Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2013)

Leftovers

Bayesian equilibrium

A Bayesian game consists of a finite set N of players, a finite set Ω of decision-relevant states (characteristics of players), and for each player $i \in N$

- a set A_i of actions
- a finite set T_i of types and a signal function $\tau_i : \Omega \to T_i$
- a probability measure p_i on Ω (prior belief) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$.
- a preference relation $\succsim_i$on the set of probability measure over $A \times \Omega$.

$a^* \in \times_{(i,t_i)} A_i$ is a Bayes-Nash equilibrium of a Bayesian game

$$\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i) \rangle$$

if it is a NE in which the set of players is the set of all pairs (i, t_i) for all $i \in N$ and $t_i \in T_i$, and for each player (i, t_i)

$$a^* \succsim_{(i,t_i)} b^* \Leftrightarrow L_i(a^*, t_i) \succsim_i L_i(b^*, t_i)$$

where $L_i(a^*, t_i)$ is a *lottery* over $A \times \Omega$ that assigns a probability $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ to

$$(a^*(j, \tau_j(\omega)))_{j \in N, \omega} \text{ if } \omega \in p_i(\tau_i^{-1}(t_i))$$

and zero otherwise.

Example: BoS with one-side imperfect information

$\underline{\omega = y}$

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

$\underline{\omega = n}$

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

Then, the expected payoffs of player 1 are given by

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	$2p$	$2(1-p)$	0
S	0	p	$1-p$	1

For any belief $p \in (0,1)$, $(B,(B,S))$ is an equilibrium (B is optimal for player 1 given the actions of the two types of player 2 and his beliefs).

Harsanyi (1973)

Consider a game $G = \langle N, (A_i), (u_i) \rangle$ and let $(\epsilon_i(a))_{i \in N, a \in A}$ be a collection of random variables with support $[-1, 1]$ where

- $\epsilon_i = (\epsilon_i(a))_{a \in A}$ is private information and has well-behaved distribution function, and $\epsilon = (\epsilon_i)_{i \in N}$ are independent.

- The payoff of each player i at the outcome a and *state* ϵ is $u_i(a) + \epsilon_i(a)$. This defines a Bayesian game $G(\epsilon)$.

For almost any game G and any collection ϵ^*, almost any $\alpha \in NE(G)$ is *approachable* – associated with the limit as $\gamma \longrightarrow 0$ of a sequence of pure strategy equilibria of the Bayesian game $G(\gamma\epsilon^*)$ (and visa versa).

A model of knowledge (OR 5.1-5.2)

Knowledge is formalized such that a player cannot know something that is false (by contrast to beliefs).

An event is common knowledge if

- all players know it,
- all players know that all players know it,
- and so on ad infinitum.

Setup

– Ω - a finite set of states of the world.

– $E \subseteq \Omega$ - an event.

– $\mathcal{P}$ - information function.

A partition of Ω, i.e., a collection of non-empty disjoint subsets of Ω whose union is Ω. The information that a player is assumed to have about the true state.

Example

$$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The information of players a and b are given by

$$\mathcal{P}^a = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9\}\}$$

and

$$\mathcal{P}^b = \{\{1, 2\}, \{3, 4, 5\}, \{6\}, \{7, 8, 9\}\}.$$

Suppose that $\omega = 2$ and consider the event

$$E = \{1, 2, 3, 4\}$$

- Does a know E?

- Does b know E?

- Does a know that b knows E?

- Does b know that a knows E?

Given $\mathcal{P}^a$ and $\mathcal{P}^b$, when $\omega = 2$ the event

$$G = \{1, 2, 3, 4, 5, 6\}$$

is common knowledge.

- a knows G,
- b knows G,
- a knows b knows G,
- b knows a knows G, and so on indefinitely.

Some definitions

– A partition $\mathcal{P}^i$ refines another partition $\mathcal{P}^j$ if every member of $\mathcal{P}^i$ is a subset of a member $\mathcal{P}^j$.

– The meet of two partitions $\mathcal{P}^i$ and $\mathcal{P}^j$, denoted by $\mathcal{P}^i \wedge \mathcal{P}^j$, is a partition of Ω such that $\mathcal{P}^i$ and $\mathcal{P}^j$ are (the only) refinements of $\mathcal{P}^i \wedge \mathcal{P}^j$.

Example (continue)

- Given $\mathcal{P}^a$ and $\mathcal{P}^b$ above, the meet $\mathcal{P}^a \wedge \mathcal{P}^b$ is the partition

$$\mathcal{P}^a \wedge \mathcal{P}^b = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}\}$$

- This is the unique partition that satisfies the conditions above.

Aumann's common knowledge

Let $\omega \in \Omega$ be the true state and fix some event $E \subseteq \Omega$. Then E is common knowledge (given ω) if and only if

$$(\mathcal{P}^a \wedge \mathcal{P}^b)(\omega) \subseteq E$$

(E is common knowledge if it contains the member of $\mathcal{P}^a \wedge \mathcal{P}^b$ that contains ω).

In the above example,

$$(\mathcal{P}^a \wedge \mathcal{P}^b)(\omega) = \{1, 2, 3, 4, 5\} \subseteq G = \{1, 2, 3, 4, 5, 6\}$$

which implies that event G is common knowledge at $\omega = 2$. The idea of the proof can be seen in Figure 1 and Figure 2.

Aumann's agreement theorem (OR 5.3)

Suppose a and b have a common (prior) probability measure p on the set of states Ω (the common prior assumption).

The posterior probabilities of event $E \subseteq \Omega$ when the state is $\omega \in \Omega$ for $i = a, b$ is given by

$$p[E|\mathcal{P}^i(\omega)] = \frac{p[E \cap \mathcal{P}^i(\omega)]}{p[\mathcal{P}^i(\omega)]}.$$

Aumann's theorem: Fix some event $E \subseteq \Omega$ and a state $\omega \in \Omega$. If $p[E|\mathcal{P}^a(\omega)]$ and $p[E|\mathcal{P}^b(\omega)]$ are common knowledge, then they must be equal. Hence, players cannot agree to disagree!

Proof

– Let $(\mathcal{P}^a \wedge \mathcal{P}^b)(\omega)$ be member of the meet of $\mathcal{P}^a$ and $\mathcal{P}^b$ that contains ω. Since a's posterior is common knowledge, there is a q such that

$$p(E|\pi) = q$$

for any $\pi \in \mathcal{P}^a \subseteq (\mathcal{P}^a \wedge \mathcal{P}^b)(\omega)$.

Proof (continue)

- Since a's posterior is common knowledge, there is a r such that

$$p(E|\rho) = r$$

for any $\rho \in \mathcal{P}^b \subseteq (\mathcal{P}^a \wedge \mathcal{P}^b)(\omega)$.

- Hence,

$$p[E|(\mathcal{P}^a \wedge \mathcal{P}^b)(\omega)] = q \text{ and } p[E|(\mathcal{P}^a \wedge \mathcal{P}^b)(\omega)] = r$$

which completes the proof.

A knowledge function

The event that a player knows an event $E \subseteq \Omega$ is given by

$$KE = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}.$$

where $K : 2^{\Omega} \rightarrow 2^{\Omega}$ (the set of all subsets of Ω to itself).

Properties of KE

i For any $E \subseteq \Omega$, $KE \subseteq E$.

ii For any $E, F \subseteq \Omega$, if $E \subseteq F$, then $KE \subseteq KF$.

iii For any $E \subseteq \Omega$, $(KE)^c \subseteq K(KE)^c$.

Why?

i If $\omega \in KE$, then $\mathcal{P}(\omega) \subseteq E$. But $\omega \in \mathcal{P}(\omega)$, so $\omega \in E$.

ii If $\omega \in KE$, then $\mathcal{P}(\omega) \subseteq E$. But then $\mathcal{P}(\omega) \subseteq F$ so $\omega \in KF$.

iii If $\omega \in (KE)^c$, then $\mathcal{P}(\omega) \nsubseteq E$.

Suppose there exists some $\omega' \in \mathcal{P}(\omega) \cap KE$. Then, $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') = \mathcal{P}(\omega) \nsubseteq E$, contradicting $\omega' \in KE$. Thus, $\mathcal{P}(\omega) \cap KE = \emptyset$, which says that $\mathcal{P}(\omega) \subseteq (KE)^c$, or $\omega \in K(KE)^c$.

If a (knowledge) function $K : 2^{\Omega} \longrightarrow 2^{\Omega}$ satisfies (i)-(iii) then there is a partition $\mathcal{P}$ of Ω such that

$$KE = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}.$$

<u>proof</u> (sketch)

- The following properties of K must be shown:

  $K\Omega = \Omega$, $KE \subseteq KKE$, and $K(E \cap F) = KE \cap KF$.

- Then, the following must be shown:

  $\omega \in KE$ if and only if $\mathcal{P}(\omega) \subseteq E$, and if $\omega \in \mathcal{P}(\omega)$ and $\omega' \in \mathcal{P}(\omega)$, then $\mathcal{P}(\omega') = \mathcal{P}(\omega)$.

Knowledge and equilibrium (an example)

States

– $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_i = [a, b] \subseteq \mathbb{R}$, and the generic element is $\omega = (\omega_1, \omega_2)$.

Signals

– $\sigma_i(\omega) = \omega_i$, $\forall \omega \in \Omega$, $i = 1, 2$ and $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$ and $\mathbf{P}_i$ has no atoms.

Actions and payoffs

$$u(a,\omega) = \begin{cases} 0 & \text{if } a = 0 \\ U(\omega_1, \omega_2) & \text{if } a = 1 \end{cases}.$$

– where $U(\omega)$ is a continuous and increasing function and actions are not weakly dominated.

Social beliefs

– An event $\{\omega_i\} \times B_{jt}$, where $\omega_j \in B_{jt} \subseteq \Omega_j$. It is common knowledge at date t that

$$\omega \in B_t(\omega) = B_{1t}(\omega) \times B_{2t}(\omega).$$

The optimal decision

- Agent i's expected payoff to action 1

$$\varphi_i(\omega_i, B_{jt}) = E[U(\omega_1, \omega_2)|\{\omega_i\} \times B_{jt}\}$$

is increasing in ω_i. The optimal strategy is the cutoff strategy

$$\begin{aligned} \omega_i &> \omega_i^*(B_{jt}) \Longrightarrow \varphi_i(\omega_i, B_{jt}) > 0, \\ \omega_i &< \omega_i^*(B_{jt}) \Longrightarrow \varphi_i(\omega_i, B_{jt}) < 0. \end{aligned}$$

where ω_i^* is the history-contingent cutoff.

- The cutoff rule implies that the set B_{jt} is an interval and that

$$B_{jt+1}(\omega) \subseteq B_{jt}(\omega) \subseteq [a, b]$$

Claim: Agents must eventually choose the same action.

– By contradiction.

  Suppose that for some B and every ω such that $B(\omega) = B$

$$E[U(\omega_1, \omega_2)|\{\omega_1\} \times B_2] > 0$$

  and

$$E[U(\omega_1, \omega_2)|B_1 \times \{\omega_2\}] < 0.$$

– The same action must be optimal for every element in the information set

$$E[U(\underline{\omega}_1, \omega_2) | \{\underline{\omega}_1\} \times B_2] \geq 0$$

and

$$E[U(\omega_1, \overline{\omega}_2) | B_1 \times \{\overline{\omega}_2\}] \leq 0,$$

where $\underline{\omega}_1 = \inf B_1(\omega)$ and $\overline{\omega}_2 = \sup B_2(\omega)$.

Then

$$U(\underline{\omega}_1, \overline{\omega}_2) \geq 0 \text{ and } U(\underline{\omega}_1, \overline{\omega}_2) \leq 0.$$

If B_i for $i = 1, 2$ is not a singleton, a contradiction. B is a singleton and $U(\omega) = 0$ if $\omega \in B$ but the set $\{\omega : U(\omega) = 0\}$ has probability zero.

An illustration

– $\sigma_i(\omega) = \omega_i$, ω_i˜$U[-1,1]$, and $U(1,\omega) = \omega_1 + \omega_2$.

– If

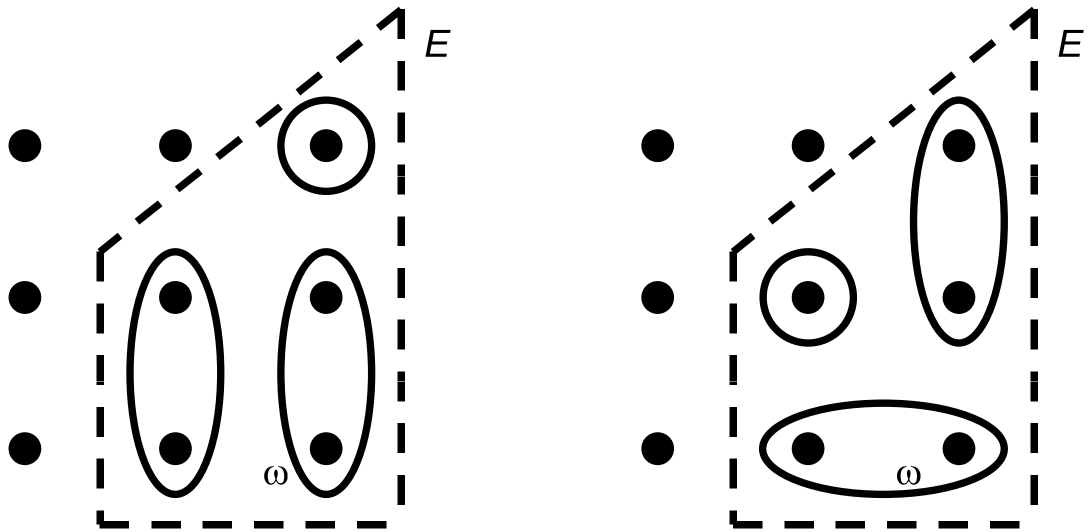
$$-\frac{t-1}{t} > \omega_1 > -\frac{t-2}{t}$$

and

$$\omega_2 > \frac{t-1}{t}$$

then $x_{1s} = 0$ and $x_{2s} = 1$ for $s < t$, and $x_{1s} = x_{2s} = 1$ for $s \geq t$.

Figure 1



a's partition *b*'s partition

Figure 2a

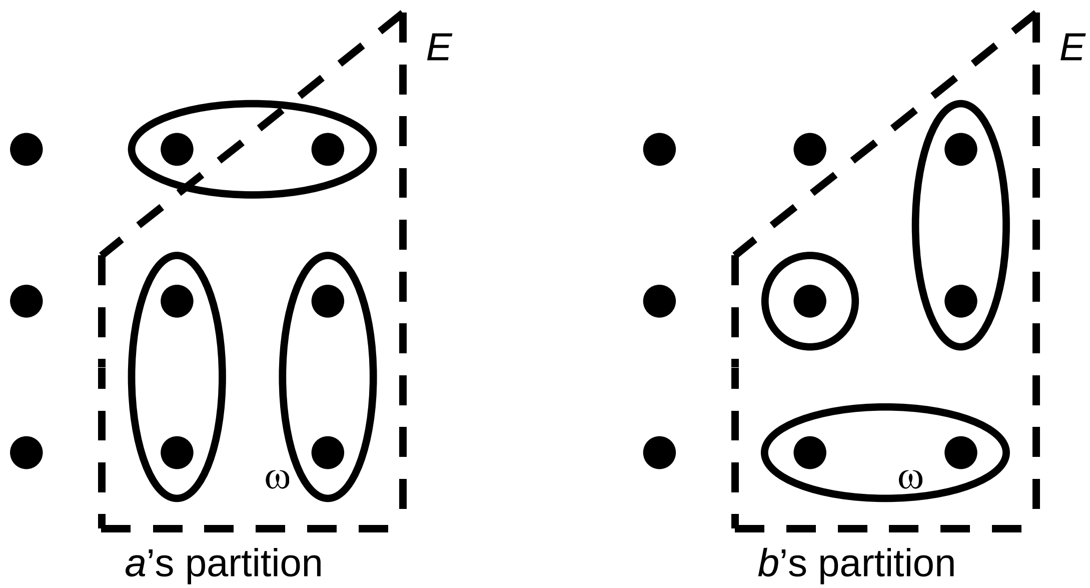
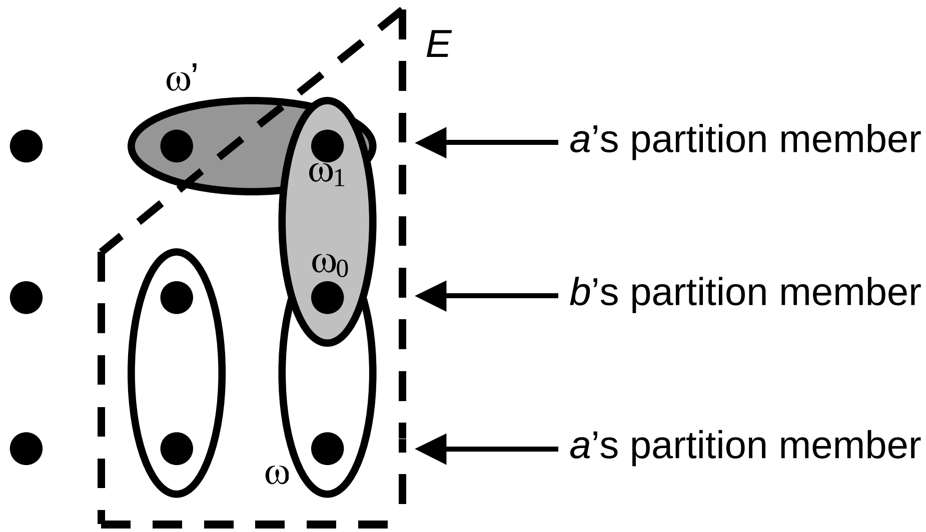


Figure 2b



Does *a* know that *b* knows that *a* knows *E*? Here the answer is no!