UC Berkeley
Haas School of Business
Game Theory
(EMBA 296 & EWMBA 211)
Summer 2015

Strategic form games

Block 2
Jun 11-12, 2015

Game theory

Adam Brandenburger:

> *There is nothing so practical as a good [game] theory. A good theory confirms the conventional wisdom that "less is more." A good theory does less because it does not give answers. At the same time, it does a lot more because it helps people organize what they know and uncover what they do not know. A good theory gives people the tools to discover what is best for them...*

Types of games

We study four groups of game theoretic models:

I strategic games

II extensive games (with perfect and imperfect information)

III repeated games

IV coalitional games

Side note I: individual preferences

Consider some (finite) set of alternatives or bundles $(x, y, z, ...)$.

- Formally, we represent the decision-maker's preferences by a binary relation $\succsim$ defined on the set of bundles.

- For any pair of bundles x and y, if the decision-maker says that x is at least as good as y, we write

$$x \succsim y$$

and say that x is *weakly preferred* to y.

<u>Bear in mind</u>: economic theory often seeks to convince you with simple examples and then gets you to extrapolate. This simple construction works in wider (and wilder circumstances).

From the weak preference relation $\succsim$ we derive two other relations on the set of alternatives:

– Strict performance relation

$$x \succ y \ \text{ if and only if } x \succsim y \text{ and not } y \succsim x.$$

The phrase $x \succ y$ is read x is *strictly preferred* to y.

– Indifference relation

$$x \sim y \ \text{ if and only if } x \succsim y \text{ and } y \succsim x.$$

The phrase $x \sim y$ is read x is *indifferent* to y.

Side note II: individual rationality

Economic theory begins with two assumptions about preferences. These assumptions are so fundamental that we can refer to them as “axioms” of decision theory.

[1] Completeness

$$x \succsim y \text{ or } y \succsim x$$

for any pair of bundles x and y.

[2] Transitivity

$$\text{if } x \succsim y \text{ and } y \succsim z \text{ then } x \succsim z$$

for any three bundles x, y and z.

Together, completeness and transitivity constitute the formal definition of *rationality* as the term is used in economics. Homo economicus (rational economic) agents are ones who

> have the ability to make choices [1], and whose choices display a logical consistency [2].

(Only) the preferences of a homo economicus can be represented, or summarized, by a *utility function*.

Strategic games

A strategic game consists of

- a set of players (decision makers)
- for each player, a set of possible actions
- for each player, preferences over the set of action profiles (outcomes).

In strategic games, players move simultaneously. A wide range of situations may be modeled as strategic games.

A two-player (finite) strategic game

The game can be described conveniently in a so-called bi-matrix. For example, a generic 2×2 (two players and two possible actions for each player) game

	L	R
T	a_1, a_2	b_1, b_2
B	c_1, c_2	d_1, d_2

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2). The two numbers in a box formed by a specific row and column are the players' payoffs given that these actions were chosen.

In this game above a_1 and a_2 are the payoffs of player 1 and player 2 respectively when player 1 is choosing strategy T and player 2 strategy L.

Applying the definition of a strategic game to the 2×2 game above yields:

- Players: $\{1, 2\}$
- Action sets: $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$
- Action profiles (outcomes):

$$A = A_1 \times A_2 = \{(T, L), (T, R), (B, L), (B, R)\}$$

- Preferences: $\succsim_1$and $\succsim_2$are given by the bi-matrix.

Classical 2×2 games

- The following simple 2×2 games represent a variety of strategic situations.

- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.

- These classical games "span" the set of almost *all* games (strategic equivalence).

Game I: Prisoner's Dilemma

	$Work$	$Goof$
$Work$	$3, 3$	$0, 4$
$Goof$	$4, 0$	$1, 1$

A situation where there are gains from cooperation but each player has an incentive to "free ride."

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.

Game II: Battle of the Sexes (BoS)

	$Ball$	$Show$
$Ball$	$2, 1$	$0, 0$
$Show$	$0, 0$	$1, 2$

Like the Prisoner's Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.

Game III-V: Coordination, Hawk-Dove, and Matching Pennies

	$Ball$	$Show$
$Ball$	2, 2	0, 0
$Show$	0, 0	1, 1

	$Dove$	$Hawk$
$Dove$	3, 3	1, 4
$Hawk$	4, 1	0, 0

	$Head$	$Tail$
$Head$	$1, -1$	$-1, 1$
$Tail$	$-1, 1$	$1, -1$

Best response and dominated actions

Action T is player 1's *best response* to action L player 2 if T is the optimal choice when 1 *conjectures* that 2 will play L.

Player 1's action T is *strictly* dominated if it is never a best response (inferior to B no matter what the other players do).

In the Prisoner's Dilemma, for example, action $Work$ is strictly dominated by action $Goof$. As we will see, a strictly dominated action is not used in any Nash equilibrium.

Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a NE is a set of actions such that all players are doing their best given the actions of the other players.

Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

		q L	$1-q$ R
p	T	pq	$p(1-q)$
$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

Let p and q be the probabilities that player 1 and 2 respectively assign to the strategy *Ball*.

Player 2 will be indifferent between using her strategy B and S when player 1 assigns a probability p such that her expected payoffs from playing B and S are the same. That is,

$$\begin{gathered} 1p + 0(1-p) = 0p + 2(1-p) \\ p = 2 - 2p \\ p^* = 2/3 \end{gathered}$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy B and probability $1 - p^* = 1/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

Similarly, player 1 will be indifferent between using her strategy B and S when player 2 assigns a probability q such that her expected payoffs from playing B and S are the same. That is,

$$2q + 0(1-q) = 0q + 1(1-q)$$
$$2q = 1 - q$$
$$q^* = 1/3$$

Hence, when player 2 assigns probability $q^* = 1/3$ to her strategy B and probability $1 - q^* = 2/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

In terms of best responses:

$$B_1(q) = \begin{cases} p = 1 & if \quad p > 1/3 \\ p \in [0, 1] & if \quad p = 1/3 \\ p = 0 & if \quad p < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} q = 1 & if \quad p > 2/3 \\ q \in [0, 1] & if \quad p = 2/3 \\ q = 0 & if \quad p < 2/3 \end{cases}$$

The BoS has two Nash equilibria in pure strategies $\{(B, B), (S, S)\}$ and one in mixed strategies $\{(2/3, 1/3)\}$. In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).

Oligopoly

- Another form of market structure is **oligopoly** – a market in which only a few firms compete with one another, and entry of new firms is impeded.

- The situation is known as the Cournot model after Antoine Augustin Cournot, a French economist, philosopher and mathematician (1801-1877).

- In the basic example, a single good is produced by two firms (the industry is a "duopoly").

Cournot's oligopoly model (1838) (Antoine Augustin Cournot, an economist, philosopher and mathematician, 1801-1877).

- A single good is produced by two firms (the industry is a "duopoly").

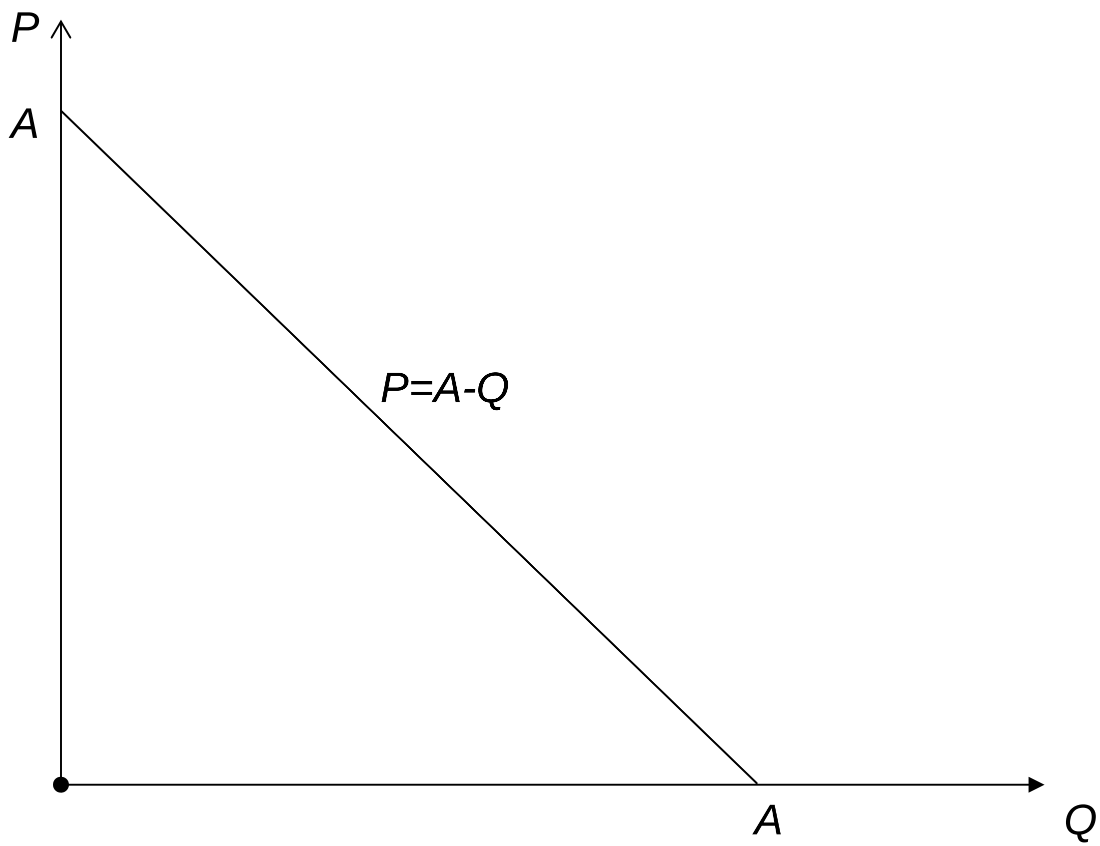
- The cost for firm $i = 1, 2$ for producing q_i units of the good is given by $c_i q_i$ ("unit cost" is constant equal to $c_i > 0$).

- If the firms' total output is $Q = q_1 + q_2$ then the market price is

  $$P = A - Q$$

  if $A \geq Q$ and zero otherwise (linear inverse demand function). We also assume that $A > c$.

The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

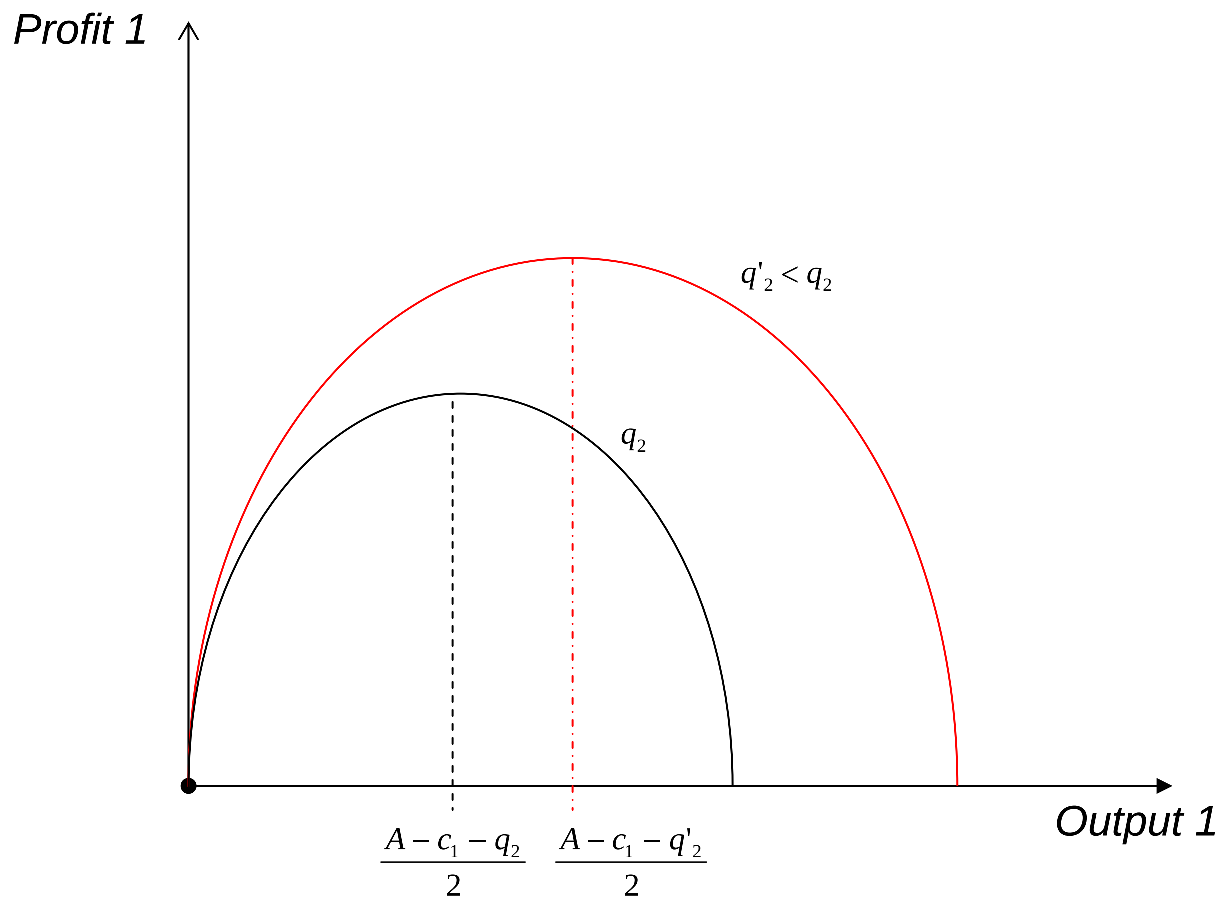
But first we need the firms payoffs (profits):

$$\begin{aligned}
\pi_1 &= Pq_1 - c_1q_1 \\
&= (A - Q)q_1 - c_1q_1 \\
&= (A - q_1 - q_2)q_1 - c_1q_1 \\
&= (A - q_1 - q_2 - c_1)q_1
\end{aligned}$$

and similarly,

$$\pi_2 = (A - q_1 - q_2 - c_2)q_2$$

Firm 1's profit as a function of its output
(given firm 2's output)



To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2.

Using calculus, we set the derivative of firm 1's profit with respect to q_1 equal to zero and solve for q_1:

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the values of q_2 and c_1.

Because firm 2's cost function is $c_2 \neq c_1$, its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

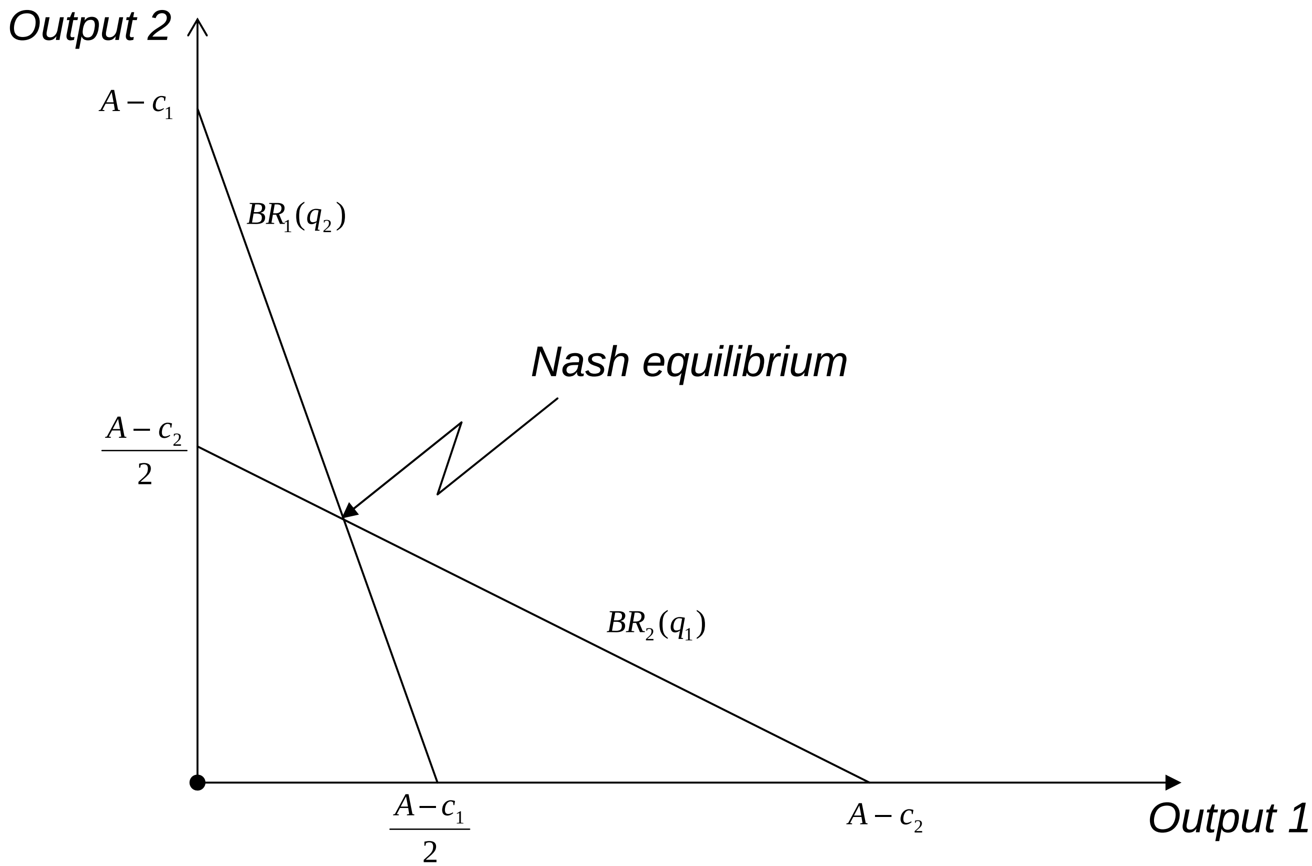
A Nash equilibrium of the Cournot's game is a pair (q_1^*, q_2^*) of outputs such that q_1^* is a best response to q_2^* and q_2^* is a best response to q_1^*.

From the figure below, we see that there is exactly one such pair of outputs

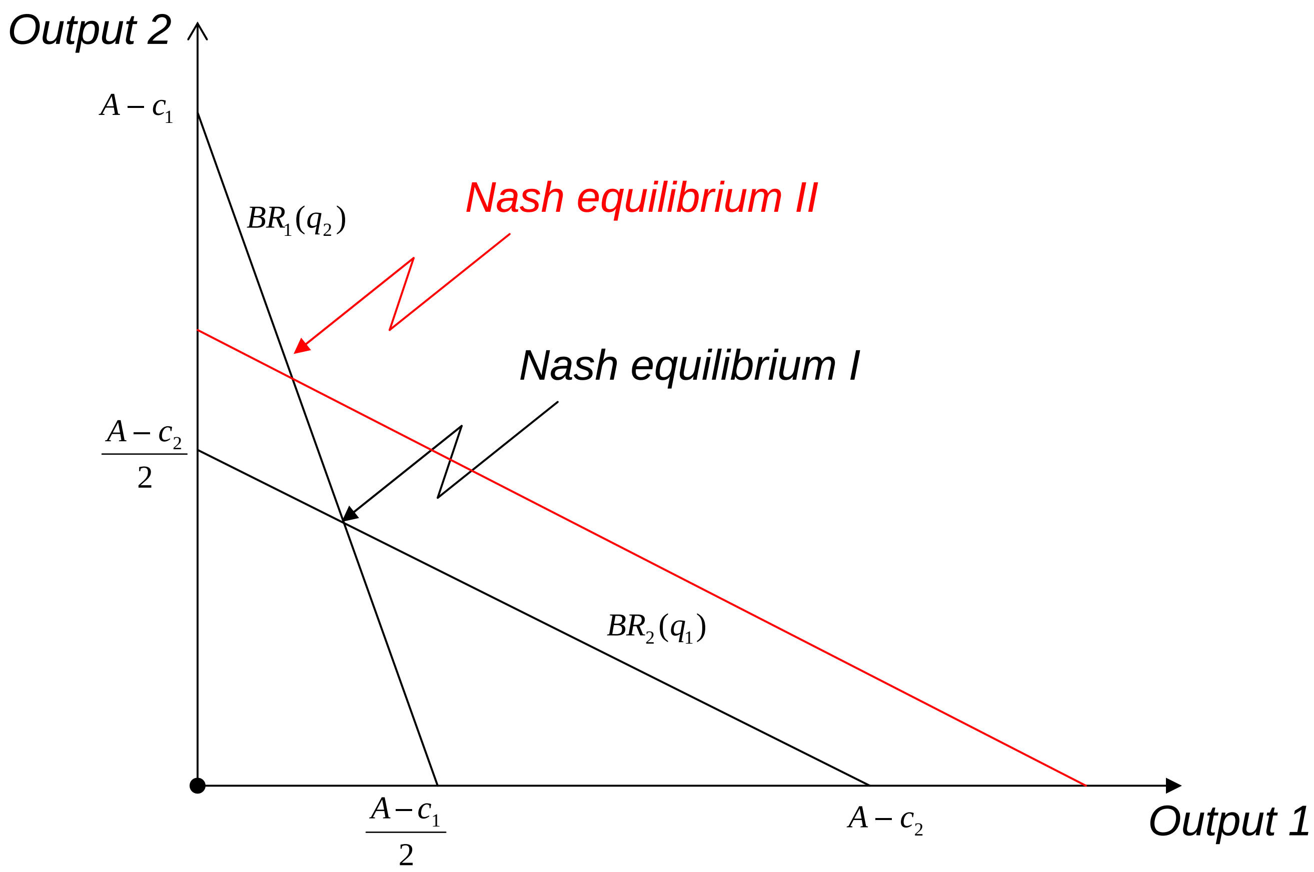
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

The best response functions in the Cournot's duopoly game



Nash equilibrium comparative statics
(a decrease in the cost of firm 2)



A question: what happens when consumers are willing to pay more (*A* increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

[1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.

[2] The relation between the equilibrium profits and the number of firms.

[1] <u>Collusive outcomes</u>: in the Cournot's duopoly game, there is a pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium.

[2] <u>Competition</u>: The price at the Nash equilibrium if the two firms have the *same* unit cost $c_1 = c_2 = c$ is given by

$$\begin{aligned} P^* &= A - q_1^* - q_2^* \\ &= \frac{1}{3}(A + 2c) \end{aligned}$$

which is above the unit cost c. But as the number of firm increases, the equilibrium price deceases, approaching c (zero profits).

Auctions
(if time permits...)

From Babylonia to eBay, auctioning has a very long history.

Babylon:

- women at marriageable age.

Athens, Rome, and medieval Europe:

- rights to collect taxes, dispose of confiscated property, lease of land and mines,

and many more...

The word “auction” comes from the Latin *augere*, meaning “to increase.”

The earliest use of the English word “auction” given by the *Oxford English Dictionary* dates from 1595 and concerns an auction “when will be sold Slaves, household goods, etc.”

In this era, the auctioneer lit a short candle and bids were valid only if made before the flame went out – Samuel Pepys (1633-1703) –

- Auctions, broadly defined, are used to allocate significant economics resources.

  Examples: works of art, government bonds, offshore tracts for oil exploration, radio spectrum, and more.

- Auctions take many forms. A game-theoretic framework enables to understand the consequences of various auction designs.

- Game theory can suggest the design likely to be most effective, and the one likely to raise the most revenues.

Types of auctions

Sequential / simultaneous

Bids may be called out sequentially or may be submitted simultaneously in sealed envelopes:

- English (or oral) – the seller actively solicits progressively higher bids and the item is soled to the highest bidder.
- Dutch – the seller begins by offering units at a “high” price and reduces it until all units are soled.
- Sealed-bid – all bids are made simultaneously, and the item is sold to the highest bidder.

First-price / second-price

The price paid may be the highest bid or some other price:

- First-price – the bidder who submits the highest bid wins and pay a price equal to her bid.
- Second-prices – the bidder who submits the highest bid wins and pay a price equal to the second highest bid.

Variants: all-pay (lobbying), discriminatory, uniform, Vickrey (William Vickrey, Nobel Laureate 1996), and more.

Private-value / common-value

Bidders can be certain or uncertain about each other's valuation:

- In <u>private-value</u> auctions, valuations differ among bidders, and each bidder is certain of her own valuation and can be certain or uncertain of every other bidder's valuation.

- In <u>common-value</u> auctions, all bidders have the same valuation, but bidders do not know this value precisely and their estimates of it vary.

First-price auction (with perfect information)

To define the game precisely, denote by v_i the value that bidder i attaches to the object. If she obtains the object at price p then her payoff is $v_i - p$.

Assume that bidders' valuations are all different and all positive. Number the bidders 1 through n in such a way that

$$v_1 > v_2 > \cdots > v_n > 0.$$

Each bidder i submits a (sealed) bid b_i. If bidder i obtains the object, she receives a payoff $v_i - b_i$. Otherwise, her payoff is zero.

Tie-breaking – if two or more bidders are in a tie for the highest bid, the winner is the bidder with the highest valuation.

In summary, a first-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.
- Actions: the set of possible bids b_i of each player i (nonnegative numbers).
- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \bar{b} & \text{if} \quad b_i = \bar{b} \text{ and } v_i > v_j \text{ if } b_j = \bar{b} \\ 0 & \text{if} \quad b_i < \bar{b} \end{cases}$$

where $\bar{b}$ is the highest bid.

The set of Nash equilibria is the set of profiles $(b_1, ..., b_n)$ of bids with the following properties:

$$\begin{array}{ll} [1] & v_2 \leq b_1 \leq v_1 \\ [2] & b_j \leq b_1 \ \underline{\text{for all}}\ j \neq 1 \\ [3] & b_j = b_1 \ \underline{\text{for some}}\ j \neq 1 \end{array}$$

It is easy to verify that all these profiles are Nash equilibria. It is harder to show that there are no other equilibria. We can easily argue, however, that there is no equilibrium in which player 1 does not obtain the object.

$\Longrightarrow$ The first-price sealed-bid auction is socially efficient, but does not necessarily raise the most revenues.

Second-price auction (with perfect information)

A second-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.

- Actions: the set of possible bids b_i of each player i (nonnegative numbers).

- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \overline{b} & \text{if} \quad b_i > \overline{b} \text{ or } b_i = \overline{b} \text{ and } v_i > v_j \text{ if } b_j = \overline{b} \\ 0 & \text{if} \quad b_i < \overline{b} \end{cases}$$

  where $\overline{b}$ is the highest bid submitted by a player other than i.

First note that for any player i the bid $b_i = v_i$ is a (weakly) dominant action (a "truthful" bid), in contrast to the first-price auction.

The second-price auction has many equilibria, but the equilibrium $b_i = v_i$ for all i is distinguished by the fact that every player's action dominates all other actions.

Another equilibrium in which player $j \neq 1$ obtains the good is that in which

[1] $b_1 < v_j$ and $b_j > v_1$
[2] $b_i = 0$ <u>for all</u> $i \neq \{1, j\}$

Common-value auctions and the winner's curse

Suppose we all participate in a sealed-bid auction for a jar of coins. Once you have estimated the amount of money in the jar, what are your bidding strategies in first- and second-price auctions?

The winning bidder is likely to be the bidder with the largest positive error (the largest overestimate).

In this case, the winner has fallen prey to the so-called the winner's curse. Auctions where the winner's curse is significant are oil fields, spectrum auctions, pay per click, and more.