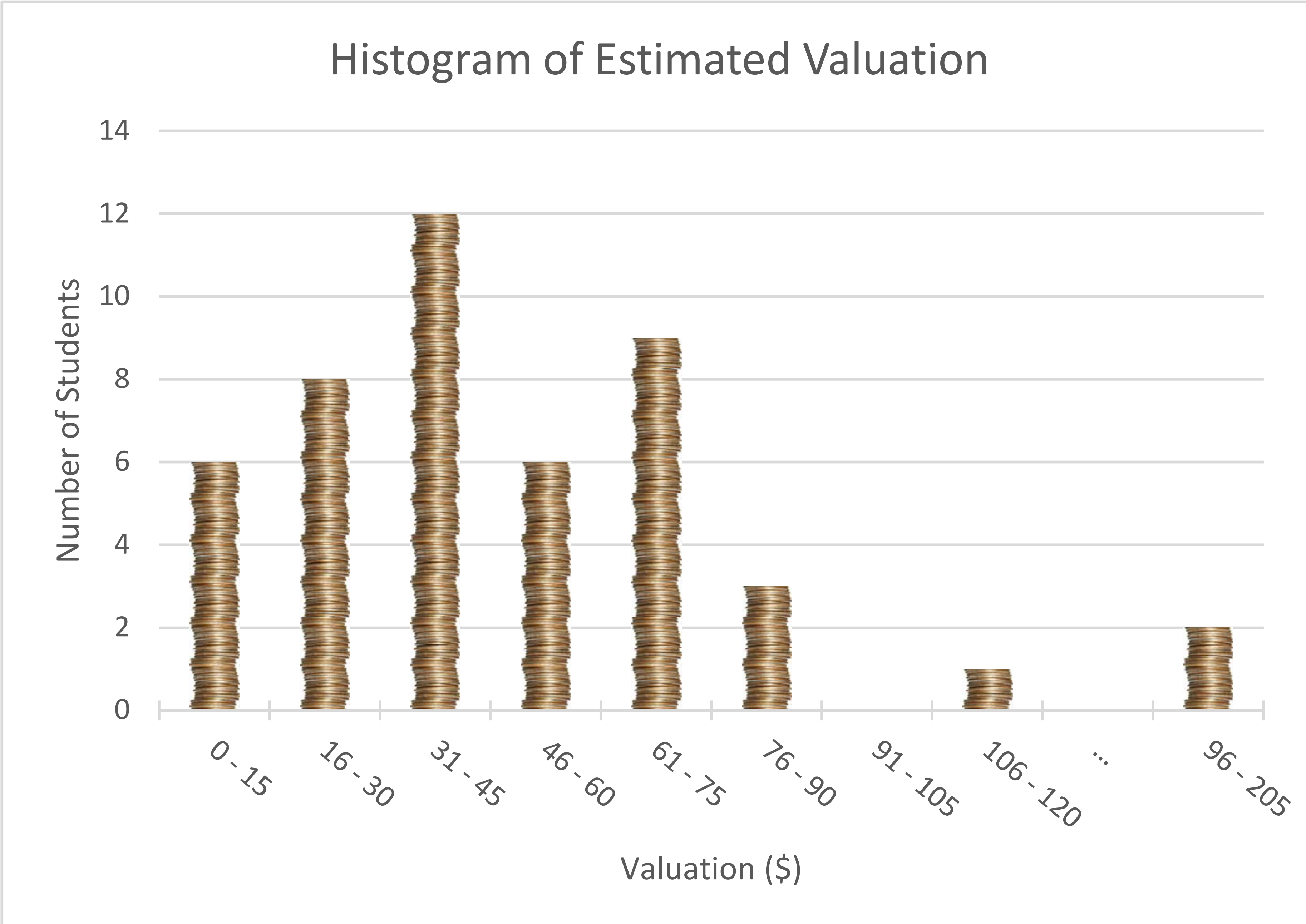
UC Berkeley
Haas School of Business
Game Theory
(EMBA 296 & EWMBA 211)
Summer 2016

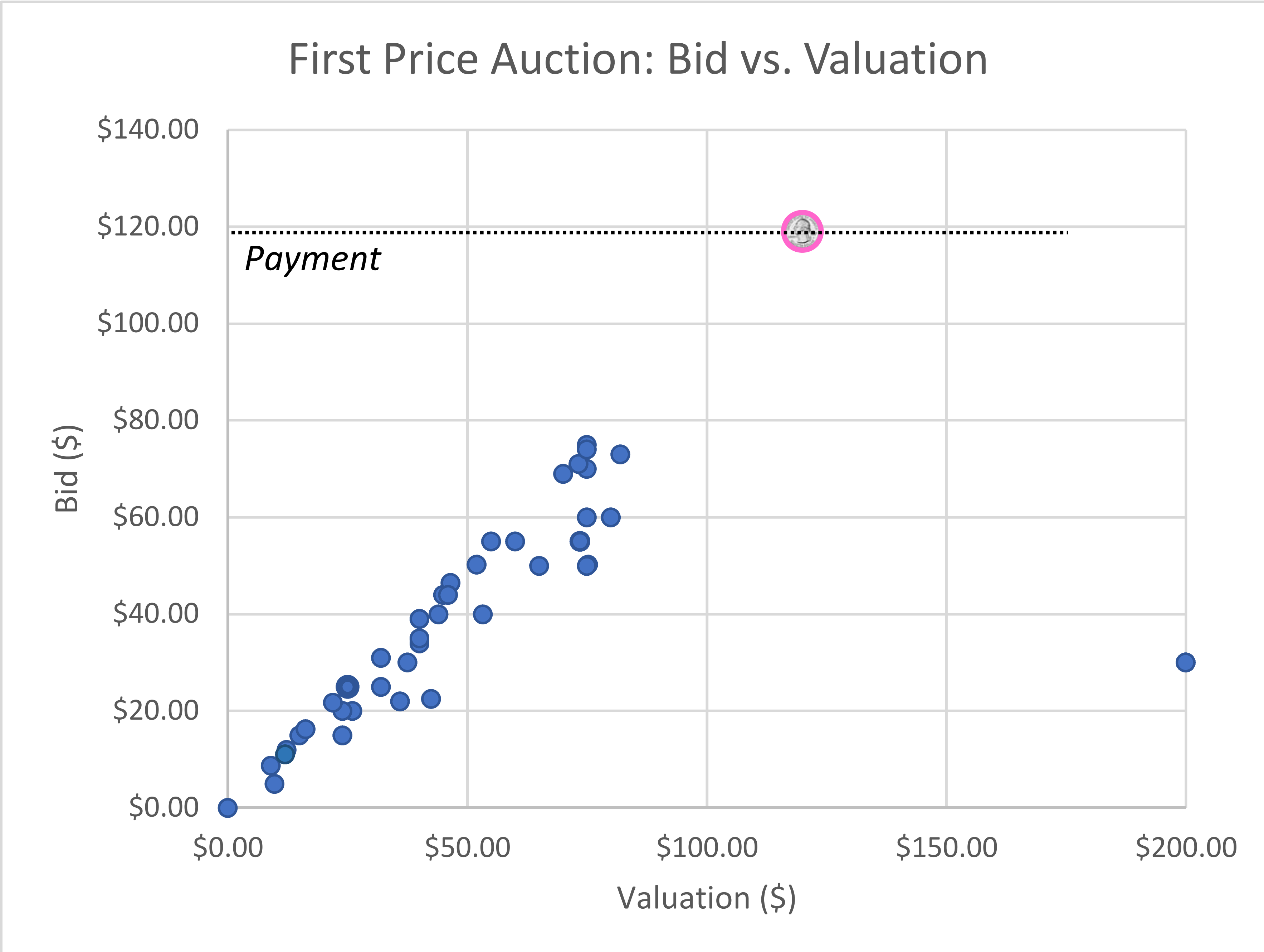
More on strategic games and extensive games with perfect information

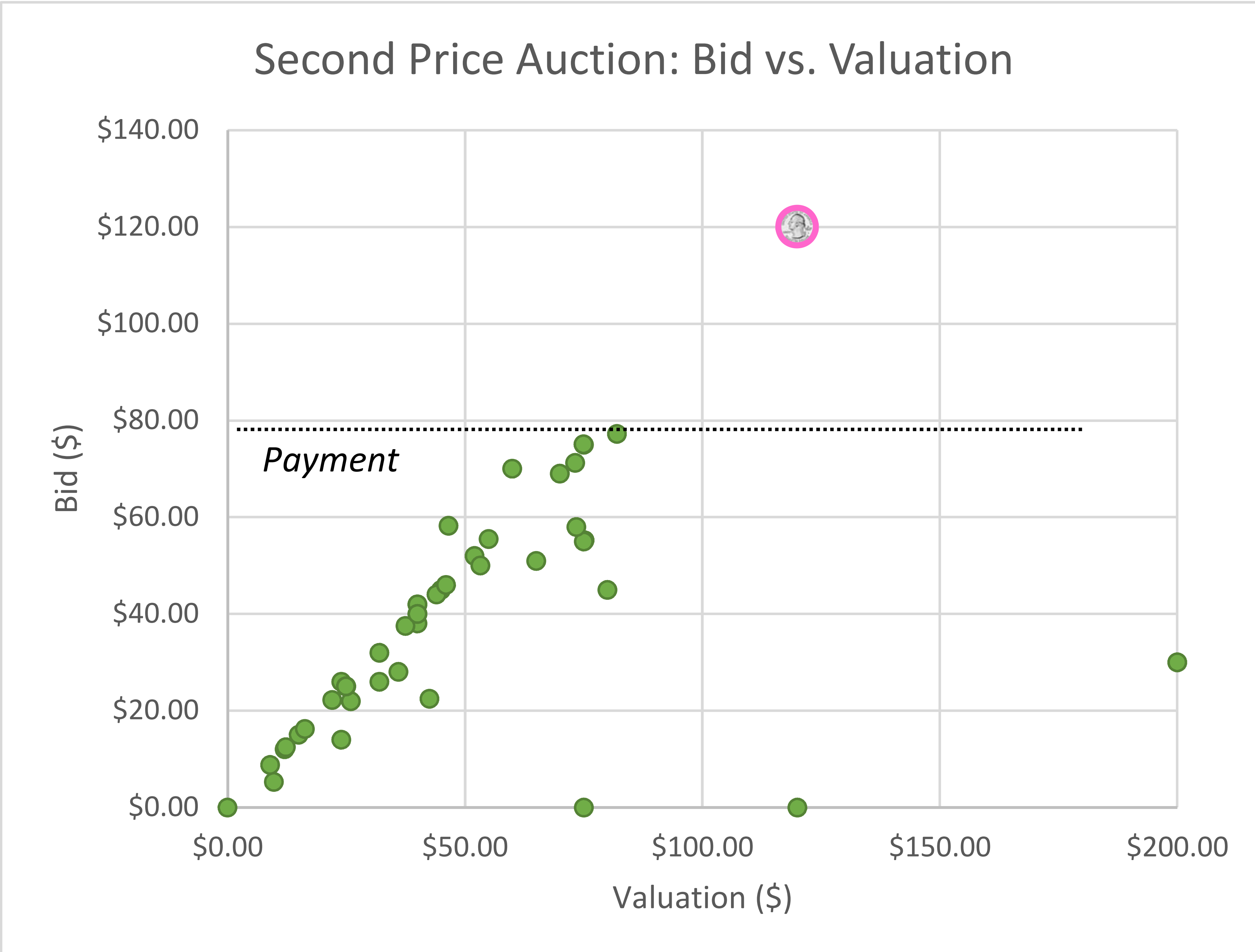
Block 2
Jun 11, 2017

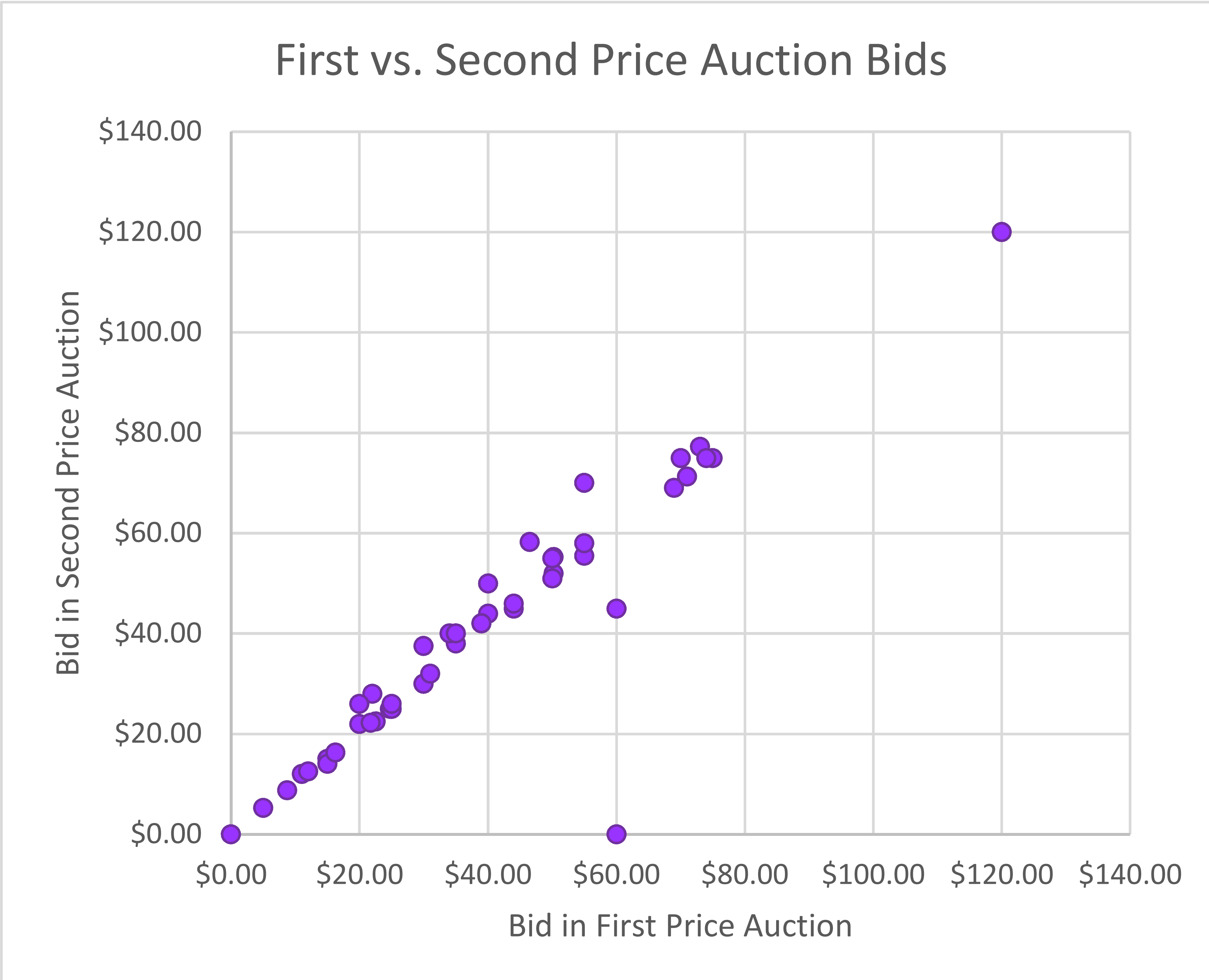
Auctions’ results

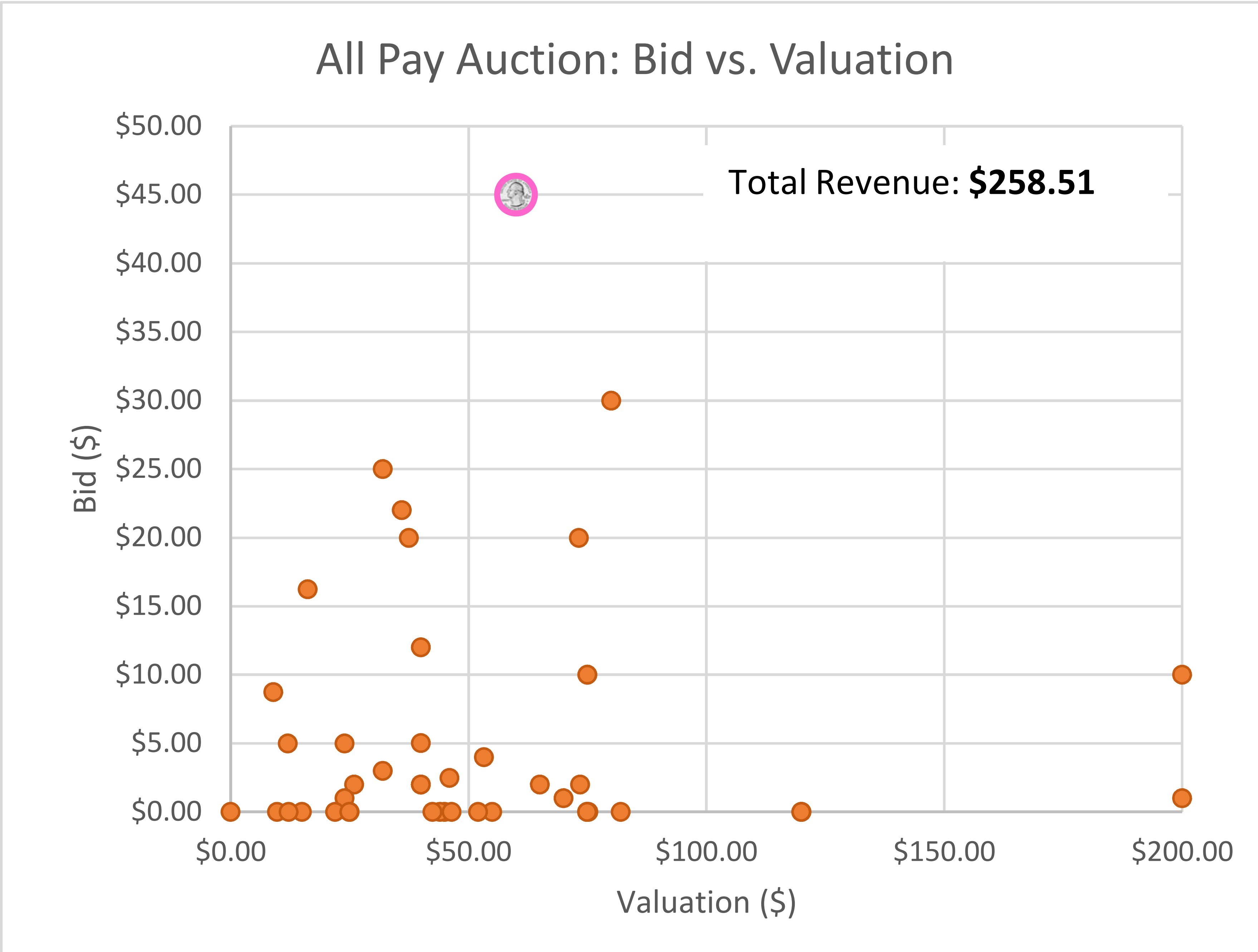


Histogram of Estimated Valuation
Number of Students
0
2
4
6
8
10
12
14
0 - 15
16 - 30
31 - 45
46 - 60
61 - 75
76 - 90
91 - 105
106 - 120
...
96 - 205
Valuation ($)


First Price Auction: Bid vs. Valuation
$140.00
$120.00
$100.00
$80.00
$60.00
$40.00
$20.00
$0.00
Payment
Bid ($)
$0.00
$50.00
$100.00
$150.00
$200.00
Valuation ($)


Second Price Auction: Bid vs. Valuation
$140.00
$120.00
$100.00
$80.00
$60.00
$40.00
$20.00
$0.00
Bid ($)
Payment
$0.00
$50.00
$100.00
$150.00
$200.00
Valuation ($)


First vs. Second Price Auction Bids
Bid in Second Price Auction
$140.00
$120.00
$100.00
$80.00
$60.00
$40.00
$20.00
$0.00
$0.00
$20.00
$40.00
$60.00
$80.00
$100.00
$120.00
$140.00
Bid in First Price Auction


All Pay Auction: Bid vs. Valuation
Total Revenue: $258.51
$50.00
$45.00
$40.00
$35.00
$30.00
$25.00
$20.00
$15.00
$10.00
$5.00
$0.00
Bid ($)
$0.00
$50.00
$100.00
$150.00
$200.00
Valuation ($)

Food for thought

LUPI

Many players simultaneously chose an integer between 1 and 99,999. Whoever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.

Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

In Morra there are two players, each of whom has four (relevant) actions, S_1G_2, S_1G_3, S_2G_3, and S_2G_4, where S_iG_j denotes the strategy (Show i, Guess j).

The payoffs in the game are as follows

	S_1G_2	S_1G_3	S_2G_3	S_2G_4
S_1G_2	$0, 0$	$2, -2$	$-3, 3$	$0, 0$
S_1G_3	$-2, 2$	$0, 0$	$0, 0$	$3, -3$
S_2G_3	$3, -3$	$0, 0$	$0, 0$	$-4, 4$
S_2G_4	$0, 0$	$-3, 3$	$4, -4$	$0, 0$

Maximal game
(sealed-bid second-price auction)

Two bidders, each of whom privately observes a signal X_i that is independent and identically distributed (i.i.d.) from a uniform distribution on $[0, 10]$.

Let $X^{\mathsf{max}} = \mathsf{max}\{X_1, X_2\}$ and assume the ex-post common value to the bidders is X^{max}.

Bidders bid in a sealed-bid second-price auction where the highest bidder wins, earns the common value X^{max} and pays the second highest bid.

Homework review

1/1 **Penalty Kick**

There are two players, 1 (kicker) and 2 (goalie). Each has two actions, $a_i \in \{L, R\}$ to denote left or right.

The kicker scores when they choose opposite directions while the goalie saves if they choose the same direction so preferences ordering over outcomes is given by

$$\begin{aligned}
(L,R) &\sim\ _1(R,L) \succ_1 (L,L) \sim_1 (R,R) \\
(L,R) &\sim\ _2(R,L) \prec_2 (L,L) \sim_2 (R,R)
\end{aligned}$$

The game can be described as follows:

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

or equivalently

	L	R
L	$0, 0$	$1, -1$
R	$1, -1$	$0, 0$

The game has a unique mixed strategy Nash equilibrium $p = q = 1/2$.

1/2 **Meeting Up**

There are two players. Each has two actions, $a_i \in \{C, S\}$ to denote Sutro or Coit. preferences ordering over outcomes is given by

$$(C,C) \sim_1 (S,S) \succ_1 (C,S) \sim_1 (S,C)$$
$$(C,C) \sim_2 (S,S) \succ_2 (C,S) \sim_2 (S,C)$$

so the game can be described as follows:

	S	C
S	1, 1	0, 0
C	0, 0	1, 1

1/5 **Public Good Contribution**

- An indivisible public project with cost 2 and 3 players, each of whom has an endowment of 1 tokens.
- The players simultaneously make a contribution to the project, which is carried out if and only if the sum of the contributions is large enough to meet its cost.
- If the project is completed, each player receives 3 tokens *plus* to the number of tokens retained from his endowment.

The set of players is $N = \{1, 2, 3\}$ and each has a strategy set $S_i = \{0, 1\}$ where 0 denotes not contributing and 1 is contributing.

The payoffs of player i denoted by v_i from a profile of strategies (s_1, s_2, s_3) is given by

$$v_i(s_1, s_2, s_3) = \begin{cases} 4 & \text{if } s_i = 0 \text{ and } s_j = 1 \text{ for both } j \neq i \\ 3 & \text{if } s_i = 1 \text{ and } s_j = 1 \text{ for some } j \neq i \\ 1 & \text{if } s_i = 0 \text{ and } s_j = 0 \text{ for both } j \neq i \\ 0 & \text{if } s_i = 1 \text{ and } s_j = 0 \text{ for both } j \neq i \end{cases}$$

- The game has the following pure-strategy equilibria:

  – There exists a pure-strategy Nash equilibrium with no player contributes.

  – Conversely, there exist multiple pure-strategy equilibria in which exactly two players contribute.

- The game also possesses mixed-strategy equilibria in which the project is completed with positive probability.

- What happens if players simultaneously make *irreversible* contributions to the project at two dates?

1/8 **Campaigning**

	P	B	N
P	0.5, 0.5	0, 1	0.3, 0.7
B	1, 0	0.5, 0.5	0.4, 0.6
N	0.7, 0.3	0.6, 0.4	0.5, 0.5

	B	N
B	0.5, 0.5	0.4, 0.6
N	0.6, 0.4	0.5, 0.5

	N
N	0.5, 0.5

1/10 **Synergies**

Two managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest.

In particular, the payoff function for player i from effort levels (e_i, e_j) is

$$v_i(e_i, e_j) = ae_i + e_ie_j - e_i^2.$$

The best response function of player i is given by

$$BR_i(e_j) = \frac{a + e_j}{2}$$

because it is the solution of the first-order condition for maximizing her payoff.

The Nash equilibrium of this game, is the solution, denoted by e_1^* and e_2^*, of

$$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2}$$

which yield $e_1^* = e_2^* = a$. Is the Nash equilibrium socially optimal?

Strategic games
(review)

A two-player (finite) strategic game

The game can be described conveniently in a so-called bi-matrix. For example, a generic 2×2 (two players and two possible actions for each player) game

	L	R
T	a_1, a_2	b_1, b_2
B	c_1, c_2	d_1, d_2

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2). The two numbers in a box formed by a specific row and column are the players' payoffs given that these actions were chosen.

In this game above a_1 and a_2 are the payoffs of player 1 and player 2 respectively when player 1 is choosing strategy T and player 2 strategy L.

Classical 2×2 games

- The following simple 2×2 games represent a variety of strategic situations.
- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.
- These classical games "span" the set of almost *all* games (strategic equivalence).

Game I: Prisoner's Dilemma

	$Work$	$Goof$
$Work$	$3, 3$	$0, 4$
$Goof$	$4, 0$	$1, 1$

A situation where there are gains from cooperation but each player has an incentive to "free ride."

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.

Game II: Battle of the Sexes (BoS)

	$Ball$	$Show$
$Ball$	$2, 1$	$0, 0$
$Show$	$0, 0$	$1, 2$

Like the Prisoner's Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.

Game III-V: Coordination, Hawk-Dove, and Matching Pennies

	Ball	*Show*
Ball	2, 2	0, 0
Show	0, 0	1, 1

	Dove	*Hawk*
Dove	3, 3	1, 4
Hawk	4, 1	0, 0

	Head	*Tail*
Head	1, −1	−1, 1
Tail	−1, 1	1, −1

Best response and dominated actions

Action T is player 1's *best response* to action L player 2 if T is the optimal choice when 1 *conjectures* that 2 will play L.

Player 1's action T is *strictly* dominated if it is never a best response (inferior to B no matter what the other players do).

In the Prisoner's Dilemma, for example, action $Work$ is strictly dominated by action $Goof$. As we will see, a strictly dominated action is not used in any Nash equilibrium.

Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a NE is a set of actions such that all players are doing their best given the actions of the other players.

Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

		q	$1-q$
		L	R
p	T	pq	$p(1-q)$
$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

Let p and q be the probabilities that player 1 and 2 respectively assign to the strategy *Ball*.

Player 2 will be indifferent between using her strategy B and S when player 1 assigns a probability p such that her expected payoffs from playing B and S are the same. That is,

$$1p + 0(1 - p) = 0p + 2(1 - p)$$
$$p = 2 - 2p$$
$$p^* = 2/3$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy B and probability $1 - p^* = 1/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

Similarly, player 1 will be indifferent between using her strategy B and S when player 2 assigns a probability q such that her expected payoffs from playing B and S are the same. That is,

$$2q + 0(1 - q) = 0q + 1(1 - q)$$
$$2q = 1 - q$$
$$q^* = 1/3$$

Hence, when player 2 assigns probability $q^* = 1/3$ to her strategy B and probability $1 - q^* = 2/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

In terms of best responses:

$$B_1(q) = \begin{cases} p = 1 & if \quad p > 1/3 \\ p \in [0,1] & if \quad p = 1/3 \\ p = 0 & if \quad p < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} q = 1 & if \quad p > 2/3 \\ q \in [0,1] & if \quad p = 2/3 \\ q = 0 & if \quad p < 2/3 \end{cases}$$

The BoS has two Nash equilibria in pure strategies $\{(B, B), (S, S)\}$ and one in mixed strategies $\{(2/3, 1/3)\}$. In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).

Three Matching Pennies games in the laboratory

		.48	.52
		a_2	b_2
.48	a_1	80, 40	40, 80
.52	b_1	40, 80	80, 40

		.16	.84
		a_2	b_2
.96	a_1	320, 40	40, 80
.04	b_1	40, 80	80, 40

		.80	.20
		a_2	b_2
.08	a_1	44, 40	40, 80
.92	b_1	40, 80	80, 40

Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^*.

Evolutionary stable strategy (ESS)

Consider a two-player payoff symmetric game

$$G = \langle \{1, 2\}, (A, A), (u_1, u_2) \rangle$$

where

$$u_1(a_1, a_2) = u_2(a_2, a_1)$$

(players exchanging a_1 and a_2).

$a^* \in A$ is ESS if and only if for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if and only if for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on ESS

[1] If a^* is an ESS then (a^*, a^*) is a NE.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an ESS.

[2] If (a^*, a^*) is a strict NE ($u(a^*, a^*) > u(a, a^*)$ for all $a \in A$) then a^* is an ESS.

Suppose a^* is not an ESS. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a NE but not a strict NE.

[3] The two-player two-action game

	a	a'
a	w, w	x, y
a'	y, x	z, z

has a strategy which is ESS.

If $w > y$ or $z > x$ then (a, a) or (a', a') are strict NE, and thus a or a' are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE (α^*, α^*) where

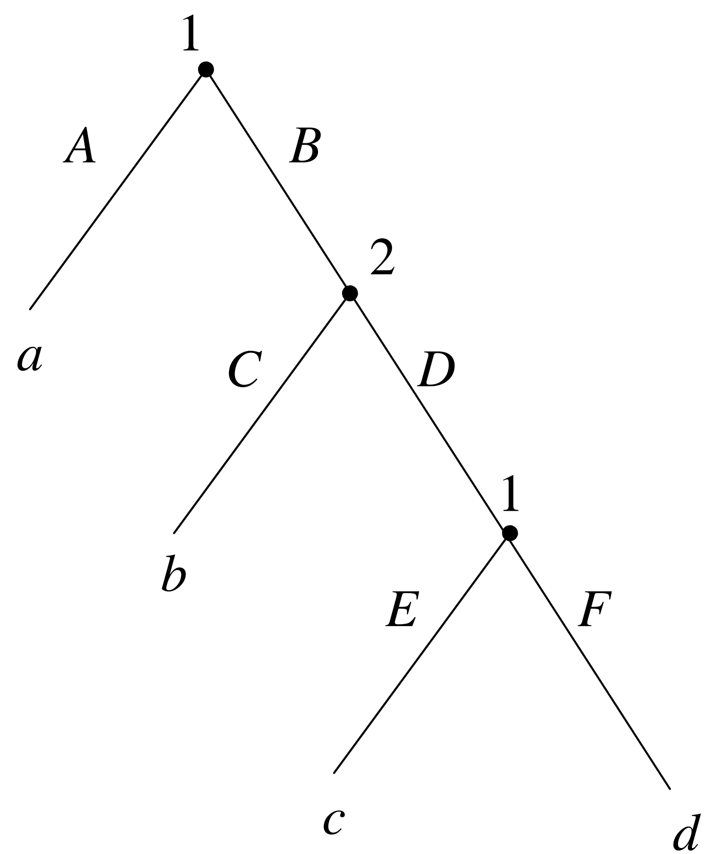
$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.

Extensive games with perfect information

Extensive games with perfect information

- The model of a strategic suppresses the sequential structure of decision making.
  - All players simultaneously choose their plan of action once and for all.
- The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly.
  - In an extensive game of perfect information all players are fully informed about all previous actions.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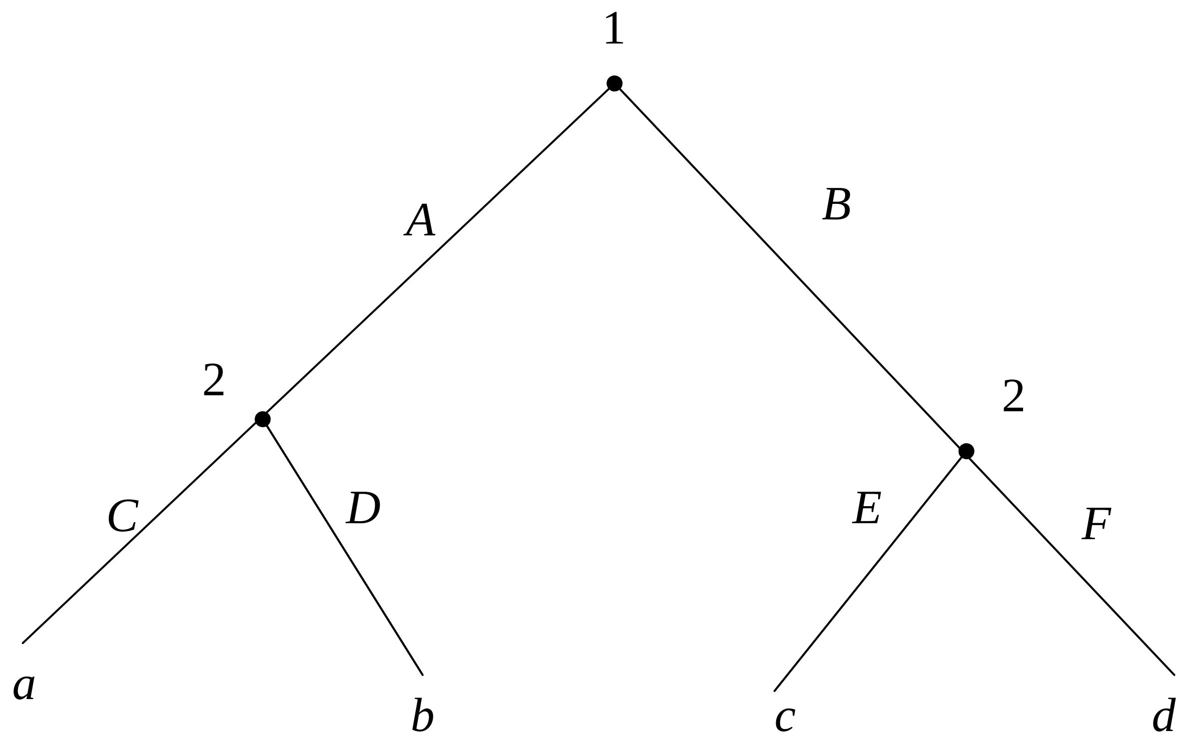
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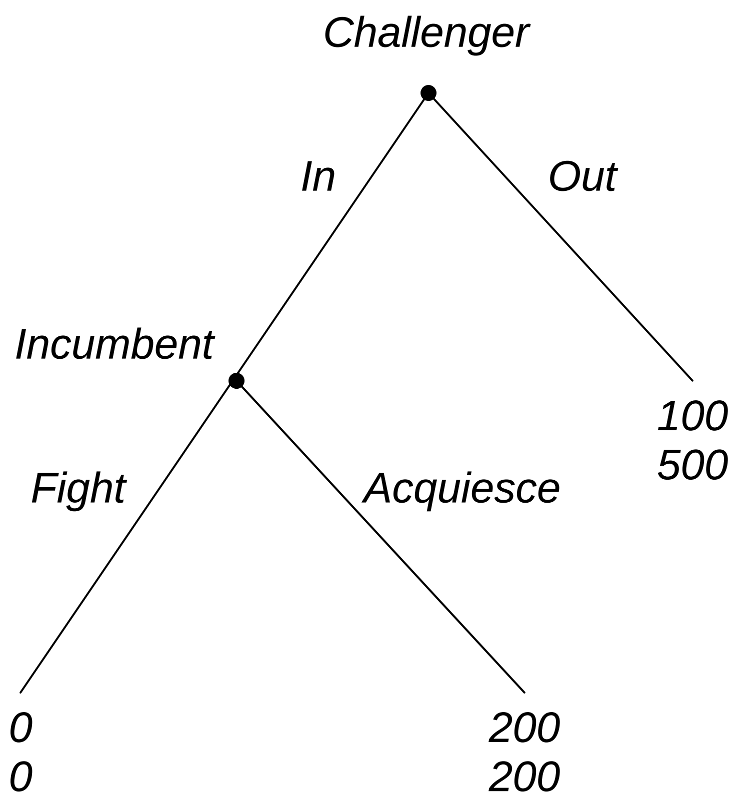
Subgame perfect equilibrium

- The notion of Nash equilibrium ignores the sequential structure of the game.

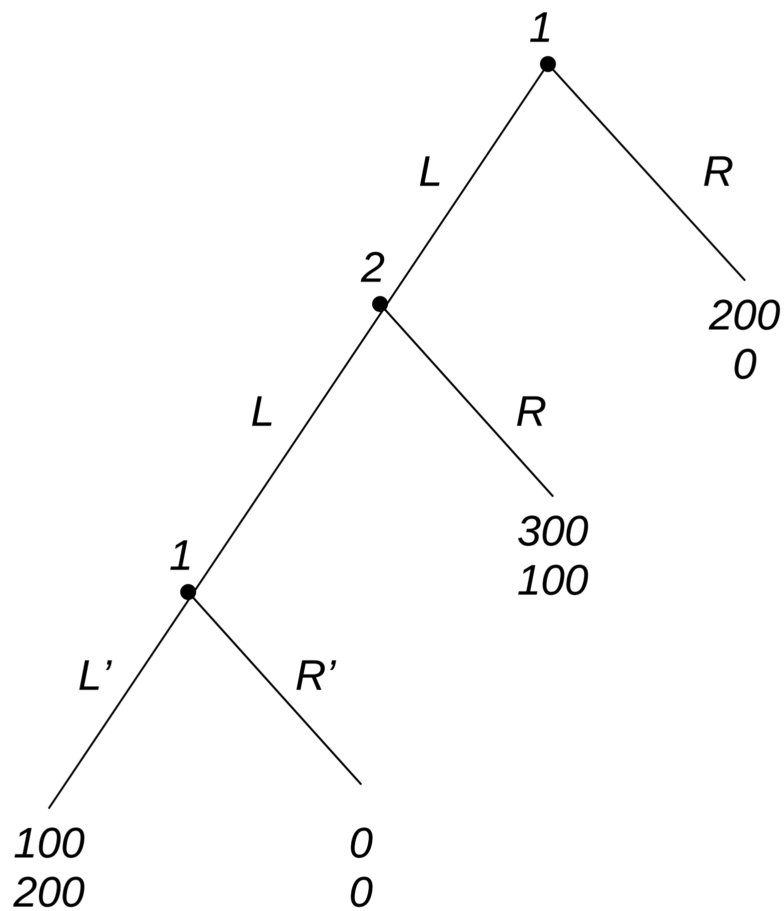
- Consequently, the steady state to which a Nash Equilibrium corresponds may not be robust.

- A *subgame perfect equilibrium* is an action profile that induces a Nash equilibrium in every *subgame* (so every subgame perfect equilibrium is also a Nash equilibrium).

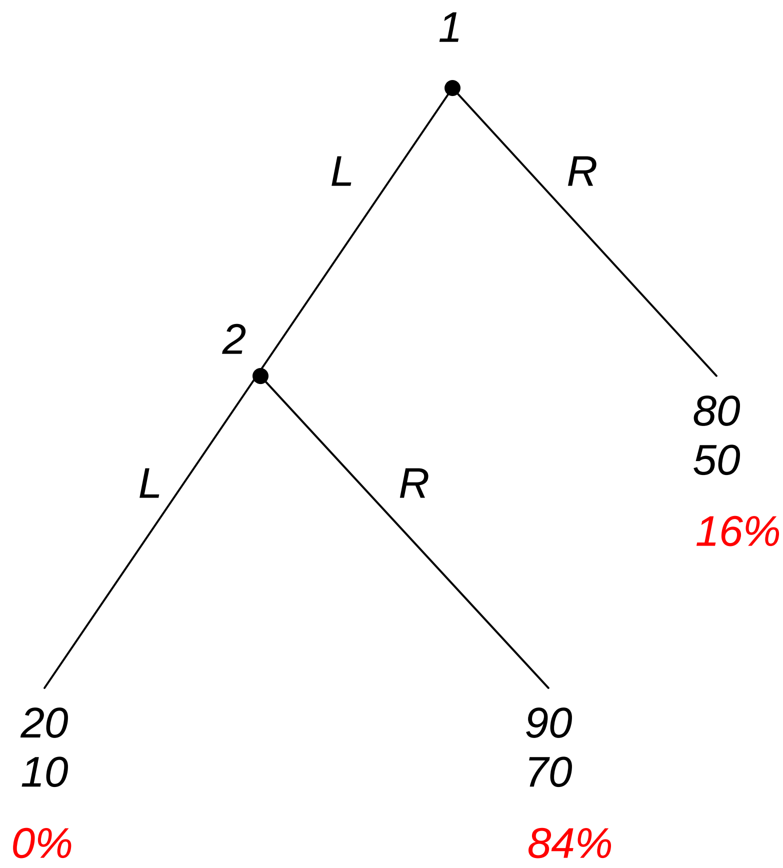
An example: entry game

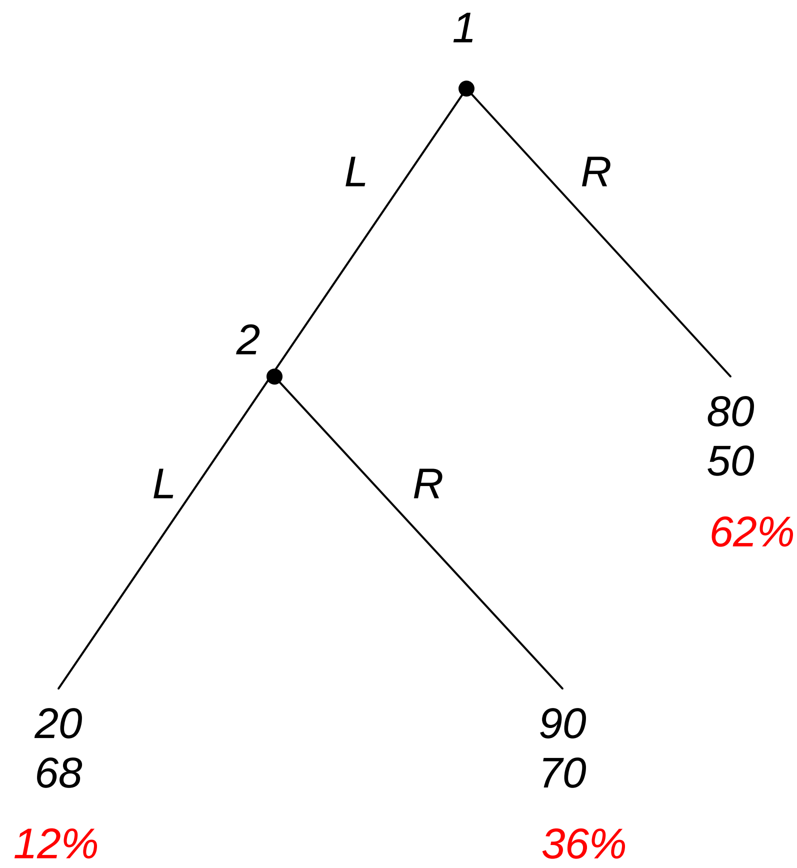


Subgame perfect and backward induction



Two entry games in the laboratory




1
L
R
2
L
R
80
50
62%
20
68
12%
90
70
36%

A review of the main ideas

We study two (out of four) groups of game theoretic models:

[1] Strategic games – all players simultaneously choose their plan of action once and for all.

[2] Extensive games (with perfect information) – players choose sequentially (and fully informed about all previous actions).

A solution (equilibrium) is a systematic description of the outcomes that may emerge in a family of games. We study two solution concepts:

[1] Nash equilibrium – a steady state of the play of a <u>strategic</u> game (no player has a profitable deviation given the actions of the other players).

[1] Subgame equilibrium – a steady state of the play of an <u>extensive</u> game (a Nash equilibrium in every <u>subgame</u> of the extensive game).

$\Longrightarrow$ Every subgame perfect equilibrium is also a Nash equilibrium.

Oligopolistic competition
(in strategic and extensive forms)

Cournot's oligopoly model (1838)

- A single good is produced by two firms (the industry is a "duopoly").

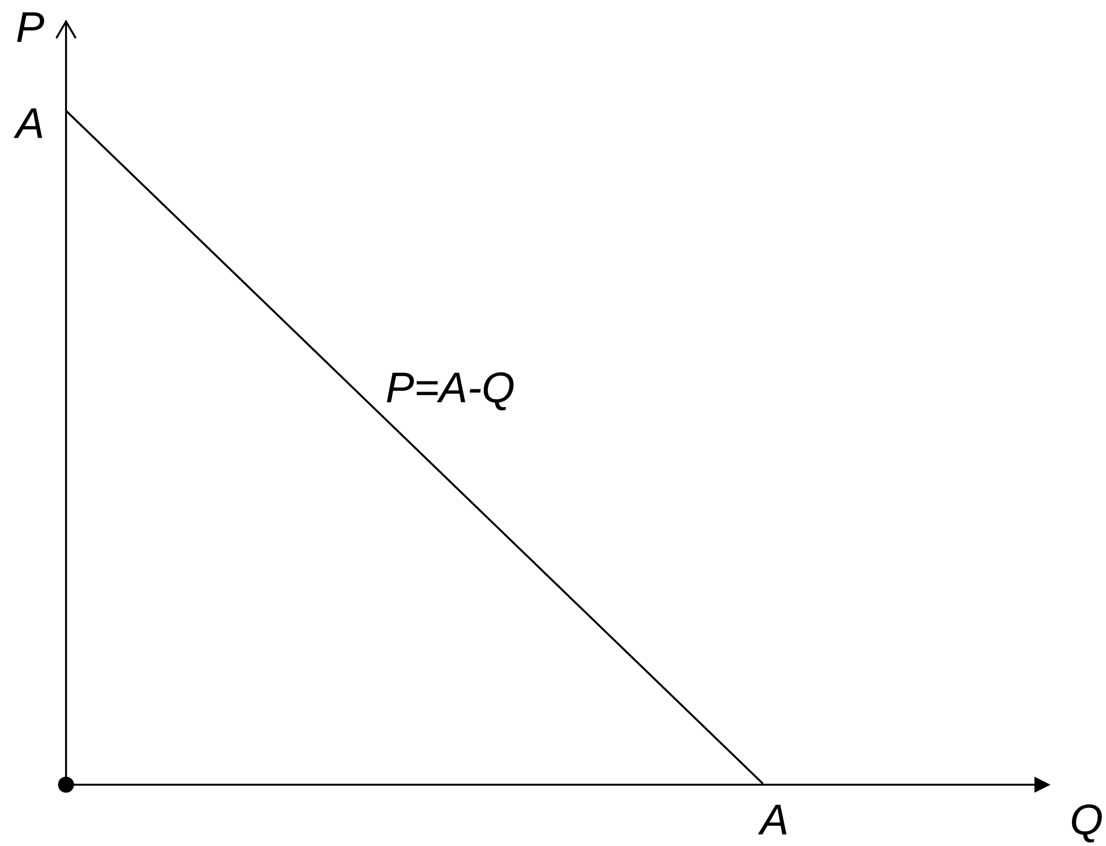
- The cost for firm $i = 1, 2$ for producing q_i units of the good is given by $c_i q_i$ ("unit cost" is constant equal to $c_i > 0$).

- If the firms' total output is $Q = q_1 + q_2$ then the market price is

  $$P = A - Q$$

  if $A \geq Q$ and zero otherwise (linear inverse demand function). We also assume that $A > c$.

The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

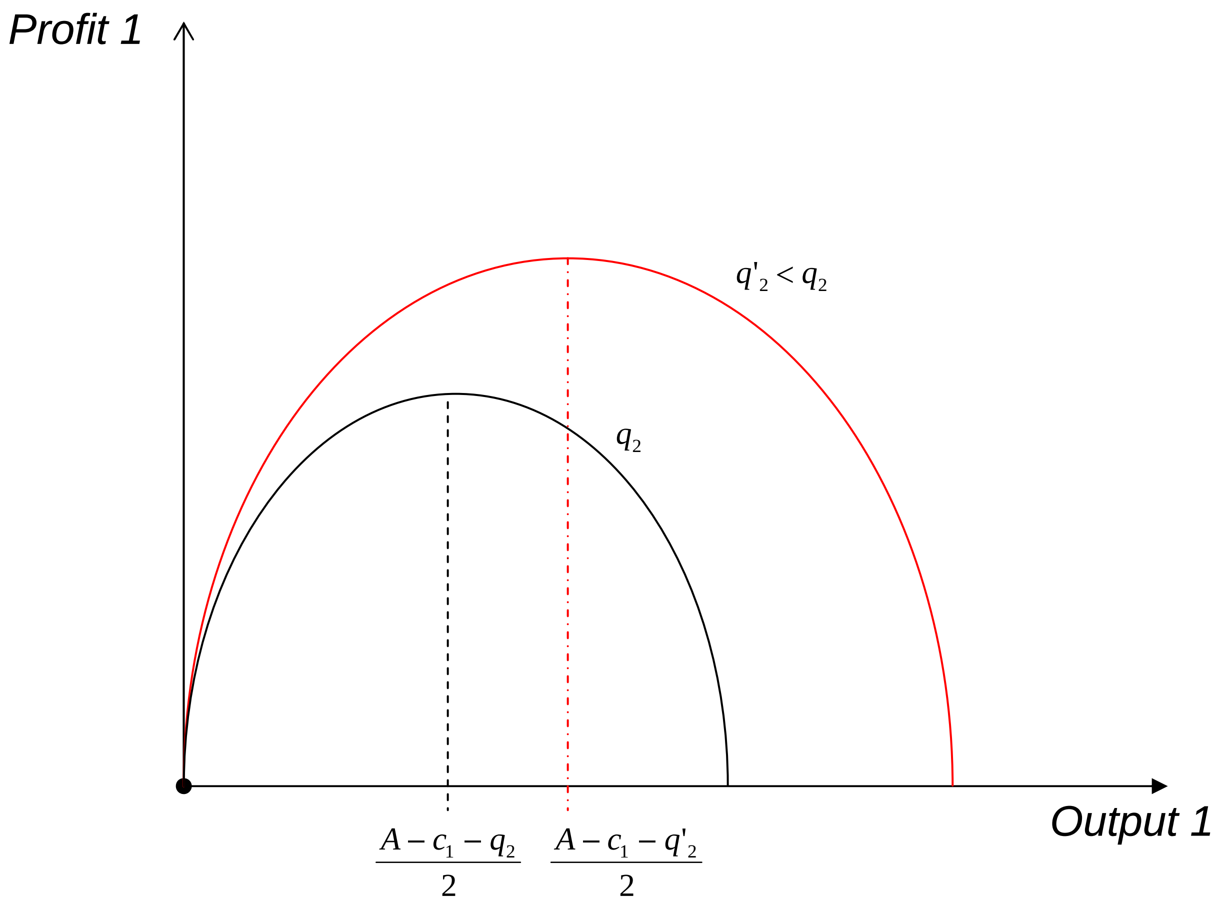
But first we need the firms payoffs (profits):

$$\begin{aligned}\pi_1 &= Pq_1 - c_1q_1 \\ &= (A-Q)q_1 - c_1q_1 \\ &= (A-q_1-q_2)q_1 - c_1q_1 \\ &= (A-q_1-q_2-c_1)q_1\end{aligned}$$

and similarly,

$$\pi_2 = (A-q_1-q_2-c_2)q_2$$

Firm 1's profit as a function of its output (given firm 2's output)



To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2.

Using calculus, we set the derivative of firm 1's profit with respect to q_1 equal to zero and solve for q_1:

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the values of q_2 and c_1.

Because firm 2's cost function is $c_2 \neq c_1$, its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

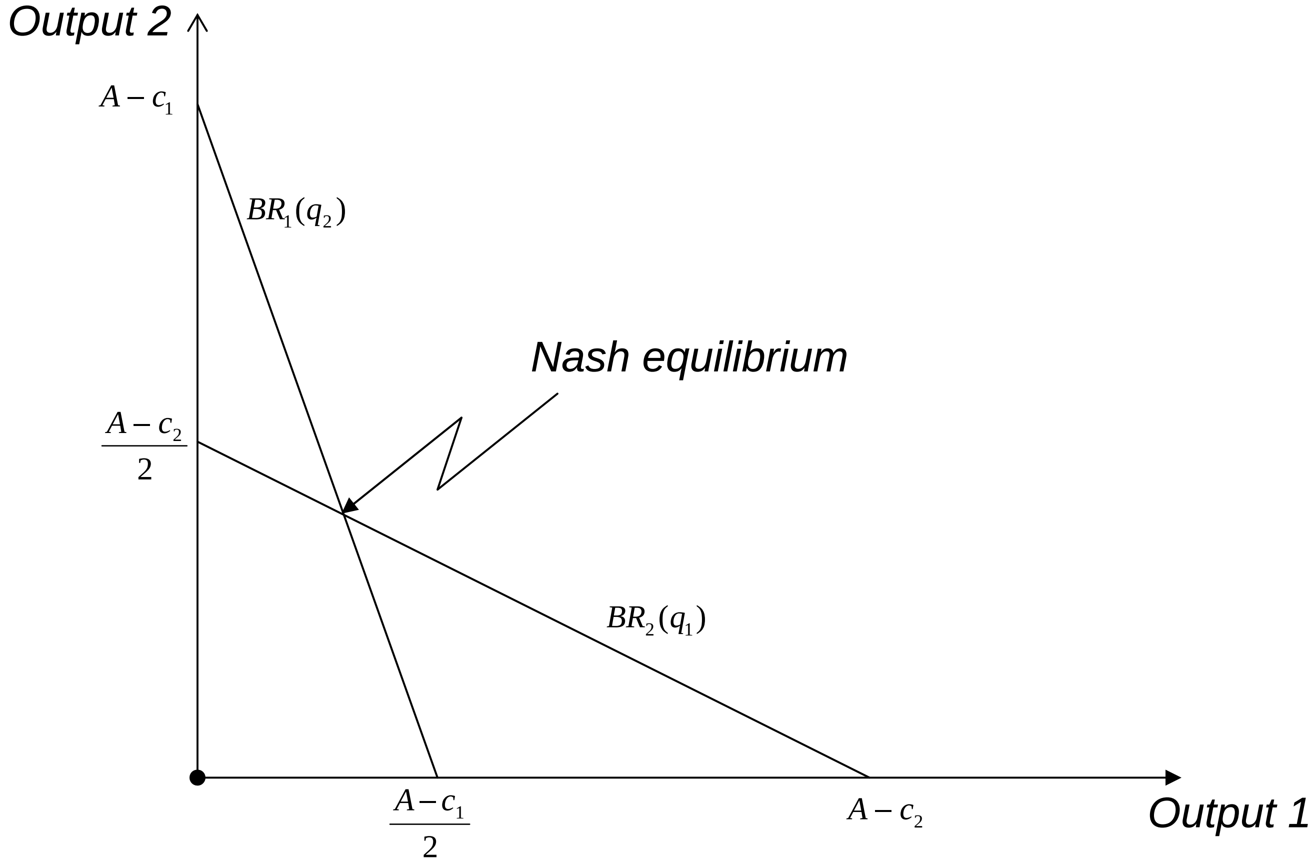
A Nash equilibrium of the Cournot's game is a pair (q_1^*, q_2^*) of outputs such that q_1^* is a best response to q_2^* and q_2^* is a best response to q_1^*.

From the figure below, we see that there is exactly one such pair of outputs

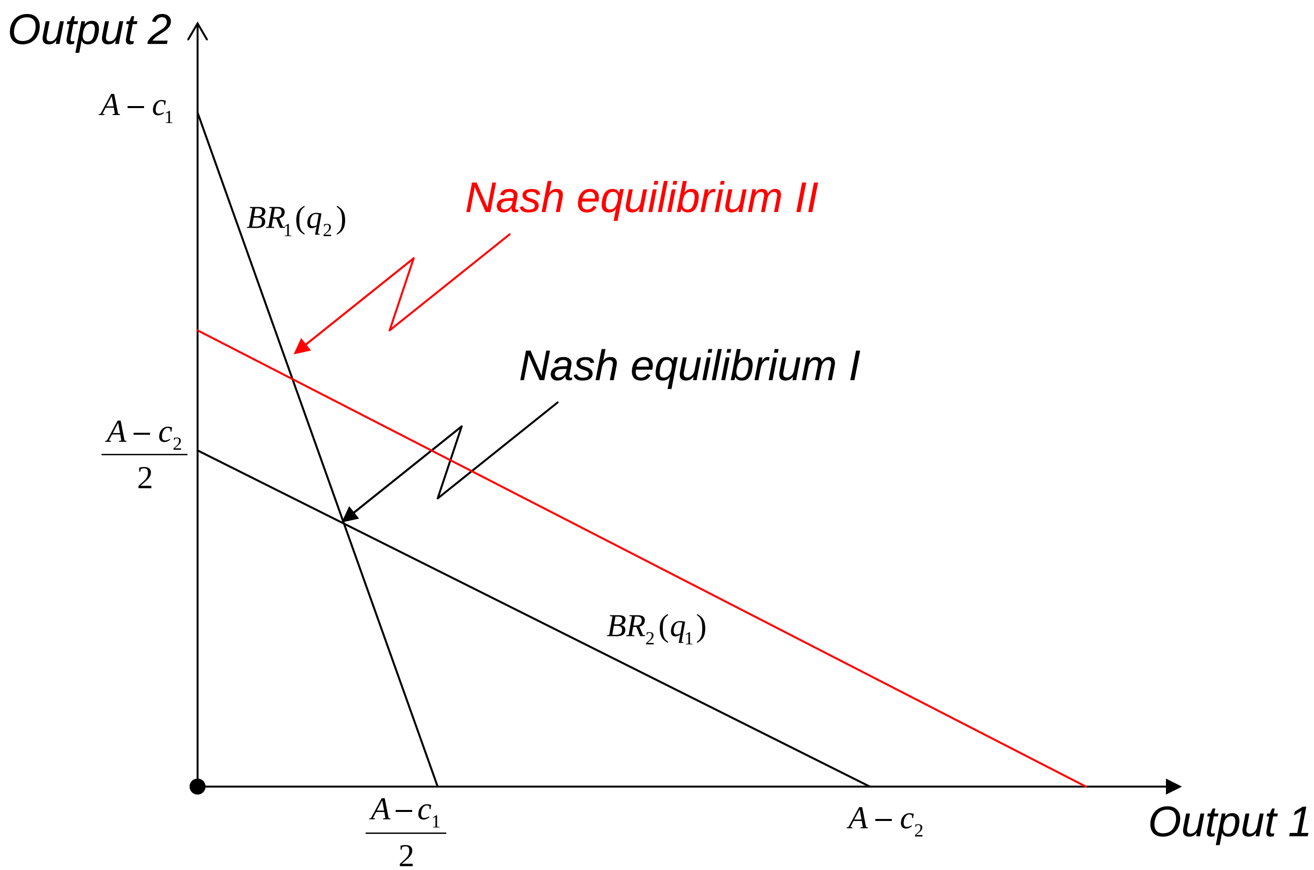
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

The best response functions in the Cournot's duopoly game



Nash equilibrium comparative statics
(a decrease in the cost of firm 2)



A question: what happens when consumers are willing to pay more (*A* increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

[1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.

[2] The relation between the equilibrium profits and the number of firms.

[1] Collusive outcomes: in the Cournot's duopoly game, there is a pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium.

[2] Competition: The price at the Nash equilibrium if the two firms have the *same* unit cost $c_1 = c_2 = c$ is given by

$$P^* = A - q_1^* - q_2^* = \frac{1}{3}(A + 2c)$$

which is above the unit cost c. But as the number of firm increases, the equilibrium price deceases, approaching c (zero profits!).

Stackelberg's duopoly model (1934)

How do the conclusions of the Cournot's duopoly game change when the firms move sequentially? Is a firm better off moving before or after the other firm?

Suppose that $c_1 = c_2 = c$ and that firm 1 moves at the start of the game. We may use backward induction to find the subgame perfect equilibrium.

– First, for *any* output q_1 of firm 1, we find the output q_2 of firm 2 that maximizes its profit. Next, we find the output q_1 of firm 1 that maximizes its profit, *given the strategy* of firm 2.

Firm 2

Since firm 2 moves after firm 1, a strategy of firm 2 is a *function* that associate an output q_2 for firm 2 for each possible output q_1 of firm 1.

We found that under the assumptions of the Cournot's duopoly game Firm 2 has a unique best response to each output q_1 of firm 1, given by

$$q_2 = \frac{1}{2}(A - q_1 - c)$$

(Recall that $c_1 = c_2 = c$).

Firm 1

Firm 1's strategy is the output q_1 the maximizes

$$\pi_1 = (A - q_1 - q_2 - c)q_1 \quad \text{subject to} \quad q_2 = \tfrac{1}{2}(A - q_1 - c)$$

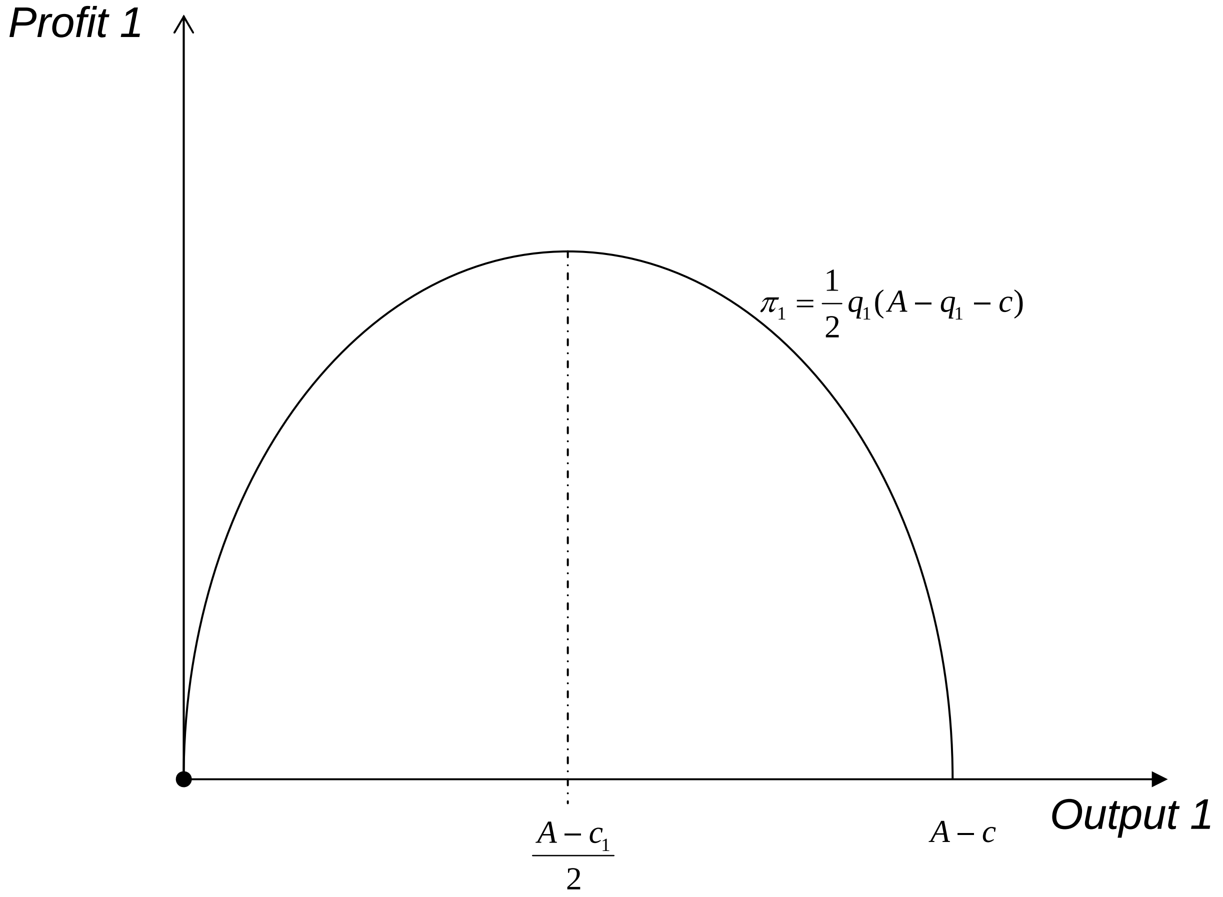
Thus, firm 1 maximizes

$$\pi_1 = (A - q_1 - (\frac{1}{2}(A - q_1 - c)) - c)q_1 = \frac{1}{2}q_1(A - q_1 - c).$$

This function is quadratic in q_1 that is zero when $q_1 = 0$ and when $q_1 = A - c$. Thus its maximizer is

$$q_1^* = \frac{1}{2}(A - c).$$

Firm 1's (first-mover) profit in Stackelberg's duopoly game



We conclude that Stackelberg's duopoly game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output

$$q_1^* = \frac{1}{2}(A - c)$$

and firm 2's output is

$$\begin{aligned} q_2^* &= \frac{1}{2}(A - q_1^* - c) \\ &= \frac{1}{2}(A - \frac{1}{2}(A - c) - c) \\ &= \frac{1}{4}(A - c). \end{aligned}$$

By contrast, in the unique Nash equilibrium of the Cournot's duopoly game under the same assumptions ($c_1 = c_2 = c$), each firm produces $\frac{1}{3}(A - c)$.

The subgame perfect equilibrium of Stackelberg's duopoly game