Appendix VIII
The generalized kinked specification

We continue to assume that state 2 has an objectively known probability $\pi_2 = \frac{1}{3}$, whereas states 1 and 3 occur with unknown probabilities π_1 and π_3. The utility of a portfolio $\mathbf{x} = (x_1, x_2, x_3)$ takes the the following form:

I. $x_2 \leq x_{\min}$

$$\alpha_1^1 u(x_2) + \alpha_2^1 u(x_{\min}) + \alpha_3^1 u(x_{\max})$$

II. $x_{\min} \leq x_2 \leq x_{\max}$

$$\alpha_1^2 u(x_{\min}) + \alpha_2^2 u(x_2) + \alpha_3^2 u(x_{\max})$$

III. $x_{\max} \leq x_2$

$$\alpha_1^3 u(x_{\min}) + \alpha_2^3 u(x_{\max}) + \alpha_3^3 u(x_2)$$

where $x_{\min} = \min\{x_1, x_3\}$ and $x_{\max} = \max\{x_1, x_3\}$. This formulation (equation 3) embeds the kinked specification (equation 1) as a special case. At the end of this note, we show that, through a suitable change of variables, the generalized kinked specification can also be interpreted as reflecting Recursive Nonexpected Utility (RNEU) where the ambiguity is modeled as an *equal* probability that $\pi_1 = \frac{2}{3}$ or $\pi_3 = \frac{2}{3}$. We begin by deriving the optimality conditions.

[1] Parameter restrictions

[1.1] Consistency

When $x_2 = x_{\min}$, consistency requires that

$$\left(\alpha_1^1 + \alpha_2^1\right) u\left(x_{\min}\right) + \alpha_3^1 u\left(x_{\max}\right) = \left(\alpha_1^2 + \alpha_2^2\right) u\left(x_{\min}\right) + \alpha_3^2 u\left(x_{\max}\right).$$

Without loss of generality we can assume that

$$\alpha_1^1 + \alpha_2^1 + \alpha_3^1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2,$$

in which case the equation preceding the last implies that

$$\left(\alpha_1^1 + \alpha_2^1\right)\left[u\left(x_{\min}\right) - u\left(x_{\max}\right)\right] = \left(\alpha_1^2 + \alpha_2^2\right)\left[u\left(x_{\min}\right) - u\left(x_{\max}\right)\right]$$

or

$$\alpha_1^1 + \alpha_2^1 = \alpha_1^2 + \alpha_2^2.$$

Similarly, when $x_2 = x_{\max}$ consistency requires that

$$\alpha_2^2 + \alpha_3^2 = \alpha_2^3 + \alpha_3^3.$$

We further normalize the coefficients so that

$$\alpha_1^j + \alpha_2^j + \alpha_3^j = 1 \text{ for all } j.$$

This leads to the following:

$$\alpha_3^1 = \alpha_3^2, \alpha_1^2 = \alpha_1^3.$$

[1.2] Reparametrization

Let

$$\alpha_1^1 = \beta_1,\ \alpha_1^1 + \alpha_2^1 = \beta_2,$$
$$\alpha_1^2 = \beta_3,\ \alpha_1^3 + \alpha_2^3 = \beta_4.$$

Using the consistency conditions, the original coefficients are reparametrized as follows:

$$\alpha_1^1 = \beta_1,\ \alpha_2^1 = \beta_2 - \beta_1,\ \alpha_3^1 = 1 - \beta_2,$$
$$\alpha_1^2 = \beta_3,\ \alpha_2^2 = \beta_2 - \beta_3,\ \alpha_3^2 = 1 - \beta_2,$$
$$\alpha_1^3 = \beta_3,\ \alpha_2^3 = \beta_4 - \beta_3,\ \alpha_3^3 = 1 - \beta_4.$$

Note that $\beta_1 \leq \beta_2 \leq 1$, $\beta_3 \leq \beta_2$ and $\beta_3 \leq \beta_4$. The utility of a portfolio $\mathbf{x} = (x_1, x_2, x_3)$ can be written with parameters $\beta_1, ..., \beta_4$:

I. $x_2 \leq x_{\min}$

$$\beta_1 u(x_2) + (\beta_2 - \beta_1)\, u(x_{\min}) + (1 - \beta_2)\, u(x_{\max})$$

II. $x_{\min} \leq x_2 \leq x_{\max}$

$$\beta_3 u(x_{\min}) + (\beta_2 - \beta_3)\, u(x_2) + (1 - \beta_2)\, u(x_{\max})$$

III. $x_{\max} \leq x_2$

$$\beta_3 u(x_{\min}) + (\beta_4 - \beta_3)\, u(x_{\max}) + (1 - \beta_4)\, u(x_2)$$

We adopt a simpler three-parameter model, in which the parameter δ measures the ambiguity attitudes, the parameter γ measures pessimism/optimism, and ρ is the coefficient of absolute risk aversion. The mapping from the two parameters δ and γ to the four parameters $\beta_1, ..., \beta_4$ is given by the equations

$$\beta_1 = \frac{1}{3} + \gamma$$
$$\beta_2 = \frac{2}{3} + \gamma + \delta$$
$$\beta_3 = \frac{1}{3} + \gamma + \delta$$
$$\beta_4 = \frac{2}{3} + \gamma,$$

with $-\frac{1}{3} < \delta, \gamma < \frac{1}{3}$ and $-\frac{1}{3} < \delta + \gamma < \frac{1}{3}$so that the decision weight attached to each payoff in equation 3 is nonnegative.

[2] Optimal solutions

By the symmetry property between x_1 and x_3, we know that $x_1 \leq x_3$ if and only if $p_1 \geq p_3$. We can use this fact to identify the price of $x_{\min}$ as $p_{\max} = \max\{p_1, p_3\}$. Similarly, we can identify the price of $x_{\max}$ as $p_{\min} = \min\{p_1, p_3\}$. For the rest of the note, we denote

$$x_i = x_{\min} \text{ and } x_j = x_{\max},$$
$$p_i = p_{\max} \text{ and } p_j = p_{\min}.$$

The maximization of the generalized kinked utility function can be broken down into three sub-problems:

- **SP1:** $x_2 \leq x_i$

$$\max_{\mathbf{x}} \left(\frac{1}{3} + \gamma\right) u(x_2) + \left(\frac{1}{3} + \delta\right) u(x_i) + \left(\frac{1}{3} - \gamma - \delta\right) u(x_j)$$
$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} = 1,\ x_j - x_i \geq 0 \text{ and } x_i - x_2 \geq 0.$$

- **SP2:** $x_i \leq x_2 \leq x_j$

$$\max_{\mathbf{x}} \left(\frac{1}{3} + \gamma + \delta\right) u(x_i) + \left(\frac{1}{3}\right) u(x_2) + \left(\frac{1}{3} - \gamma - \delta\right) u(x_j)$$
$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} = 1,\ x_j - x_2 \geq 0 \text{ and } x_2 - x_i \geq 0.$$

- **SP3:** $x_j \leq x_2$

$$\max_{\mathbf{x}} \left(\frac{1}{3} + \gamma + \delta\right) u(x_i) + \left(\frac{1}{3} - \delta\right) u(x_j) + \left(\frac{1}{3} - \gamma\right) u(x_2)$$
$$\text{s.t. } \mathbf{p} \cdot \mathbf{x} = 1,\ x_j - x_i \geq 0, \text{ and } x_2 - x_j \geq 0.$$

We adopt the CARA utility function $u(x) = -\frac{1}{\rho}e^{-\rho x}$. Instead of characterizing the exact conditions of prices and model parameters that tell which sub-problem the optimal solution of demands belongs to, we can adopt the following two-step algorithm computing a (globally) optimal demand:

Step 1 Given a price vector $\mathbf{p}$ and parameter values (δ, γ, ρ), compute a (locally) optimal solution in each of the three sub-problems.

Step 2 Compare the utilities of locally optimal solutions of three sub-problems and choose one yielding the highest utility as a (globally) optimal solution of demand.

In what follows, we characterize optimal demand with conditions on parameters in each sub-problem. Due to the fact that the CARA utility function generates a boundary solution for certain price vectors, we first set up the Lagrangian function for optimal solutions without the non-negativity condition of demand and impose that condition later, for computational ease.

[2.1] SP1: $x_2 \leq x_i$

The Lagrangian function without the non-negativity condition of demand is given by

$$\begin{aligned}\mathcal{L}(\mathbf{x}) = & \left(\frac{1}{3}+\gamma\right) u(x_2) + \left(\frac{1}{3}+\delta\right) u(x_i) + \left(\frac{1}{3}-\gamma-\delta\right) u(x_j) \\ & +\lambda_1 (x_i - x_2) + \lambda_2 (x_j - x_i) + \mu (1 - p_1 x_1 - p_2 x_2 - p_3 x_3).\end{aligned}$$

The necessary conditions for the maximization problem are given by

$$\begin{aligned}\mathcal{L}_2(\mathbf{x}) &= \left(\frac{1}{3}+\gamma\right) \exp\{-\rho x_2\} - \lambda_1 - \mu p_2 = 0, \\ \mathcal{L}_i(\mathbf{x}) &= \left(\frac{1}{3}+\delta\right) \exp\{-\rho x_i\} + \lambda_1 - \lambda_2 - \mu p_i = 0, \\ \mathcal{L}_j(\mathbf{x}) &= \left(\frac{1}{3}-\gamma-\delta\right) \exp\{-\rho x_j\} + \lambda_2 - \mu p_j = 0, \\ \lambda_1 (x_i - x_2) &= 0 = \lambda_2 (x_j - x_i), \lambda_1 \geq 0, \lambda_2 \geq 0, \\ x_i - x_2 &\geq 0, x_j - x_i \geq 0, \\ 1 &= p_1 x_1 + p_2 x_2 + p_3 x_3, \mu > 0.\end{aligned}$$

[2.1.1] $\lambda_1 > 0$ and $\lambda_2 > 0$

This implies that $x_i^* = x_2^* = x_j^*$. Then the optimal demand is given by

$$x_1^* = x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3}.$$

For the parameter conditions leading to this solution, we need to check the following:

$$\begin{aligned}\left(\frac{1}{3}+\gamma\right) \exp(-\rho x_2) &> \mu p_2, \\ \left(\frac{1}{3}-\gamma-\delta\right) \exp(-\rho x_j) &< \mu p_j, \\ \left(\frac{2}{3}+\gamma+\delta\right) \exp(-\rho x_i) &> \mu (p_2 + p_i), \\ \left(\frac{2}{3}-\gamma\right) \exp(-\rho x_j) &< \mu (p_1 + p_3),\end{aligned}$$

which yields the following inequality conditions under the optimal solution:

$$\begin{aligned}\ln\left(\frac{p_2}{p_j}\right) &< \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}-\gamma-\delta}\right), \\ \ln\left(\frac{p_2}{p_1+p_3}\right) &< \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right), \\ \ln\left(\frac{p_2+p_i}{p_j}\right) &< \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right).\end{aligned}$$

[2.1.2] $\lambda_1 = 0$ and $\lambda_2 > 0$

This implies that $x_1^* = x_3^* > x_2^*$. The solution without non-negativity condition is given by

$$x_2^* = \frac{1}{p_1+p_2+p_3} - \frac{(p_1+p_3)}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_1+p_3}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right)\right],$$
$$x_1^* = x_3^* = \frac{1}{p_1+p_2+p_3} + \frac{p_2}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_1+p_3}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right)\right].$$

The inequality conditions for this solution are given by

$$\ln\left(\frac{p_2}{p_1+p_3}\right) > \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right),$$
$$\ln\left(\frac{p_i}{p_j}\right) < \ln\left(\frac{\frac{1}{3}+\delta}{\frac{1}{3}-\gamma-\delta}\right).$$

If $x_2^* \geq 0$, then the optimal demand is

$$x_2^* = \frac{1}{p_1+p_2+p_3} - \frac{(p_1+p_3)}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_1+p_3}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right)\right],$$
$$x_1^* = x_3^* = \frac{1}{p_1+p_2+p_3} + \frac{p_2}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_1+p_3}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{2}{3}-\gamma}\right)\right].$$

If $x_2^* < 0$, then the optimal demand is given by

$$x_2^* = 0 \text{ and } x_1^* = x_3^* = \frac{1}{p_2+p_3}.$$

[2.1.3] $\lambda_1 > 0$ and $\lambda_2 = 0$

This implies that $x_2^* = x_i^* < x_j^*$. The solution without non-negativity condition is given by

$$x_2^* = x_i^* = \frac{1}{p_1+p_2+p_3} - \frac{p_j}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2+p_i}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right],$$
$$x_j^* = \frac{1}{p_1+p_2+p_3} + \frac{p_2+p_i}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2+p_i}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right].$$

The inequality condition for this solution is given by

$$\ln\left(\frac{p_2+p_i}{p_j}\right) > \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right),$$
$$\ln\left(\frac{p_2}{p_i}\right) < \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}+\delta}\right).$$

If $x_2^* = x_i^* \geq 0$, the optimal demand will be the same as above:

$$x_2^* = x_i^* = \frac{1}{p_1+p_2+p_3} - \frac{p_j}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2+p_i}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right],$$
$$x_j^* = \frac{1}{p_1+p_2+p_3} + \frac{p_2+p_i}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2+p_i}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right].$$

If $x_2^* = x_i^* < 0$, the optimal demand will be

$$x_2^* = x_i^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$

[2.1.4] $\lambda_1 = 0$ and $\lambda_2 = 0$

This implies that $x_j^* > x_i^* > x_2^*$. The solution without non-negativity condition is given by

$$\begin{aligned}
x_2^* = {} & \frac{1}{p_1+p_2+p_3} - \frac{p_i}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_i}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}+\delta}\right)\right] \\
& - \frac{p_j}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}-\gamma-\delta}\right)\right], \\
x_i^* = {} & \frac{1}{p_1+p_2+p_3} + \frac{p_2+p_j}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_i}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}+\delta}\right)\right] \\
& - \frac{p_j}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}-\gamma-\delta}\right)\right], \\
x_j^* = {} & \frac{1}{p_1+p_2+p_3} - \frac{p_i}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_i}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}+\delta}\right)\right] \\
& + \frac{p_2+p_i}{\rho\,(p_1+p_2+p_3)}\left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}+\gamma}{\frac{1}{3}-\gamma-\delta}\right)\right].
\end{aligned}$$

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: (i) $x_2^* < x_i^* < 0$, (ii) $x_2^* < 0$ and $x_i^* > 0$.

(i) $x_2^* < x_i^* < 0$

The optimal solution is then given by

$$x_j^* = \frac{1}{p_j} \text{ and } x_2^* = x_i^* = 0.$$

(ii) $x_2^* < 0$ and $x_i^* > 0$

The solution to the problem by imposing that $x_2^* = 0$ is given by

$$x_i' = \frac{1}{p_1 + p_3} - \frac{p_j}{\rho(p_1 + p_3)} \left[\ln\left(\frac{p_i}{p_j}\right) - \ln\left(\frac{\frac{1}{3} + \delta}{\frac{1}{3} - \gamma - \delta}\right) \right],$$
$$x_j' = \frac{1}{p_1 + p_3} + \frac{p_i}{\rho(p_1 + p_3)} \left[\ln\left(\frac{p_i}{p_j}\right) - \ln\left(\frac{\frac{1}{3} + \delta}{\frac{1}{3} - \gamma - \delta}\right) \right].$$

If $x_i' \geq 0$, then the solution with $x_2^* = 0$ is the optimal one in the original problem with the non-negativity condition of demands:

$$x_2^* = 0, x_i^* = x_i' \text{ and } x_j^* = x_j'.$$

If $x_i' < 0$, then the optimal solution is given by

$$x_2^* = x_i^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$

[2.2] SP2: $x_i \leq x_2 \leq x_j$

The Lagrangian function without the non-negativity condition of demand is given by

$$\mathcal{L}(\mathbf{x}) = \left(\frac{1}{3} + \gamma + \delta\right) u(x_i) + \left(\frac{1}{3}\right) u(x_2) + \left(\frac{1}{3} - \gamma - \delta\right) u(x_j)$$
$$+\lambda_1 (x_j - x_2) + \lambda_2 (x_2 - x_i) + \mu (1 - p_1 x_1 - p_2 x_2 - p_3 x_3).$$

The necessary conditions for the maximization problem are given by

$$\mathcal{L}_i(\mathbf{x}) = \left(\frac{1}{3} + \gamma + \delta\right) \exp(-\rho x_i) - \lambda_2 - \mu p_i = 0,$$
$$\mathcal{L}_2(\mathbf{x}) = \left(\frac{1}{3}\right) \exp(-\rho x_2) - \lambda_1 + \lambda_2 - \mu p_2 = 0,$$
$$\mathcal{L}_j(\mathbf{x}) = \left(\frac{1}{3} - \gamma - \delta\right) \exp(-\rho x_j) + \lambda_1 - \mu p_j = 0,$$
$$0 = \lambda_2 (x_2 - x_i) = \lambda_1 (x_j - x_2), \lambda_1 \geq 0, \lambda_2 \geq 0,$$
$$x_j - x_2 \geq 0, x_2 - x_i \geq 0,$$
$$\mu > 0, 1 - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0.$$

[2.2.1] $\lambda_1 > 0$ and $\lambda_2 > 0$

This implies that $x_i^* = x_2^* = x_j^*$. Thus, the optimal demand is given by

$$x_1^* = x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3}.$$

We need to check the following parameter conditions for the optimal demand:

$$\left(\frac{1}{3}+\gamma+\delta\right)\exp\{-\rho x_i\} > \mu p_i,$$
$$\left(\frac{1}{3}-\gamma-\delta\right)\exp\{-\rho x_j\} < \mu p_j,$$
$$\left(\frac{2}{3}+\gamma+\delta\right)\exp\{-\rho x_2\} > \mu\left(p_i+p_2\right),$$
$$\left(\frac{2}{3}-\gamma-\delta\right)\exp\{-\rho x_2\} < \mu\left(p_2+p_j\right).$$

Then we have the following inequality conditions for model parameters:

$$\ln\left(\frac{p_i}{p_j}\right) < \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right),$$
$$\ln\left(\frac{p_i}{p_2+p_j}\right) < \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{2}{3}-\gamma-\delta}\right),$$
$$\ln\left(\frac{p_i+p_2}{p_j}\right) < \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right).$$

[2.2.2] $\lambda_1 = 0$ and $\lambda_2 > 0$

This implies that $x_2^* = x_i^* < x_j^*$. The optimal demand without the non-negativity condition is given by

$$x_2^* = x_i^* = \frac{1}{p_1+p_2+p_3} - \frac{p_j}{\rho\left(p_1+p_2+p_3\right)}\left[\ln\left(\frac{p_i+p_2}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right],$$
$$x_j^* = \frac{1}{p_1+p_2+p_3} + \frac{p_2+p_i}{\rho\left(p_1+p_2+p_3\right)}\left[\ln\left(\frac{p_i+p_2}{p_j}\right) - \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right)\right].$$

The parameter condition for this solution is given by

$$\ln\left(\frac{p_i+p_2}{p_j}\right) > \ln\left(\frac{\frac{2}{3}+\gamma+\delta}{\frac{1}{3}-\gamma-\delta}\right),$$
$$\ln\left(\frac{p_i}{p_2}\right) < \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{1}{3}}\right).$$

If $x_2^* = x_i^* \geq 0$, then the above solution is the optimal one from the original maximization problem. Otherwise, the optimal solution with the non-negativity condition is given by

$$x_2^* = x_i^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$

[2.2.3] $\lambda_1 > 0$ and $\lambda_2 = 0$

This implies that $x_j^* = x_2^* > x_i^*$. The optimal demand without the non-negativity condition is given by

$$x_j^* = x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_i}{p_2 + p_j}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right) \right],$$

$$x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{p_2 + p_j}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_i}{p_2 + p_j}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right) \right].$$

The parameter condition for this solution is given by

$$\ln\left(\frac{p_i}{p_2 + p_j}\right) > \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right),$$

$$\ln\left(\frac{p_2}{p_j}\right) < \ln\left(\frac{\frac{1}{3}}{\frac{1}{3} - \gamma - \delta}\right).$$

If $x_i^* \geq 0$, the optimal demand from the original problem will be the same as above. Otherwise, the optimal demand with the non-negativity condition is

$$x_i^* = 0 \text{ and } x_2^* = x_j^* = \frac{1}{p_2 + p_j}.$$

[2.2.4] $\lambda_1 = 0$ and $\lambda_2 = 0$

This implies that $x_j^* > x_2^* > x_i^*$. The optimal solution without the non-negativity condition is given by

$$x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_2 + p_j)}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3}}\right) \right]$$
$$- \frac{p_j}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}}{\frac{2}{3} - \gamma - \delta}\right) \right],$$

$$x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3}}\right) \right]$$
$$- \frac{p_j}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}}{\frac{2}{3} - \gamma - \delta}\right) \right],$$

$$x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3}}\right) \right]$$
$$+ \frac{p_i + p_2}{\rho\,(p_1 + p_2 + p_3)} \left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}}{\frac{2}{3} - \gamma - \delta}\right) \right].$$

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: *(i)* $x_i^* < x_2^* < 0$, *(ii)* $x_i^* < 0$ and $x_2^* > 0$.

(i) $x_i^* < x_2^* < 0$

The optimal solution is then given by

$$x_i^* = x_2^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$

(ii) $x_i^* < 0$ and $x_2^* > 0$

By imposing that $x_i^* = 0$, we have the new solution as

$$\begin{aligned} x_2' &= \frac{1}{p_2 + p_j} - \frac{p_j}{\rho\left(p_2 + p_j\right)}\left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}}{\frac{2}{3} - \gamma - \delta}\right)\right], \\ x_j' &= \frac{1}{p_2 + p_j} + \frac{p_2}{\rho\left(p_2 + p_j\right)}\left[\ln\left(\frac{p_2}{p_j}\right) - \ln\left(\frac{\frac{1}{3}}{\frac{2}{3} - \gamma - \delta}\right)\right]. \end{aligned}$$

If $x_2' \geq 0$, then the optimal demand from the original problem will be

$$x_i^* = 0, x_2^* = x_2' \text{ and } x_j^* = x_j'.$$

If $x_2' < 0$, then the optimal demand will be

$$x_i^* = x_2^* = 0 \text{ and } x_j^* = \frac{1}{p_j}.$$

[2.3] SP3: $x_j \leq x_2$

The Lagrangian function without the non-negativity condition is given by

$$\begin{aligned} \mathcal{L}\left(\mathbf{x}\right) = &\left(\frac{1}{3} + \gamma + \delta\right) u\left(x_i\right) + \left(\frac{1}{3} - \delta\right) u\left(x_j\right) + \left(\frac{1}{3} - \gamma\right) u\left(x_2\right) \\ &+ \lambda_1\left(x_2 - x_j\right) + \lambda_2\left(x_j - x_i\right) + \mu\left(1 - p_1 x_1 - p_2 x_2 - p_3 x_3\right). \end{aligned}$$

The necessary conditions for the maximization problem are given by

$$\begin{aligned} \mathcal{L}_i\left(\mathbf{x}\right) &= \left(\frac{1}{3} + \gamma + \delta\right)\exp\left(-\rho x_i\right) - \lambda_2 - \mu p_i = 0, \\ \mathcal{L}_j\left(\mathbf{x}\right) &= \left(\frac{1}{3} - \delta\right)\exp\left(-\rho x_j\right) - \lambda_1 + \lambda_2 - \mu p_j = 0, \\ \mathcal{L}_2\left(\mathbf{x}\right) &= \left(\frac{1}{3} - \gamma\right)\exp\left(-\rho x_2\right) + \lambda_1 - \mu p_2 = 0, \\ 0 &= \lambda_1\left(x_2 - x_j\right) = \lambda_2\left(x_j - x_i\right), \lambda_1, \lambda_2 \geq 0, \\ \mu &> 0 \text{ and } 1 - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0. \end{aligned}$$

[2.3.1] $\lambda_1 > 0$ and $\lambda_2 > 0$

This implies that $x_2^* = x_j^* = x_i^*$. The optimal solution from the original problem is then given by

$$x_1^* = x_2^* = x_3^* = \frac{1}{p_1 + p_2 + p_3}.$$

The parameter conditions for this solution are given by

$$\ln\left(\frac{p_i}{p_2}\right) < \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3} - \gamma}\right),$$
$$\ln\left(\frac{p_i}{p_2 + p_j}\right) < \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right),$$
$$\ln\left(\frac{p_1 + p_3}{p_2}\right) < \ln\left(\frac{\frac{2}{3} + \gamma}{\frac{1}{3} - \gamma}\right).$$

[2.3.2] $\lambda_1 = 0$ and $\lambda_2 > 0$

This implies that $x_j^* = x_i^* < x_2^*$. The optimal solution without the non-negativity condition is given by

$$x_1^* = x_3^* = \frac{1}{p_1 + p_2 + p_3} - \frac{p_2}{\rho\,(p_1 + p_2 + p_3)}\left[\ln\left(\frac{p_1 + p_3}{p_2}\right) - \ln\left(\frac{\frac{2}{3} + \gamma}{\frac{1}{3} - \gamma}\right)\right],$$
$$x_2^* = \frac{1}{p_1 + p_2 + p_3} + \frac{(p_1 + p_3)}{\rho\,(p_1 + p_2 + p_3)}\left[\ln\left(\frac{p_1 + p_3}{p_2}\right) - \ln\left(\frac{\frac{2}{3} + \gamma}{\frac{1}{3} - \gamma}\right)\right].$$

The parameter conditions for this solution are given by

$$\ln\left(\frac{p_1 + p_3}{p_2}\right) > \ln\left(\frac{\frac{2}{3} + \gamma}{\frac{1}{3} - \gamma}\right),$$
$$\ln\left(\frac{p_i}{p_j}\right) < \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{1}{3} - \delta}\right).$$

If $x_1^* = x_3^* \geq 0$, then the optimal solution from the original problem is the same as above. Otherwise, the optimal demand with the non-negativity condition is given by

$$x_1^* = x_3^* = 0 \text{ and } x_2^* = \frac{1}{p_2}.$$

[2.3.3] $\lambda_1 > 0$ and $\lambda_2 = 0$

This implies that $x_2^* = x_j^* > x_i^*$. The optimal demand without the non-negativity condition is given by

$$x_i^* = \frac{1}{p_1 + p_2 + p_3} - \frac{(p_2 + p_j)}{\rho\,(p_1 + p_2 + p_3)}\left[\ln\left(\frac{p_i}{p_2 + p_j}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right)\right],$$
$$x_2^* = x_j^* = \frac{1}{p_1 + p_2 + p_3} + \frac{p_i}{\rho\,(p_1 + p_2 + p_3)}\left[\ln\left(\frac{p_i}{p_2 + p_j}\right) - \ln\left(\frac{\frac{1}{3} + \gamma + \delta}{\frac{2}{3} - \gamma - \delta}\right)\right].$$

The parameter condition for this solution is given by

$$\ln\left(\frac{p_i}{p_2+p_j}\right) > \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{2}{3}-\gamma-\delta}\right),$$
$$\ln\left(\frac{p_j}{p_2}\right) < \ln\left(\frac{\frac{1}{3}-\delta}{\frac{1}{3}-\gamma}\right).$$

If $x_i^* \geq 0$, then the optimal demand from the original problem is the same as above. Otherwise, the optimal demand with the non-negativity condition is given by

$$x_i^* = 0 \text{ and } x_2^* = x_j^* = \frac{1}{p_2+p_j}.$$

[2.3.4] $\lambda_1 = 0$ and $\lambda_2 = 0$

The conditions imply that $x_2^* > x_j^* > x_i^*$. The optimal demand without the non-negativity condition is given by

$$\begin{aligned}
x_2 = &\frac{1}{p_1+p_2+p_3} + \frac{p_i}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{1}{3}-\gamma}\right)\right] \\
&+ \frac{p_j}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_j}{p_2}\right) - \ln\left(\frac{\frac{1}{3}-\delta}{\frac{1}{3}-\gamma}\right)\right], \\
x_j = &\frac{1}{p_1+p_2+p_3} + \frac{p_i}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{1}{3}-\gamma}\right)\right] \\
&- \frac{(p_2+p_i)}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_j}{p_2}\right) - \ln\left(\frac{\frac{1}{3}-\delta}{\frac{1}{3}-\gamma}\right)\right], \\
x_i = &\frac{1}{p_1+p_2+p_3} - \frac{(p_2+p_j)}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_i}{p_2}\right) - \ln\left(\frac{\frac{1}{3}+\gamma+\delta}{\frac{1}{3}-\gamma}\right)\right] \\
&+ \frac{p_j}{\rho(p_1+p_2+p_3)}\left[\ln\left(\frac{p_j}{p_2}\right) - \ln\left(\frac{\frac{1}{3}-\delta}{\frac{1}{3}-\gamma}\right)\right].
\end{aligned}$$

If the non-negativity condition for each asset is satisfied, then the above solution is the optimal demand from the problem with the non-negativity condition of demands. Otherwise, we need to further refine the problem by setting an asset violating the non-negativity condition to be zero. There are two cases to consider: (*i*) $x_i^* < x_j^* < 0$, (*ii*) $x_i^* < 0$ and $x_j^* > 0$.

(*i*) $x_i^* < x_j^* < 0$

Then the optimal solution from the original problem is given by

$$x_1^* = x_3^* = 0 \text{ and } x_2^* = \frac{1}{p_2}.$$

(*ii*) $x_i^* < 0$ and $x_j^* > 0$

By imposing that $x_i^* = 0$, we have the following new solution as

$$x_2' = \frac{1}{p_2 + p_j} + \frac{p_j}{\rho\,(p_2 + p_j)}\left[\ln\left(\frac{p_j}{p_2}\right) - \ln\left(\frac{\frac{1}{3} - \delta}{\frac{1}{3} - \gamma}\right)\right],$$
$$x_j' = \frac{1}{p_2 + p_j} - \frac{p_2}{\rho\,(p_2 + p_j)}\left[\ln\left(\frac{p_j}{p_2}\right) - \ln\left(\frac{\frac{1}{3} - \delta}{\frac{1}{3} - \gamma}\right)\right].$$

If $x_j' \geq 0$, then the optimal demand from the original problem is given by

$$x_i^* = 0, x_j^* = x_j' \text{ and } x_2^* = x_2'.$$

If $x_j' < 0$, then the optimal demand from the original problem is given by

$$x_1^* = x_3^* = 0 \text{ and } x_2^* = \frac{1}{p_2}.$$

[2.4] Non-uniqueness of the optimal demand

Finally we note that when $\delta < 0$ and/or $\gamma < 0$, the optimal demand is not unique when $p_k = p_{k'}$ for some $k \neq k' = 1, 2, 3$ because the generalized kinked utility function is not quasi-convex everywhere. Nevertheless, the utility function is not quasi-convex in each sub-problem. The above characterization of the optimal demands incorporates the cases of non-uniqueness.

[3] Recursive Nonexpected Utility (RNEU)

Finally, we show that the generalized kinked specification can also be interpreted as reflecting a special case of RNEU where there is an equal probability that $\pi_1 = \frac{2}{3}$ or $\pi_3 = \frac{2}{3}$. Consider the following two-stage recursive Rank-Dependent Utility (RDU) model. Given a fixed underlying distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, the first-stage rank-dependent expected utility $V_{\boldsymbol{\pi}}$ is given by

$$V_{(\frac{2}{3},\frac{1}{3},0)}(\mathbf{x}) = [1 - w(\tfrac{1}{3})] \max\{u(x_1), u(x_2)\} + w(\tfrac{1}{3}) \min\{u(x_1), u(x_2)\},$$
$$V_{(0,\frac{1}{3},\frac{2}{3})}(\mathbf{x}) = [1 - w(\tfrac{1}{3})] \max\{u(x_2), u(x_3)\} + w(\tfrac{1}{3}) \min\{u(x_2), u(x_3)\}.$$

The second stage takes the rank-dependent expectation of the first-stage rank-dependent expected utilities:

$$\begin{aligned} U(\mathbf{x}) = {} & [1 - w(\tfrac{1}{2})] \max\left\{V_{(\frac{2}{3},\frac{1}{3},0)}(\mathbf{x}), V_{(0,\frac{1}{3},\frac{2}{3})}(\mathbf{x})\right\} \\ & + w(\tfrac{1}{2}) \min\left\{V_{(\frac{2}{3},\frac{1}{3},0)}(\mathbf{x}), V_{(0,\frac{1}{3},\frac{2}{3})}(\mathbf{x})\right\}, \end{aligned}$$

and the decision weights can be expressed as follows:

$$\begin{aligned} \beta_1 &= w(\tfrac{1}{3}), \\ \beta_2 - \beta_1 &= w(\tfrac{1}{2})[1 - w(\tfrac{1}{3})], \\ \beta_3 &= w(\tfrac{1}{2})w(\tfrac{1}{3}), \\ \beta_4 - \beta_3 &= [1 - w(\tfrac{1}{2})]w(\tfrac{1}{3}). \end{aligned}$$

Now consider the three relevant cases:

I. $x_2 \leq x_{\min}$

$$\begin{aligned} U(\mathbf{x}) &= [1 - w(\tfrac{1}{2})] \left\{[1 - w(\tfrac{1}{3})]u(x_{\max}) + w(\tfrac{1}{3})u(x_2)\right\} \\ &\quad + w(\tfrac{1}{2}) \left\{[1 - w(\tfrac{1}{3})]u(x_{\min}) + w(\tfrac{1}{3})u(x_2)\right\}. \end{aligned}$$

Rearranging,

$$U(\mathbf{x}) = \beta_1 u(x_2) + (\beta_2 - \beta_3)\, u(x_{\min}) + (1 - \beta_2)u(x_{\max}).$$

II. $x_{\min} \leq x_2 \leq x_{\max}$

$$\begin{aligned} U(\mathbf{x}) &= [1 - w(\tfrac{1}{2})] \left\{[1 - w(\tfrac{1}{3})]u(x_{\max}) + w(\tfrac{1}{3})u(x_2)\right\} \\ &\quad + w(\tfrac{1}{2}) \left\{[1 - w(\tfrac{1}{3})]u(x_2) + w(\tfrac{1}{3})u(x_{\min})\right\}. \end{aligned}$$

Rearranging,

$$U(\mathbf{x}) = \beta_3 u(x_{\min}) + (\beta_2 - \beta_3)u(x_2) + (1 - \beta_2)u(x_{\max}).$$

III. $x_{\max} \leq x_2$

$$\begin{aligned} U(\mathbf{x}) &= [1 - w(\tfrac{1}{2})] \left\{[1 - w(\tfrac{1}{3})]u(x_2) + w(\tfrac{1}{3})u(x_{\max})\right\} \\ &\quad + w(\tfrac{1}{2}) \left\{[1 - w(\tfrac{1}{3})]u(x_2) + w(\tfrac{1}{3})u(x_{\min})\right\}. \end{aligned}$$

Rearranging,

$$U(\mathbf{x}) = \beta_3 u(x_{\min}) + (\beta_4 - \beta_3)\, u(x_{\max}) + (1 - \beta_4)u(x_2).$$