

Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2013)

Extensive games with perfect information
OR 6 and 7, FT 3, 4 and 11

Perfect information

A finite extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ consists of

– A set N of players.

– A set H of sequences (histories) where $\emptyset \in H$ and for any $L < K$

$$(a^k)_{k=1}^K \in H \implies (a^k)_{k=1}^L \in H.$$

– A player function $P : H \setminus Z \rightarrow N$ where $h \in Z \subseteq H$ if $(h, a) \notin H$.

– A preference relation \succsim_i on Z for each player $i \in N$.

Strategies, outcomes and Nash equilibrium

A strategy

$$s_i : h \rightarrow A(h) \text{ for every } h \in H \setminus Z \text{ such that } P(h) = i.$$

A Nash equilibrium of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile $(s_i^*)_{i \in N}$ such that for any $i \in N$

$$O(s^*) \succsim_i O(s_i, s_{-i}^*) \quad \forall s_i$$

where $O(s) = (a^1, \dots, a^K) \in Z$ such that

$$s_{P(a^1, \dots, a^k)}(a^1, \dots, a^k) = a^{k+1}$$

for any $0 \leq k < K$ (an outcome).

The (reduced) strategic form

$G = \langle N, (S_i), (\succsim'_i) \rangle$ is the strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S_i is player i 's strategy set in Γ and \succsim'_i is defined by

$$s \succsim'_i s' \Leftrightarrow O(s) \succsim'_i O(s') \quad \forall s, s' \in \times_{i \in N} S_i$$

$G = \langle N, (S'_i), (\succsim''_i) \rangle$ is the reduced strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S'_i contains one member of *equivalent* strategies in S_i , that is,

$$s_i, s'_i \in S_i \text{ are equivalent if } (s_i, s_{-i}) \sim'_j (s'_i, s_{-i}) \quad \forall j \in N,$$

and \succsim''_i defined over $\times_{j \in N} S'_j$ and induced by \succsim'_i .

Subgames and subgame perfection

A subgame of Γ that follows the history h is the game $\Gamma(h)$

$$\langle N, H |_h, P |_h, (\succsim_i |_h) \rangle$$

where for each $h' \in H_h$

$$(h, h') \in H, P |_h(h') = P(h, h') \text{ and } h' \succsim_i |_h h'' \Leftrightarrow (h, h') \succsim_i (h, h'').$$

$s^* \in \times_{i \in N} S_i$ is a subgame perfect equilibrium (SPE) of Γ if

$$O_h(s_i^* |_h, s_{-i}^* |_h) \succsim_i |_h O_h(s_i |_h, s_{-i}^* |_h)$$

for each $i \in N$ and $h \in H \setminus Z$ for which $P(h) = i$ and for any $s_i |_h$.

Thus, the equilibrium of the full game must induce on equilibrium on every subgame.

Backward induction and Kuhn's theorems

Let Γ be a finite extensive game with perfect information

- Γ has a *SPE* (Kuhn's theorem).

The proof is by backward induction (Zermelo, 1912) which is also an algorithm for calculating the set of *SPE*.

- Γ has a unique *SPE* if there is no $i \in N$ such that $z \sim_i z'$ for any $z, z' \in Z$.
- Γ is dominance solvable if $z \sim_i z' \implies \exists i \in N$ then $z \sim_j z' \forall j \in N$ (but elimination of weakly dominated strategies in G may eliminate the *SPE* in Γ).

Forward induction

- Backward induction cannot always ensure a self-enforcing equilibrium (Ben-Porath and Dekel; 1988, 1992).
- In an extensive game with simultaneous moves, players interpret a deviation as a signal about future play.
- The concept of iterated weak dominance can be used to capture forward and backward induction.

(Van Damme 1989) A solution concept S is consistent with forward induction in the class $\Gamma = \langle \{1, 2\}, H, P, (\succsim_i) \rangle$ if there is no equilibrium in S such that player i

- can ensure that a proper (outside-option) subgame of Γ is reached by deviating, and
- according to S , $O(s^*) \succsim_i O(s)$ and $O(s') \succsim_i O(s^*)$ for one s' and all $s \neq s'$.

Thus a deviation gives a clear signal how the deviator intends to play in the future.

Bargaining

In the strategic approach, the players bargain over a pie of size 1.

An agreement is a pair (x_1, x_2) where x_i is player i 's share of the pie. The set of possible agreements is

$$X = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\}$$

Player i prefers $x \in X$ to $y \in X$ if and only if $x_i > y_i$.

The bargaining protocol

The players can take actions only at times in the (infinite) set $T = \{0, 1, 2, \dots\}$. In each $t \in T$ player i , proposes an agreement $x \in X$ and $j \neq i$ either accepts (Y) or rejects (N).

If x is accepted (Y) then the bargaining ends and x is implemented. If x is rejected (N) then the play passes to period $t + 1$ in which j proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement (D). The only asymmetry is that player 1 is the first to make an offer.

Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

- an extensive game of perfect information with the structure given above, and
- player i 's preference ordering \succsim_i over $(X \times T) \cup \{D\}$ is complete and transitive.

Preferences over $X \times T$ are represented by $\delta_i^t u_i(x_i)$ for any $0 < \delta_i < 1$ where u_i is increasing and concave.

Assumptions on preferences

A1 Disagreement is the worst outcome

For any $(x, t) \in X \times T$,

$$(x, t) \succsim_i D$$

for each i .

A2 Pie is desirable

– For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t) \text{ if and only if } x_i > y_i.$$

A3 Time is valuable

For any $t \in T$, $s \in T$ and $x \in X$

$$(x, t) \succsim_i (x, s) \text{ if } t < s$$

and with strict preferences if $x_i > 0$.

A4 Preference ordering is continuous

Let $\{(x_n, t)\}_{n=1}^{\infty}$ and $\{(y_n, s)\}_{n=1}^{\infty}$ be members of $X \times T$ for which

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y.$$

Then, $(x, t) \succsim_i (y, s)$ whenever $(x_n, t) \succsim_i (y_n, s)$ for all n .

A2-A4 imply that for any outcome (x, t) either there is a unique $y \in X$ such that

$$(y, 0) \sim_i (x, t)$$

or

$$(y, 0) \succ_i (x, t)$$

for every $y \in X$.

Note \succsim_i satisfies **A2-A4** iff it can be represented by a continuous function

$$U_i : [0, 1] \times T \rightarrow \mathbb{R}$$

that is increasing (decreasing) in the first (second) argument.

A5 Stationarity

For any $t \in T$, $x \in X$ and $y \in X$

$(x, t) \succ_i (y, t + 1)$ if and only if $(x, 0) \succ_i (y, 1)$.

If \succ_i satisfies **A2-A5** then for every $\delta \in (0, 1)$ there exists a continuous increasing function $u_i : [0, 1] \rightarrow \mathbb{R}$ (not necessarily concave) such that

$$U_i(x_i, t) = \delta_i^t u_i(x_i).$$

Present value

Define $v_i : [0, 1] \times T \rightarrow [0, 1]$ for $i = 1, 2$ as follows

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succsim_i (x, t) \text{ for all } y \in X. \end{cases}$$

We call $v_i(x_i, t)$ player i 's present value of (x, t) and note that

$$(y, t) \succsim_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$

If \succsim_i satisfies **A2-A4**, then for any $t \in T$ $v_i(\cdot, t)$ is continuous, non decreasing and increasing whenever $v_i(x_i, t) > 0$.

Further, $v_i(x_i, t) \leq x_i$ for every $(x, t) \in X \times T$ and with strict whenever $x_i > 0$ and $t \geq 1$.

With **A5**, we also have that

$$v_i(v_i(x_i, 1), 1) = v_i(x_i, 2)$$

for any $x \in X$.

Delay

A6 Increasing loss to delay

$x_i - v_i(x_i, \mathbf{1})$ is an increasing function of x_i .

If u_i is differentiable then under **A6** in any representation $\delta_i^t u_i(x_i)$ of \succsim_i

$$\delta_i u_i'(x_i) < u_i'(v_i(x_i, \mathbf{1}))$$

whenever $v_i(x_i, \mathbf{1}) > 0$.

This assumption is weaker than concavity of u_i which implies

$$u_i'(x_i) < u_i'(v_i(x_i, \mathbf{1})).$$

The single crossing property of present values

If \succsim_i for each i satisfies **A2-A6**, then there exist a unique pair $(x^*, y^*) \in X \times X$ such that

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

– For every $x \in X$, let $\psi(x)$ be the agreement for which

$$\psi_1(x) = v_1(x_1, 1)$$

and define $H : X \rightarrow \mathbb{R}$ by

$$H(x) = x_2 - v_2(\psi_2(x), 1).$$

- The pair of agreements x and $y = \psi(x)$ satisfies also $x_2 = v_2(\psi_2(x), 1)$ iff $H(x) = 0$.
- Note that $H(0, 1) \geq 0$ and $H(1, 0) \leq 0$, H is a continuous function, and

$$H(x) = [v_1(x_1, 1) - x_1] + [1 - v_1(x_1, 1) - v_2(1 - v_1(x_1, 1), 1)].$$

- Since $v_1(x_1, 1)$ is non decreasing in x_1 , and both terms are decreasing in x_1 , H has a unique zero by **A6**.

Examples

[1] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = \delta_i^t x_i$$

where $\delta_i \in (0, 1)$, and $U_i(D) = 0$.

[2] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = x_i - c_i t$$

where $c_i > 0$, and $U_i(D) = -\infty$ (constant cost of delay).

Although **A6** is violated, when $c_1 \neq c_2$ there is a unique pair $(x, y) \in X \times X$ such that $y_1 = v_1(x_1, 1)$ and $x_2 = v_2(y_2, 1)$.

Strategies

Let X^t be the set of all sequences $\{x^0, \dots, x^{t-1}\}$ of members of X .

A strategy of player 1 (2) is a sequence of functions

$$\sigma = \{\sigma^t\}_{t=0}^{\infty}$$

such that $\sigma^t : X^t \rightarrow X$ if t is even (odd), and $\sigma^t : X^{t+1} \rightarrow \{Y, N\}$ if t is odd (even).

The way of representing a player's strategy is closely related to the notion of automation.

Nash equilibrium

For any $\bar{x} \in X$, the outcome $(\bar{x}, 0)$ is a *NE* when players' preference satisfy **A1-A6**.

To see this, consider the stationary strategy profile

Player 1	proposes	\bar{x}
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	\bar{x}
	accepts	$x_2 \geq \bar{x}_2$

This is an example for a pair of one-state automata.

The set of outcomes generated in the Nash equilibrium includes also delays (agreements in period 1 or later).

Subgame perfect equilibrium

Any bargaining game of alternating offers in which players' preferences satisfy **A1-A6** has a unique *SPE* which is the solution of the following equations

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Note that if $y_1^* > 0$ and $x_2^* > 0$ then

$$(y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1).$$

The equilibrium strategy profile is given by

Player 1	proposes	x^*
	accepts	$y_1 \geq y_1^*$
Player 2	proposes	y^*
	accepts	$x_1 \leq x_1^*$

The unique outcome is that player 1 proposes x^* in period 0 and player 2 accepts.

Step 1 (x^*, y^*) is a *SPE*

Player 1:

- proposing x^* at t^* leads to an outcome (x^*, t^*) . Any other strategy generates either

$$(x, t) \text{ where } x_1 \leq x_1^* \text{ and } t \geq t^*$$

or

$$(y^*, t) \text{ where } t \geq t^* + 1$$

or D .

- Since $x_1^* > y_1^*$ it follows from **A1-A3** that (x^*, t^*) is a best response.

Player 2:

- accepting x^* at t^* leads to an outcome (x^*, t^*) . Any other strategy generates either

$$(y, t) \text{ where } y_2 \leq y_2^* \text{ and } t \geq t^* + 1$$

or

$$(x^*, t) \text{ where } t \geq t^*$$

or D .

– By **A1-A3** and **A5**

$$(x^*, t^*) \succsim_2 (y^*, t^* + 1)$$

and thus accepting x^* at t^* , which leads to the outcome (x^*, t^*) , is a best response.

Note that similar arguments apply to a subgame starting with an offer of player 2.

Step 2 (x^*, y^*) is the unique SPE

Let G_i be a subgame starting with an offer of player i and define

$$M_i = \sup\{v_i(x_i, t) : (x, t) \in SPE(G_i)\},$$

and

$$m_i = \inf\{v_i(x_i, t) : (x, t) \in SPE(G_i)\}.$$

It suffices to show that

$$M_1 = m_1 = x_1^* \text{ and } M_2 = m_2 = y_2^*.$$

It follows that the present value for player 1 (2) of every SPE of G_1 (G_2) is x_1^* (y_2^*).

First, we argue that in every *SPE* of G_1 and G_2 the first offer is accepted because

$$v_1(y_1^*, 1) \leq y_1^* < x_1^* \text{ and } v_2(x_2^*, 1) \leq x_2^* < y_2^*$$

(after a rejection, the present value for player 1 is less than x_1^* and for player 2 is less than y_2^*).

It remains to show that

$$m_2 \geq 1 - v_1(M_1, 1) \tag{1}$$

and

$$M_1 \leq 1 - v_2(m_2, 1). \tag{2}$$

[1] and the fact that $m_2 \leq y_2^*$ imply that the pair $(M_1, 1 - m_2)$ lies below the line

$$y_1 = v_1(x_1, 1),$$

and [2] and the fact that $M_1 \leq x_1^*$ imply that this pair lies to the left of the line

$$x_2 = v_2(y_2, 1).$$

Thus,

$$M_1 = x_1^* \text{ and } m_2 = y_2^*,$$

and with the role of the players reversed, the same argument shows that $M_2 = y_2^*$ and $m_1 = x_1^*$.

Properties of Rubinstein's model

[1] Delay (without uncertainty)

Subgame perfection alone cannot rule out delay. In Rubinstein's model delay is closely related to the existence of multiple equilibria.

The uniqueness proof relies only on **A1-A3** and **A6**. When both players have the same constant cost of delay (**A6** is violated), there are multiple equilibria.

If the cost of delay is small enough, in some of these equilibria, agreement is not reached immediately. Any other conditions that guarantees a unique solution can be used instead of **A6**.

An example

Assume that $X = \{a, b, c\}$ where $a_1 > b_1 > c_1$, the ordering \succsim_i satisfies **A1-A3** and **A5** for $i = 1, 2$, and if $(x, t) \succ (y, t)$ then $(x, t+1) \succ (y, t)$.

Then, for each $\bar{x} \in X$, the pair of strategies in which each player insists on \bar{x}

Player 1	proposes	\bar{x}
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	\bar{x}
	accepts	$x_2 \geq \bar{x}_2$

is a subgame perfect equilibrium.

An example of a subgame perfect equilibrium in which agreement is reached in period 1 is given by

		A	B	C
Player 1	proposes	a	b	c
	accepts		a and b	$a, b,$ and c
Player 2	proposes		b	c
	accepts	c	b and c	c

where A is the initial state, B and C are absorbing states, and if player 2 rejects a (b or c) then the state changes to B (C).

The outcome is that player 1 offers a in period 0, player 2 rejects and proposes b in period 1 which player 1 accepts.

[2] Patience

The ordering \succsim'_1 is *less patient than* \succsim_1 if

$$v'_1(x_1, \mathbf{1}) \leq v_1(x_1, \mathbf{1})$$

for all $x \in X$ (with constant cost of delay $\delta'_1 \leq \delta_1$).

The model predicts that when a player becomes less patient his negotiated share of the pie decreases.

[3] Asymmetry

The structure of the model is asymmetric only in one respect: player 1 is the first to make an offer.

Recall that with constant discount rates the equilibrium condition implies that

$$y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^*$$

so that

$$x^* = \left(\frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \text{ and } y^* = \left(\frac{\delta_1(1 - \delta_2)}{1 - \delta_1 \delta_2}, \frac{1 - \delta_1}{1 - \delta_1 \delta_2} \right).$$

Thus, if $\delta_1 = \delta_2 = \delta$ ($v_1 = v_2$) then

$$x^* = \left(\frac{1}{1 + \delta}, \frac{\delta}{1 + \delta} \right) \text{ and } y^* = \left(\frac{\delta}{1 + \delta}, \frac{1}{1 + \delta} \right)$$

so player 1 obtains more than half of the pie.

By shrinking the length of a period by considering a sequence of games indexed by Δ in which $u_i = \delta_i^{\Delta t} x_i$ we have

$$\lim_{\Delta \rightarrow 0} x^*(\Delta) = \lim_{\Delta \rightarrow 0} y^*(\Delta) = \left(\frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2} \right)$$

(l'Hôpital's rule).

Models in which players have outside options

Suppose player 2 has the option of terminating. In this event the outcome worth b to him and 0 to player 1.

- Case I: can quit only after he rejected an offer, then the game has unique subgame perfect equilibrium.
- Case II: can quit only after player 1 rejected an offer or after any rejections, then the game has multiple equilibria.

- b is small: ($b < \delta/(1-\delta)$ when $\delta_1 = \delta_2 = \delta$) no effect on the outcome of the game (not a credible threat).
- b is large: in case I there is a unique subgame perfect equilibrium in which the payoff pair is $(1-b, b)$; in case II there are multiple equilibria.

The “ingredients” of the proofs are the same as in the proof of Rubinstein (omitted).

Rubinstein's model with three players

Suppose that the ordering \succsim_i satisfies **A1-A6** for $i = 1, 2, 3$; and agreement requires the approval of all three players.

Then, if $v_i(1, 1) \geq 1/2$ for $i = 1, 2, 3$ then for every partition x^* there is a subgame perfect equilibrium in which immediate agreement is reached on the partition x^* (Shaked 1987).

A subgame perfect equilibrium where there is an immediate agreement on x^* is given by

		x^*	e^j
i	proposes	x^*	e^j
	accepts	$x_i \geq v_i(x_i^*, \mathbf{1})$	$x_i \geq 0$
j	proposes	x^*	e^j
	accepts	$x_j \geq v_j(x_j^*, \mathbf{1})$	$x_j \geq v_j(\mathbf{1}, \mathbf{1})$

where e^j is the j^{th} unit vector, and if player i proposes $x_i > x_i^*$ go to state e^j where $j \neq i$ is the player with the lower index for whom $x_j < 1/2$.

The main force holding together the equilibrium is that one of the players is “rewarded” for rejecting a deviant offer – after his rejection, she/he obtains all the pie.

The only stationary subgame perfect equilibrium has a form similar to the unique equilibrium of the two-player game. With a common discount factor δ , this equilibrium leads to the division

$$(\xi, \delta\xi, \delta^2\xi) \text{ where } \xi + \delta\xi + \delta^2\xi = 1.$$

Other routs may be taken in order to isolate a unique outcome in the three-player game.