

**UC Berkeley  
Haas School of Business  
Game Theory  
(EMBA 296 & EW MBA 211)  
Summer 2016**

**More on strategic games and extensive games with perfect information**

**Block 2  
Jun 12, 2016**

**Food for thought**

## LUPI

Many players simultaneously chose an integer between 1 and 99,999. Whoever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.

## Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

## Maximal game (sealed-bid second-price auction)

Two bidders, each of whom privately observes a signal  $X_i$  that is independent and identically distributed (i.i.d.) from a uniform distribution on  $[0, 10]$ .

Let  $X^{\max} = \max\{X_1, X_2\}$  and assume the ex-post common value to the bidders is  $X^{\max}$ .

Bidders bid in a sealed-bid second-price auction where the highest bidder wins, earns the common value  $X^{\max}$  and pays the second highest bid.

**Strategic games  
(review)**

## A two-player (finite) strategic game

The game can be described conveniently in a so-called bi-matrix. For example, a generic  $2 \times 2$  (two players and two possible actions for each player) game

	<i>L</i>	<i>R</i>
<i>T</i>	$a_1, a_2$	$b_1, b_2$
<i>B</i>	$c_1, c_2$	$d_1, d_2$

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2). The two numbers in a box formed by a specific row and column are the players' payoffs given that these actions were chosen.

In this game above  $a_1$  and  $a_2$  are the payoffs of player 1 and player 2 respectively when player 1 is choosing strategy  $T$  and player 2 strategy  $L$ .

## Classical $2 \times 2$ games

- The following simple  $2 \times 2$  games represent a variety of strategic situations.
- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.
- These classical games “span” the set of almost *all* games (strategic equivalence).



## Game I: Prisoner's Dilemma

	<i>Work</i>	<i>Goof</i>
<i>Work</i>	3, 3	0, 4
<i>Goof</i>	4, 0	1, 1

A situation where there are gains from cooperation but each player has an incentive to “free ride.”

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.

## Game II: Battle of the Sexes (BoS)

	<i>Ball</i>	<i>Show</i>
<i>Ball</i>	2, 1	0, 0
<i>Show</i>	0, 0	1, 2

Like the Prisoner's Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.

### Game III-V: Coordination, Hawk-Dove, and Matching Pennies

	<i>Ball</i>	<i>Show</i>
<i>Ball</i>	2, 2	0, 0
<i>Show</i>	0, 0	1, 1

	<i>Dove</i>	<i>Hawk</i>
<i>Dove</i>	3, 3	1, 4
<i>Hawk</i>	4, 1	0, 0

	<i>Head</i>	<i>Tail</i>
<i>Head</i>	1, -1	-1, 1
<i>Tail</i>	-1, 1	1, -1

## Best response and dominated actions

Action  $T$  is player 1's *best response* to action  $L$  player 2 if  $T$  is the optimal choice when 1 *conjectures* that 2 will play  $L$ .

Player 1's action  $T$  is *strictly* dominated if it is never a best response (inferior to  $B$  no matter what the other players do).

In the Prisoner's Dilemma, for example, action *Work* is strictly dominated by action *Goof*. As we will see, a strictly dominated action is not used in any Nash equilibrium.

## Nash equilibrium

Nash equilibrium ( $NE$ ) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a  $NE$  is a set of actions such that all players are doing their best given the actions of the other players.

## Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

		$q$	$1 - q$
		$L$	$R$
$p$	$T$	$pq$	$p(1 - q)$
$1 - p$	$B$	$(1 - p)q$	$(1 - p)(1 - q)$

Let  $p$  and  $q$  be the probabilities that player 1 and 2 respectively assign to the strategy *Ball*.

Player 2 will be indifferent between using her strategy  $B$  and  $S$  when player 1 assigns a probability  $p$  such that her expected payoffs from playing  $B$  and  $S$  are the same. That is,

$$\begin{aligned}1p + 0(1 - p) &= 0p + 2(1 - p) \\ p &= 2 - 2p \\ p^* &= 2/3\end{aligned}$$

Hence, when player 1 assigns probability  $p^* = 2/3$  to her strategy  $B$  and probability  $1 - p^* = 1/3$  to her strategy  $S$ , player 2 is indifferent between playing  $B$  or  $S$  any mixture of them.

Similarly, player 1 will be indifferent between using her strategy  $B$  and  $S$  when player 2 assigns a probability  $q$  such that her expected payoffs from playing  $B$  and  $S$  are the same. That is,

$$\begin{aligned}2q + 0(1 - q) &= 0q + 1(1 - q) \\2q &= 1 - q \\q^* &= 1/3\end{aligned}$$

Hence, when player 2 assigns probability  $q^* = 1/3$  to her strategy  $B$  and probability  $1 - q^* = 2/3$  to her strategy  $S$ , player 2 is indifferent between playing  $B$  or  $S$  any mixture of them.



In terms of best responses:

$$B_1(q) = \begin{cases} p = 1 & \text{if } p > 1/3 \\ p \in [0, 1] & \text{if } p = 1/3 \\ p = 0 & \text{if } p < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} q = 1 & \text{if } p > 2/3 \\ q \in [0, 1] & \text{if } p = 2/3 \\ q = 0 & \text{if } p < 2/3 \end{cases}$$

The *BoS* has two Nash equilibria in pure strategies  $\{(B, B), (S, S)\}$  and one in mixed strategies  $\{(2/3, 1/3)\}$ . In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).

## Three Matching Pennies games in the laboratory

		.48	.52
		$a_2$	$b_2$
.48	$a_1$	80, 40	40, 80
.52	$b_1$	40, 80	80, 40

		.16	.84			.80	.20
		$a_2$	$b_2$			$a_2$	$b_2$
.96	$a_1$	320, 40	40, 80	.08	$a_1$	44, 40	40, 80
.04	$b_1$	40, 80	80, 40	.92	$b_1$	40, 80	80, 40

## Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

$\varepsilon$  of players consists of mutants taking action  $a$  while others take action  $a^*$ .

## Evolutionary stable strategy (*ESS*)

Consider a two-player payoff symmetric game

$$G = \langle \{1, 2\}, (A, A), (u_1, u_2) \rangle$$

where

$$u_1(a_1, a_2) = u_2(a_2, a_1)$$

(players exchanging  $a_1$  and  $a_2$ ).

$a^* \in A$  is *ESS* if and only if for any  $a \in A$ ,  $a \neq a^*$  and  $\varepsilon > 0$  sufficiently small

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if and only if for any  $a \neq a^*$  either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

## Three results on *ESS*

[1] If  $a^*$  is an *ESS* then  $(a^*, a^*)$  is a *NE*.

Suppose not. Then, there exists a strategy  $a \in A$  such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for  $\varepsilon$  small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus  $a^*$  is not an *ESS*.

[2] If  $(a^*, a^*)$  is a strict  $NE$  ( $u(a^*, a^*) > u(a, a^*)$  for all  $a \in A$ ) then  $a^*$  is an  $ESS$ .

Suppose  $a^*$  is not an  $ESS$ . Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so  $(a^*, a^*)$  can be a  $NE$  but not a strict  $NE$ .

[3] The two-player two-action game

	$a$	$a'$
$a$	$w, w$	$x, y$
$a'$	$y, x$	$z, z$

has a strategy which is *ESS*.

If  $w > y$  or  $z > x$  then  $(a, a)$  or  $(a', a')$  are strict *NE*, and thus  $a$  or  $a'$  are *ESS*.

If  $w < y$  and  $z < x$  then there is a unique symmetric mixed strategy *NE*  $(\alpha^*, \alpha^*)$  where

$$\alpha^*(a) = (z - x) / (w - y + z - x)$$

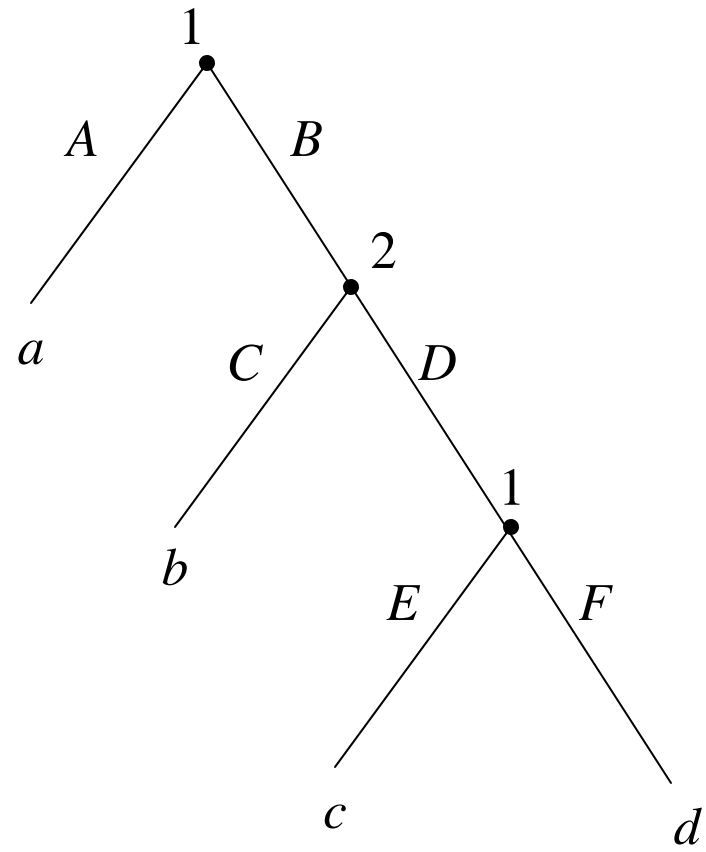
and  $u(\alpha^*, \alpha) > u(\alpha, \alpha)$  for any  $\alpha \neq \alpha^*$ .

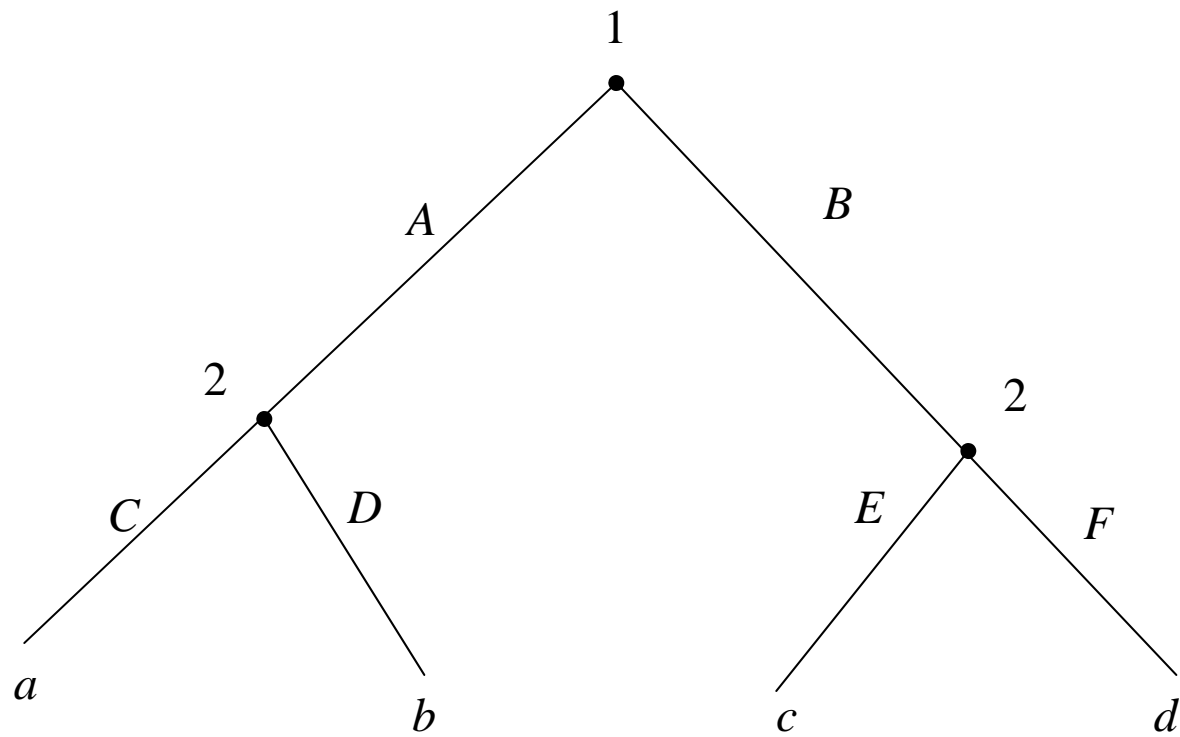


## **Extensive games with perfect information**

## **Extensive games with perfect information**

- The model of a strategic suppresses the sequential structure of decision making.
  - All players simultaneously choose their plan of action once and for all.
- The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly.
  - In an extensive game of perfect information all players are fully informed about all previous actions.

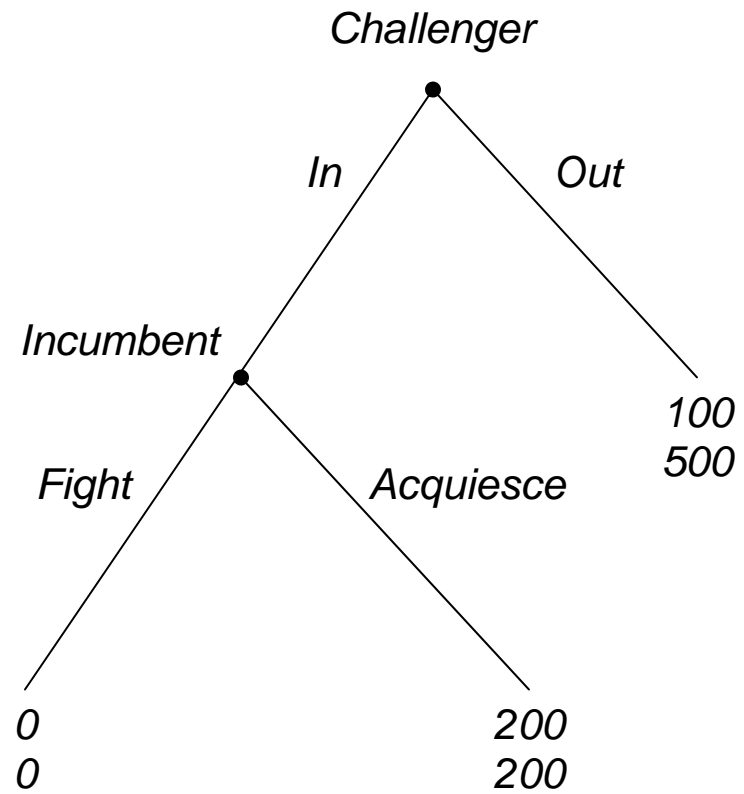




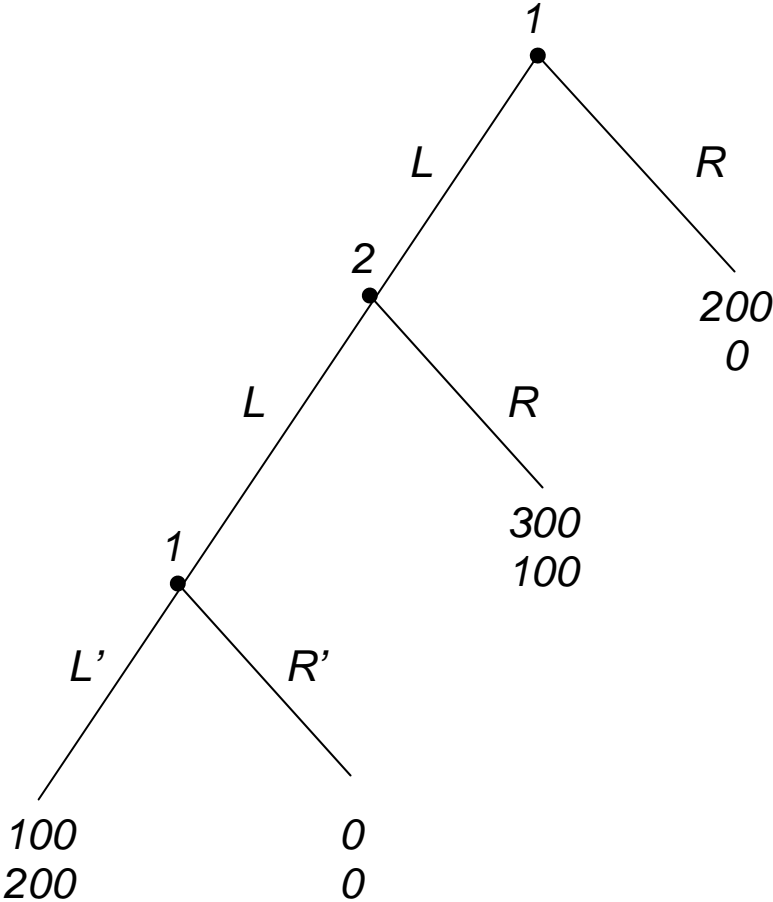
## Subgame perfect equilibrium

- The notion of Nash equilibrium ignores the sequential structure of the game.
- Consequently, the steady state to which a Nash Equilibrium corresponds may not be robust.
- A *subgame perfect equilibrium* is an action profile that induces a Nash equilibrium in every *subgame* (so every subgame perfect equilibrium is also a Nash equilibrium).

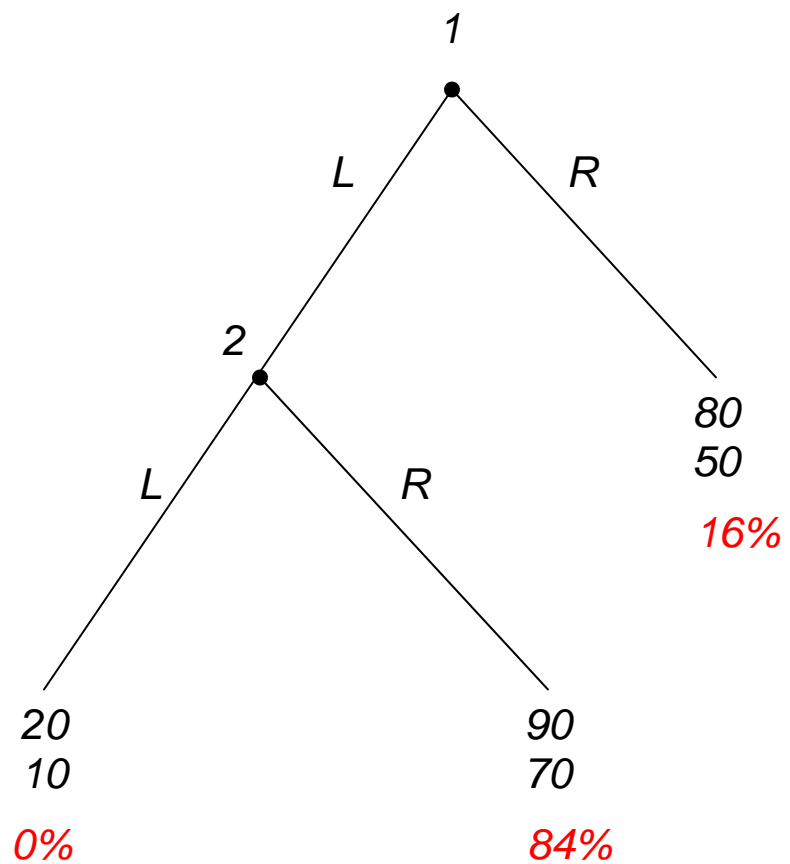
## An example: entry game



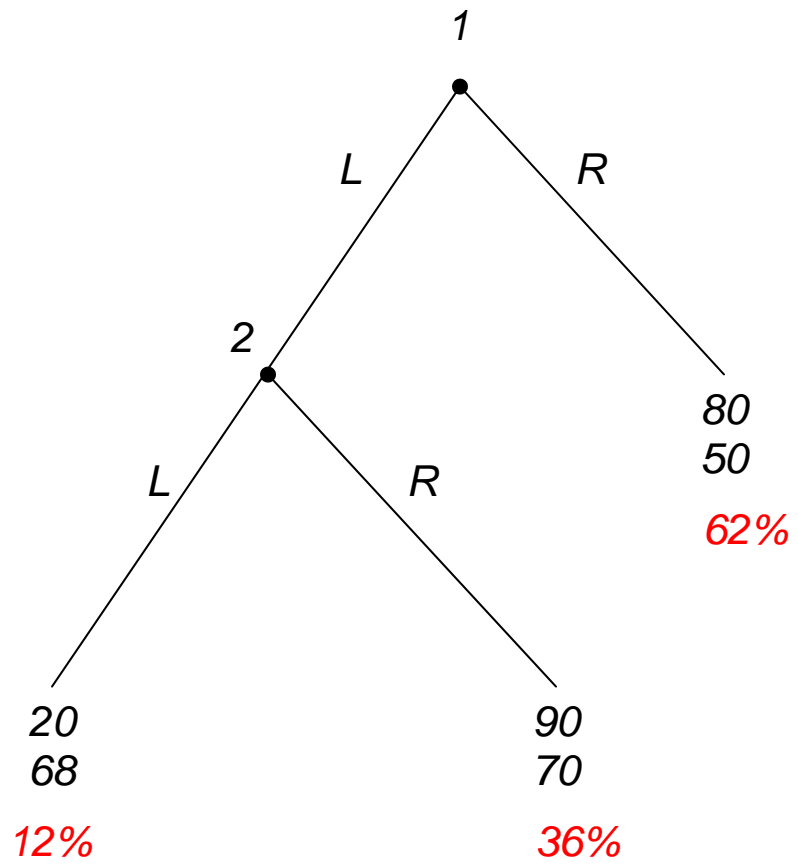
# Subgame perfect and backward induction



## Two entry games in the laboratory







## A review of the main ideas

We study two (out of four) groups of game theoretic models:

- [1] Strategic games – all players simultaneously choose their plan of action once and for all.
- [2] Extensive games (with perfect information) – players choose sequentially (and fully informed about all previous actions).

A solution (equilibrium) is a systematic description of the outcomes that may emerge in a family of games. We study two solution concepts:

- [1] Nash equilibrium – a steady state of the play of a strategic game (no player has a profitable deviation given the actions of the other players).
- [1] Subgame equilibrium – a steady state of the play of an extensive game (a Nash equilibrium in every subgame of the extensive game).

⇒ Every subgame perfect equilibrium is also a Nash equilibrium.

**Oligopolistic competition  
(in strategic and extensive forms)**

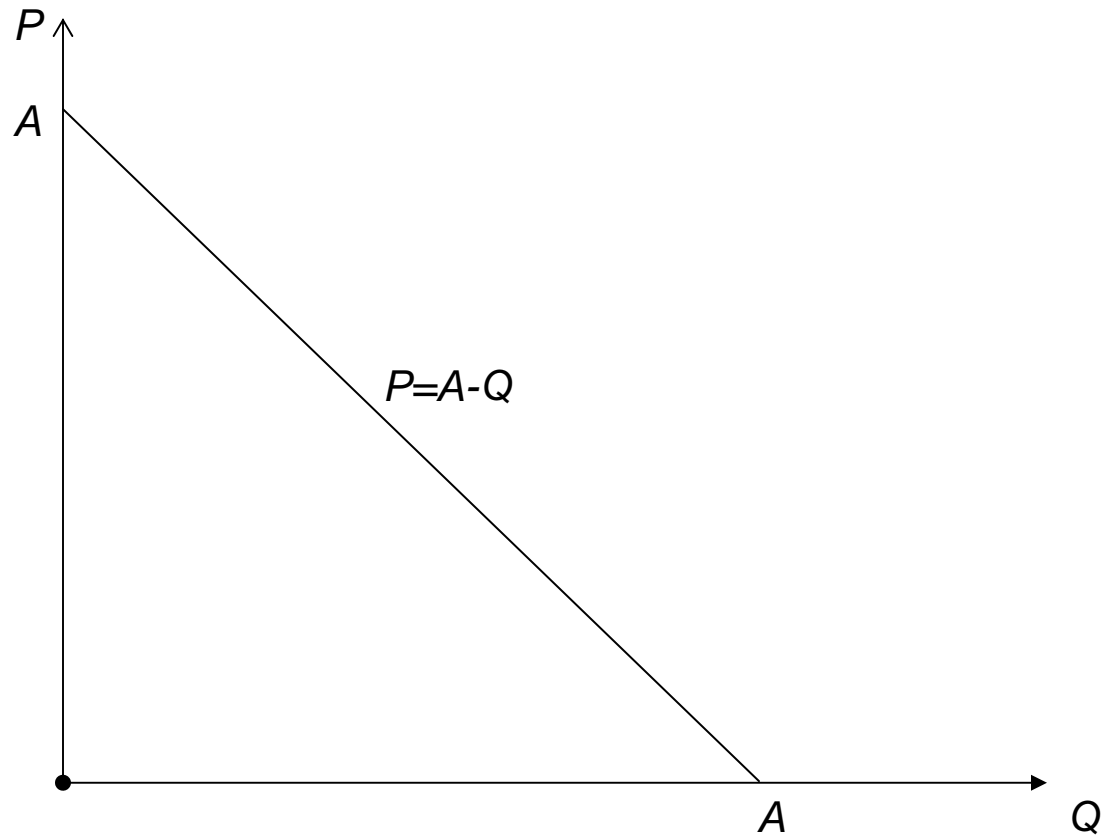
## Cournot's oligopoly model (1838)

- A single good is produced by two firms (the industry is a “duopoly”).
- The cost for firm  $i = 1, 2$  for producing  $q_i$  units of the good is given by  $c_i q_i$  (“unit cost” is constant equal to  $c_i > 0$ ).
- If the firms' total output is  $Q = q_1 + q_2$  then the market price is

$$P = A - Q$$

if  $A \geq Q$  and zero otherwise (linear inverse demand function). We also assume that  $A > c$ .

## The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

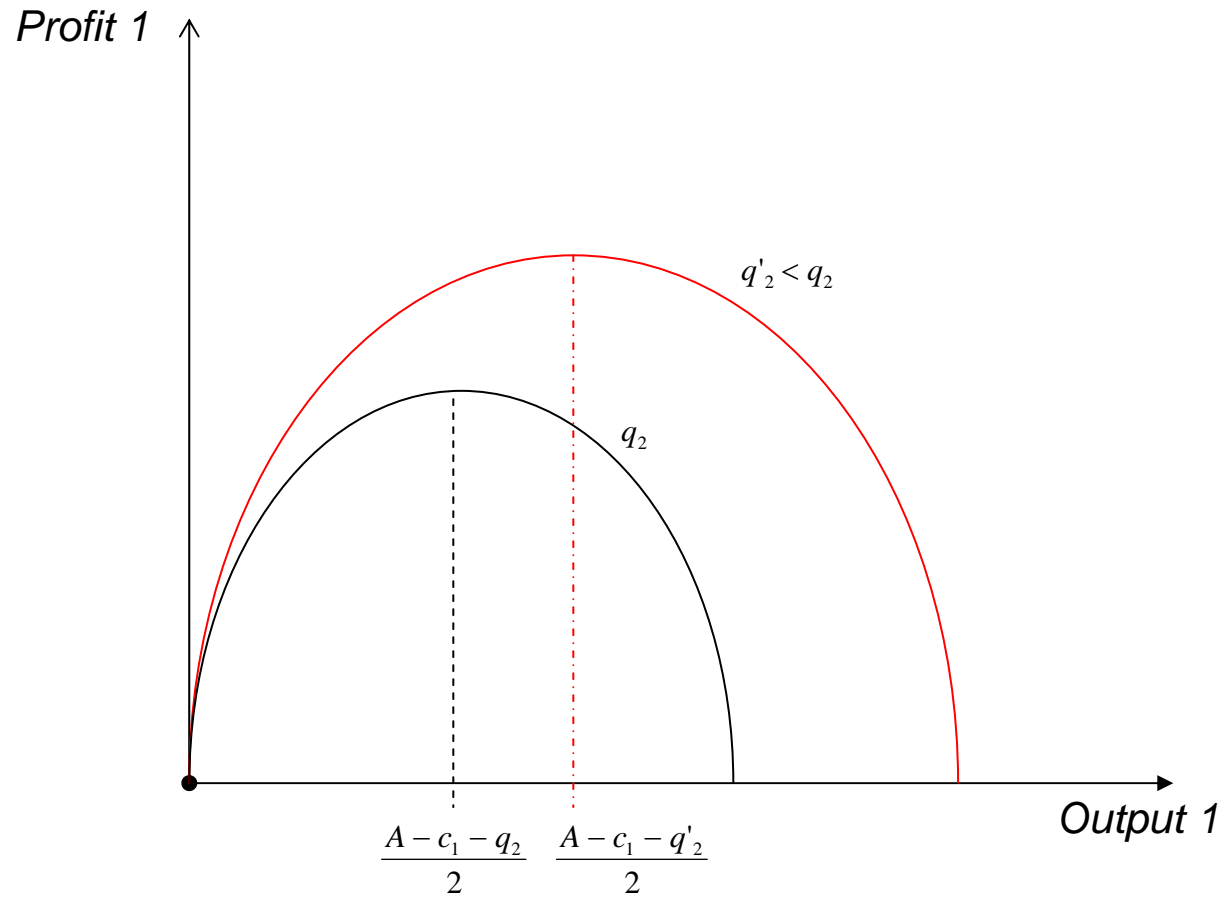
But first we need the firms payoffs (profits):

$$\begin{aligned}\pi_1 &= Pq_1 - c_1q_1 \\ &= (A - Q)q_1 - c_1q_1 \\ &= (A - q_1 - q_2)q_1 - c_1q_1 \\ &= (A - q_1 - q_2 - c_1)q_1\end{aligned}$$

and similarly,

$$\pi_2 = (A - q_1 - q_2 - c_2)q_2$$

**Firm 1's profit as a function of its output  
(given firm 2's output)**





To find firm 1's best response to any given output  $q_2$  of firm 2, we need to study firm 1's profit as a function of its output  $q_1$  for given values of  $q_2$ .

Using calculus, we set the derivative of firm 1's profit with respect to  $q_1$  equal to zero and solve for  $q_1$ :

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output  $q_2$  of firm 2 depends on the values of  $q_2$  and  $c_1$ .

Because firm 2's cost function is  $c_2 \neq c_1$ , its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

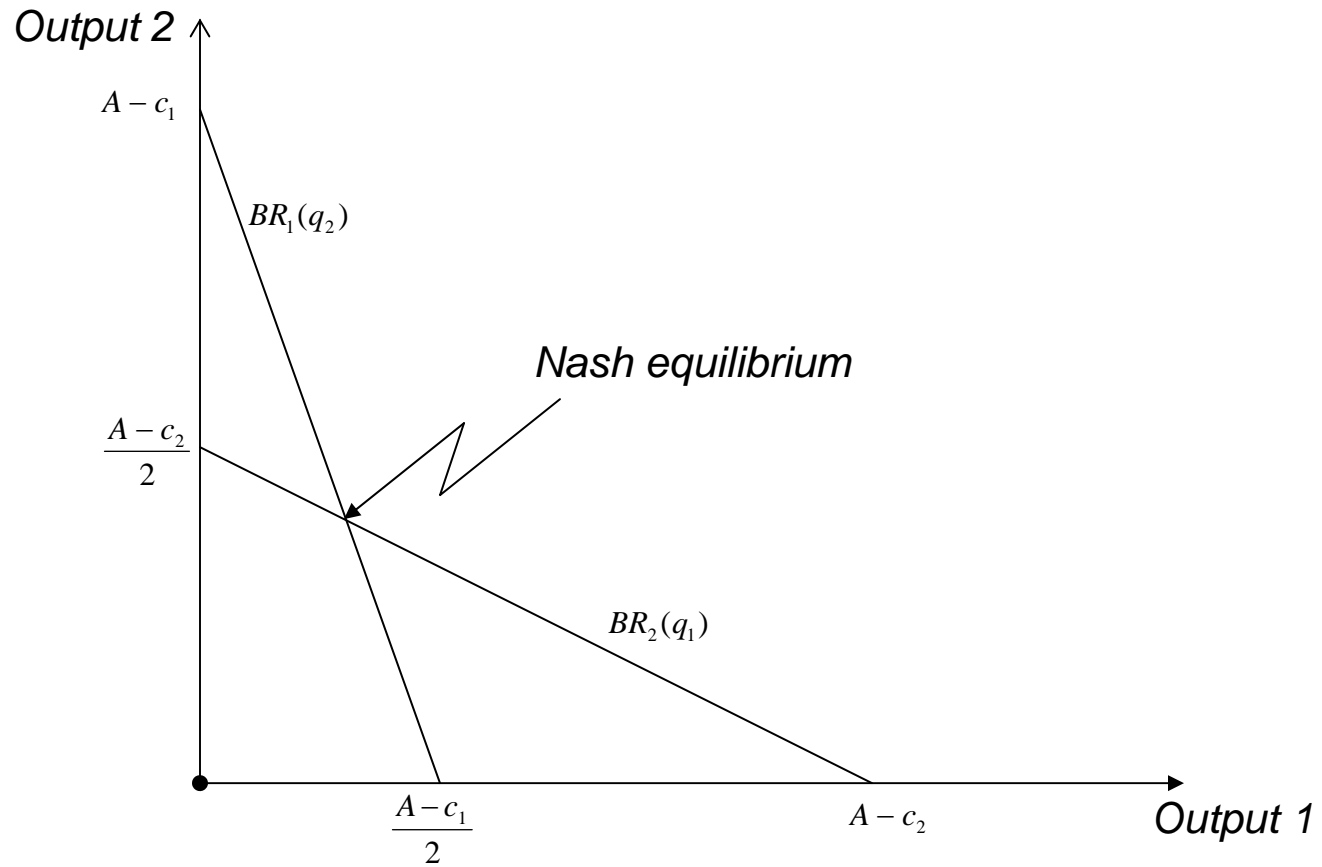
A Nash equilibrium of the Cournot's game is a pair  $(q_1^*, q_2^*)$  of outputs such that  $q_1^*$  is a best response to  $q_2^*$  and  $q_2^*$  is a best response to  $q_1^*$ .

From the figure below, we see that there is exactly one such pair of outputs

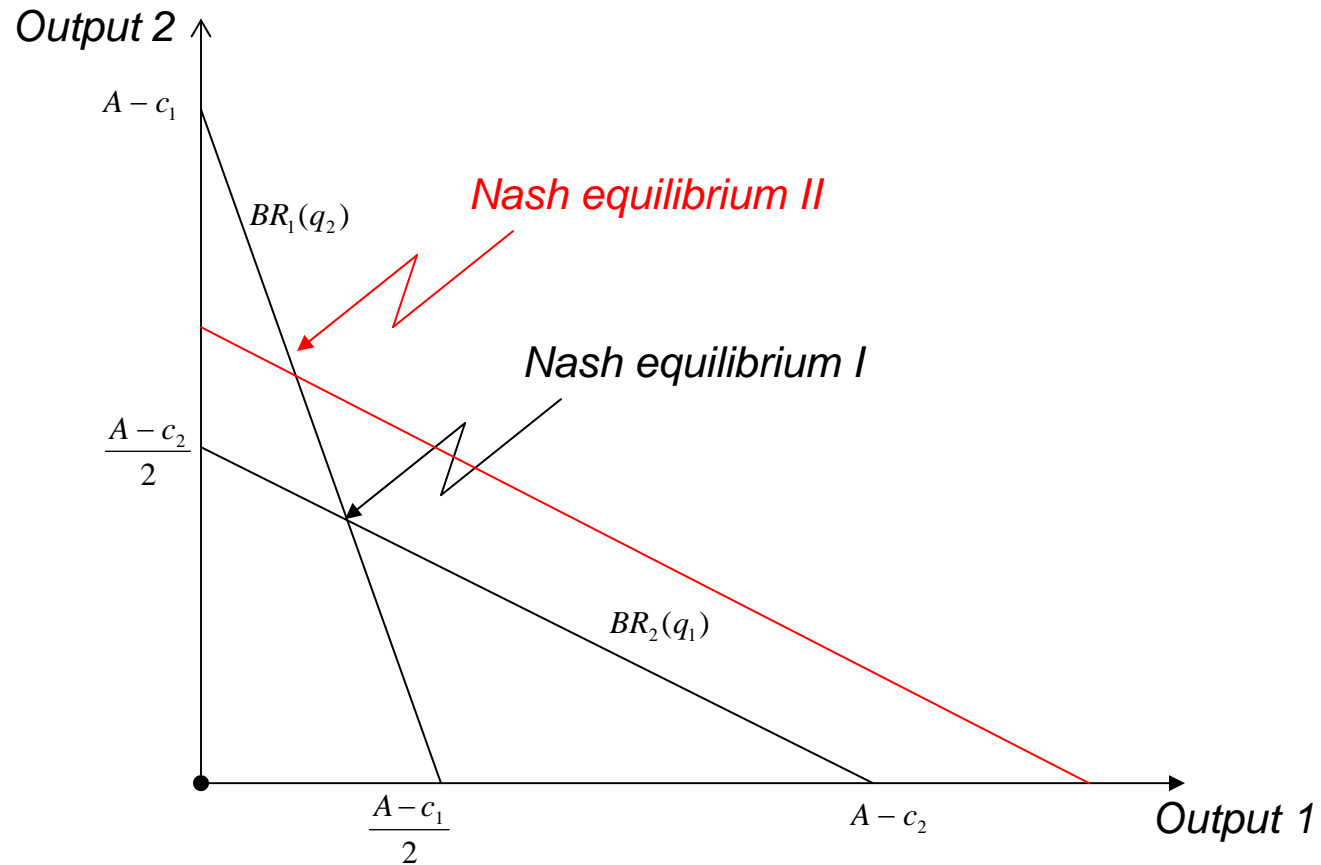
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

### The best response functions in the Cournot's duopoly game



**Nash equilibrium comparative statics  
(a decrease in the cost of firm 2)**



A question: what happens when consumers are willing to pay more ( $A$  increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

- [1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.
- [2] The relation between the equilibrium profits and the number of firms.

- [1] Collusive outcomes: in the Cournot's duopoly game, there is a pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium.
- [2] Competition: The price at the Nash equilibrium if the two firms have the *same* unit cost  $c_1 = c_2 = c$  is given by

$$\begin{aligned} P^* &= A - q_1^* - q_2^* \\ &= \frac{1}{3}(A + 2c) \end{aligned}$$

which is above the unit cost  $c$ . But as the number of firm increases, the equilibrium price decreases, approaching  $c$  (zero profits!).

## Stackelberg's duopoly model (1934)

How do the conclusions of the Cournot's duopoly game change when the firms move sequentially? Is a firm better off moving before or after the other firm?

Suppose that  $c_1 = c_2 = c$  and that firm 1 moves at the start of the game. We may use backward induction to find the subgame perfect equilibrium.

- First, for *any* output  $q_1$  of firm 1, we find the output  $q_2$  of firm 2 that maximizes its profit. Next, we find the output  $q_1$  of firm 1 that maximizes its profit, *given the strategy* of firm 2.

## Firm 2

Since firm 2 moves after firm 1, a strategy of firm 2 is a *function* that associate an output  $q_2$  for firm 2 for each possible output  $q_1$  of firm 1.

We found that under the assumptions of the Cournot's duopoly game Firm 2 has a unique best response to each output  $q_1$  of firm 1, given by

$$q_2 = \frac{1}{2}(A - q_1 - c)$$

(Recall that  $c_1 = c_2 = c$ ).



## Firm 1

Firm 1's strategy is the output  $q_1$  the maximizes

$$\pi_1 = (A - q_1 - q_2 - c)q_1 \quad \text{subject to} \quad q_2 = \frac{1}{2}(A - q_1 - c)$$

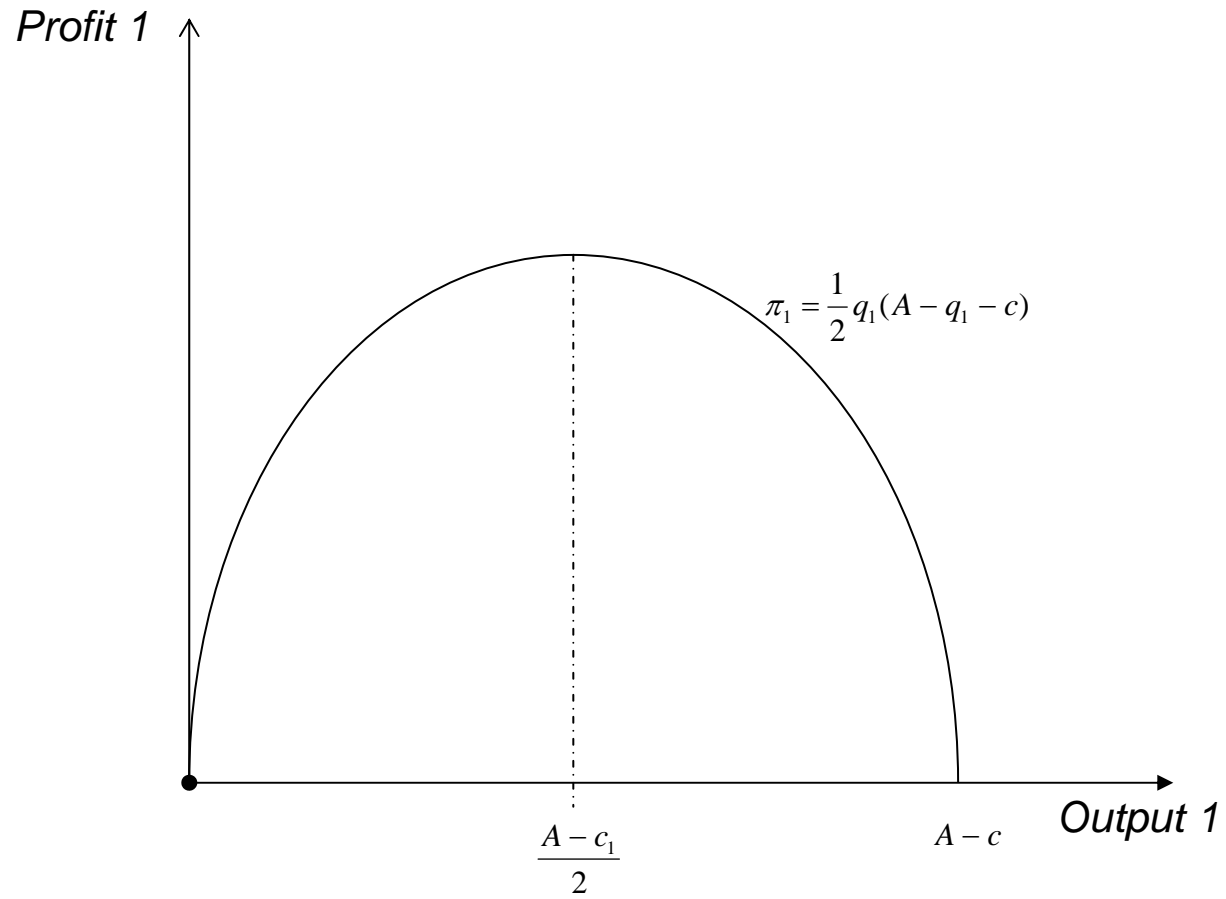
Thus, firm 1 maximizes

$$\pi_1 = (A - q_1 - (\frac{1}{2}(A - q_1 - c)) - c)q_1 = \frac{1}{2}q_1(A - q_1 - c).$$

This function is quadratic in  $q_1$  that is zero when  $q_1 = 0$  and when  $q_1 = A - c$ . Thus its maximizer is

$$q_1^* = \frac{1}{2}(A - c).$$

### Firm 1's (first-mover) profit in Stackelberg's duopoly game



We conclude that Stackelberg's duopoly game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output

$$q_1^* = \frac{1}{2}(A - c)$$

and firm 2's output is

$$\begin{aligned} q_2^* &= \frac{1}{2}(A - q_1^* - c) \\ &= \frac{1}{2}\left(A - \frac{1}{2}(A - c) - c\right) \\ &= \frac{1}{4}(A - c). \end{aligned}$$

By contrast, in the unique Nash equilibrium of the Cournot's duopoly game under the same assumptions ( $c_1 = c_2 = c$ ), each firm produces  $\frac{1}{3}(A - c)$ .

## The subgame perfect equilibrium of Stackelberg's duopoly game

