} \frac{p(x)}{1 - p(x')} u(x)$$

$$\shortparallel$$

$$\sum_{\forall x} p(x)u(x).$$

This establishes (†) by induction, since X is finite.

! We also need to establish the uniqueness result: if u and u' represent $\succ$ in the sense of $(*)$ then then each is a positive affine transformation of the other.■

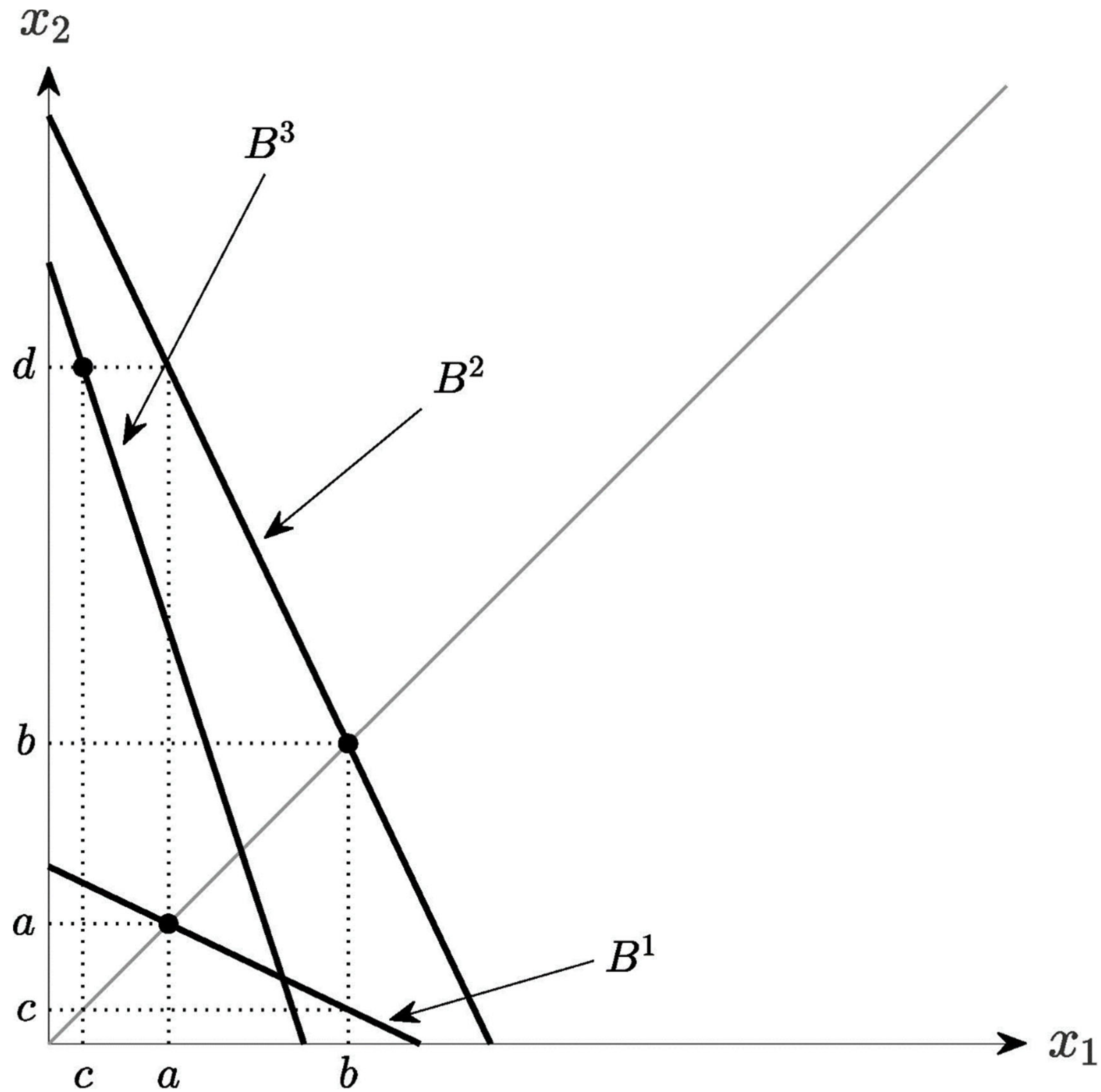
Concluding remarks

- vNM EUT lies at the very heart of economics—a normative guide for choice and also as a descriptive model of how individuals choose.

- But much of the evidence of "anomalies" in choice behavior suggests that EUT may not the right model of choice under risk.

- non-EUT theories of choice under risk that relax the independence axiom but adhere to the fundamental/conventional axioms.

A simple violation of EU-rationalizability



EU requires that

$$\begin{aligned} U(a,a) &= 2u(a) \geq u(b) + u(c) \\ U(b,b) &= 2u(b) \geq u(a) + u(d) \end{aligned}$$

b/c $(a,a)R^D(b,c)$ and $(b,b)R^D(a,d)$.

But rearranging yields

$$u(a) + u(b) \geq u(c) + u(d)$$

which contradicts that $(c,d)R^D(a,b)$.

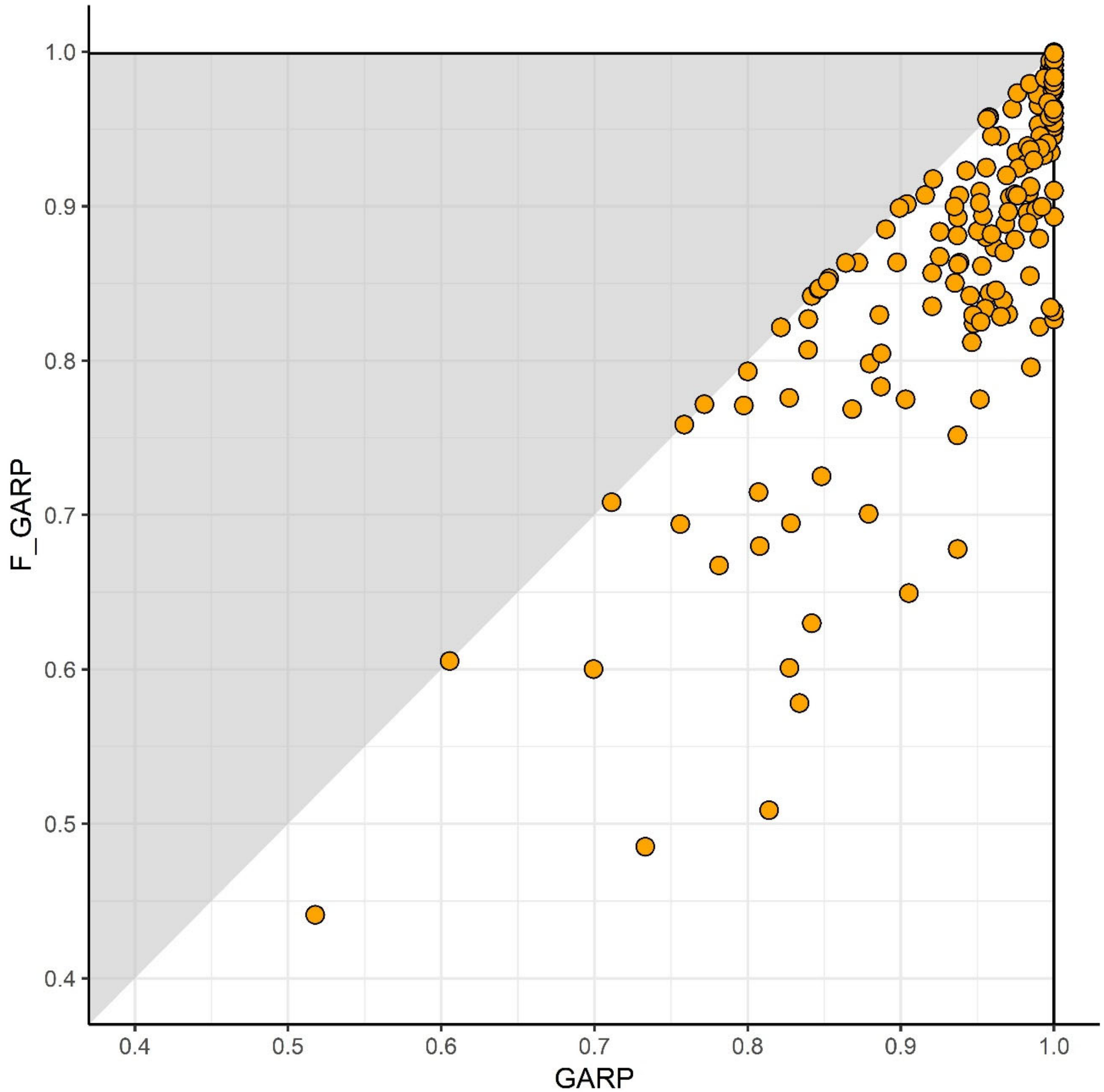
– e^* – maximizing any utility function (GARP).

– $e^{**} \leq e^*$ – maximizing a monotonic utility function (GARP+FOSD).

– $e^{***} \leq e^{**}$ – maximizing an expected utility function (GARP+FOSD+EU).

$\Rightarrow$ For all non-EU theories, which number well into double figures (Starmer, 2000), including stochastic reference dependence (Kőszegi and Rabin, 2006 & 2007):

$$e^{***} < e^{**} = e^* = 1.$$



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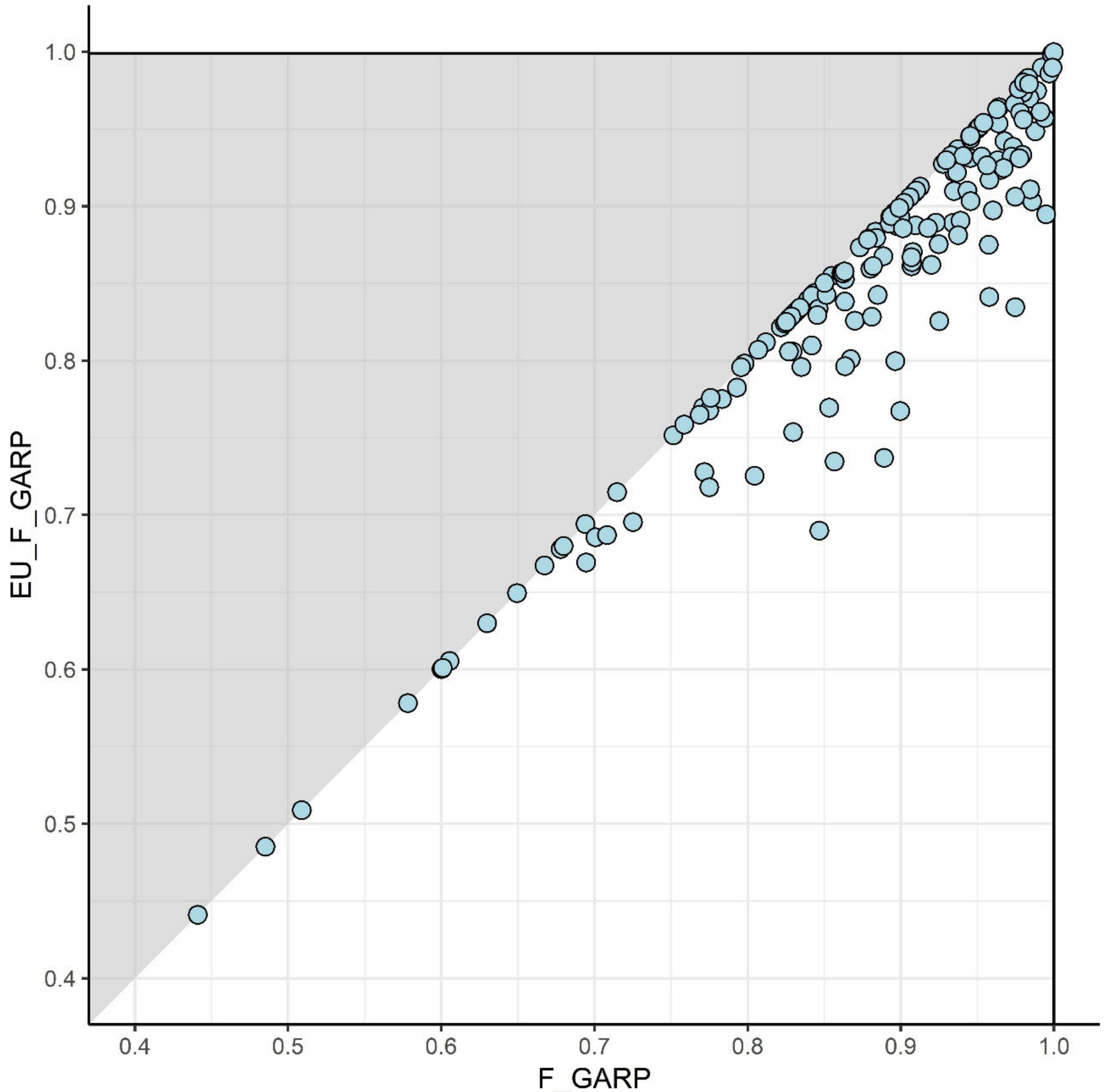

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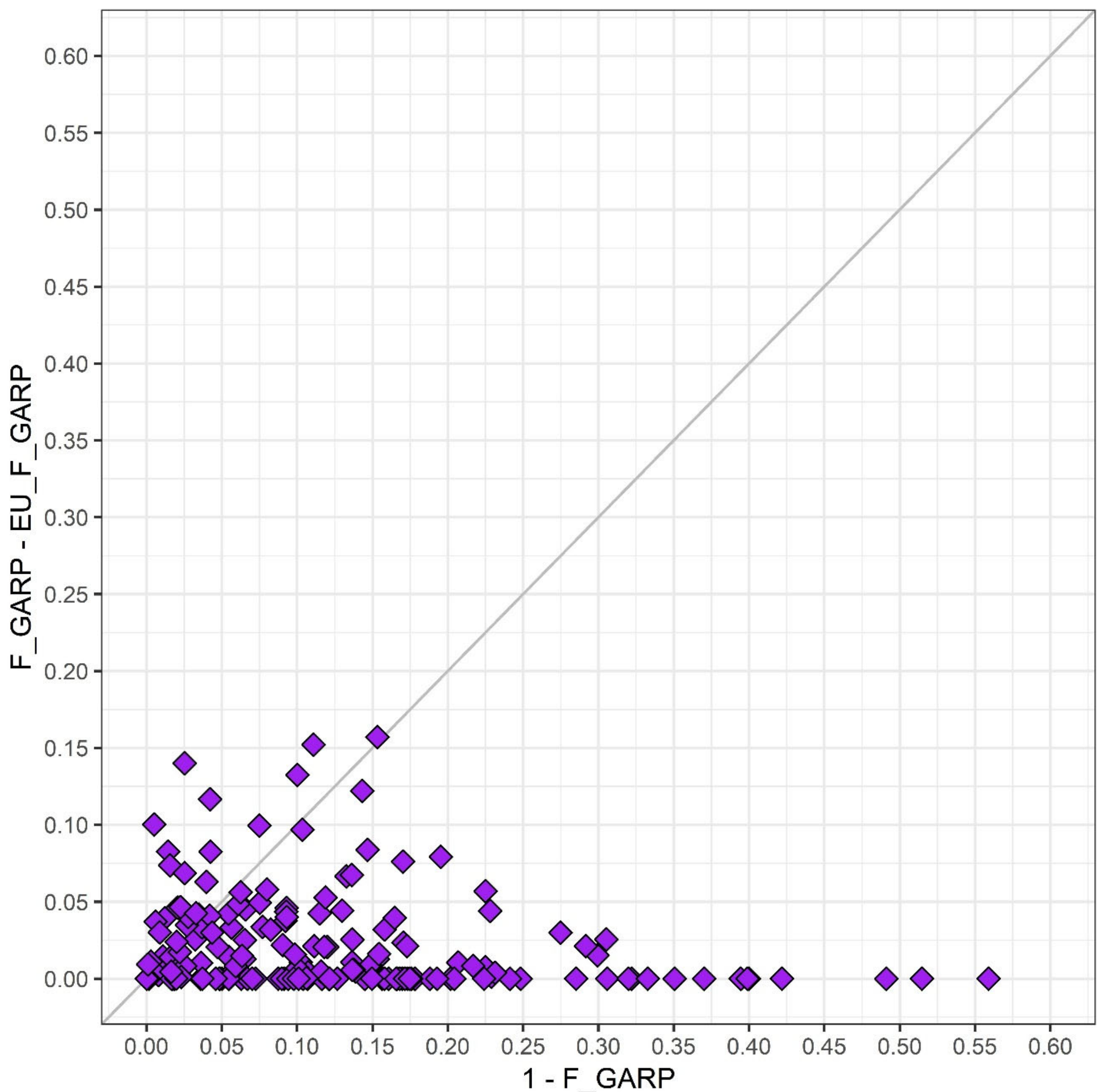

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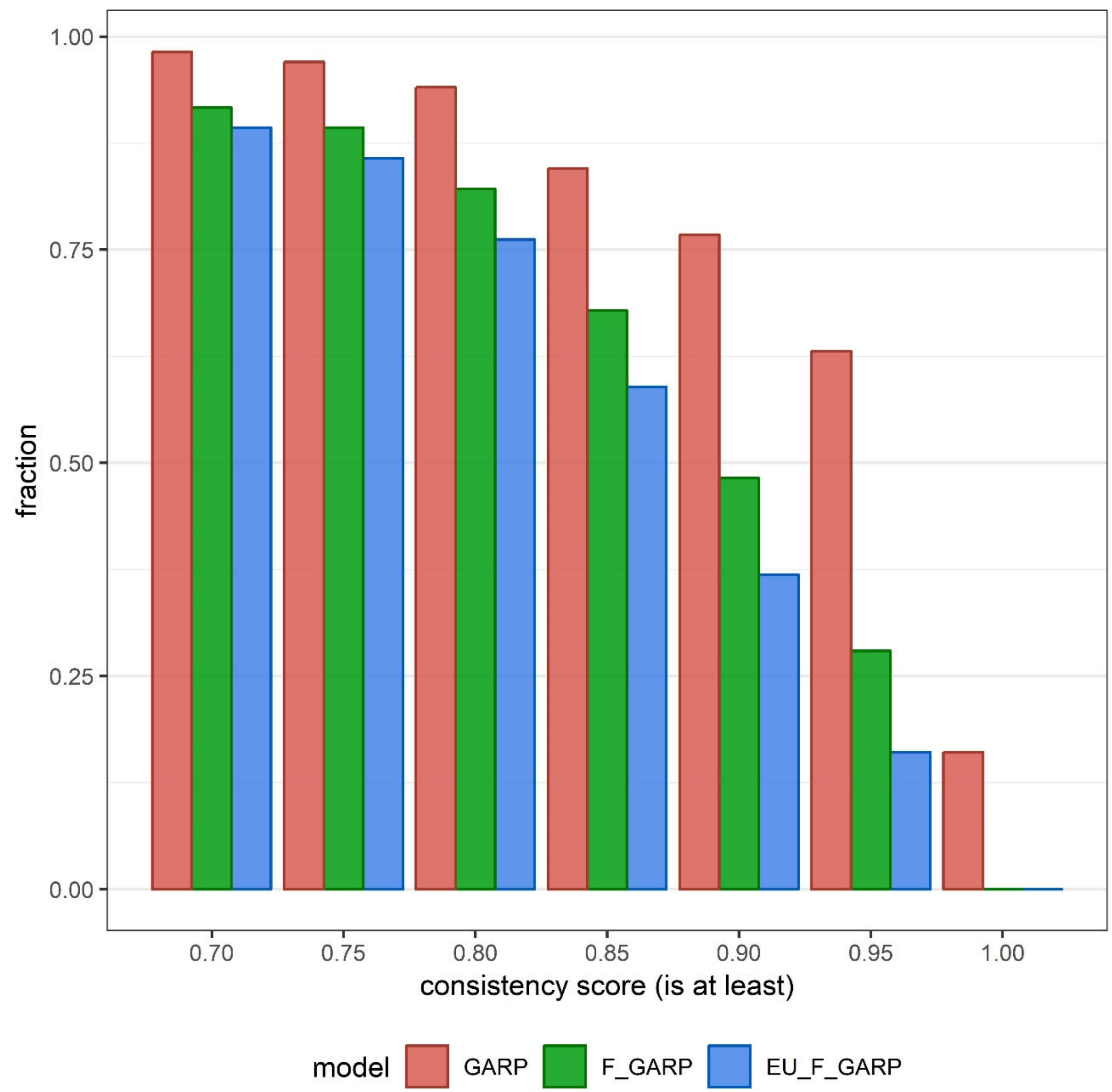

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