UC Berkeley
Haas School of Business
Game Theory
(EMBA 296 & EWMBA 211)
Spring 2025

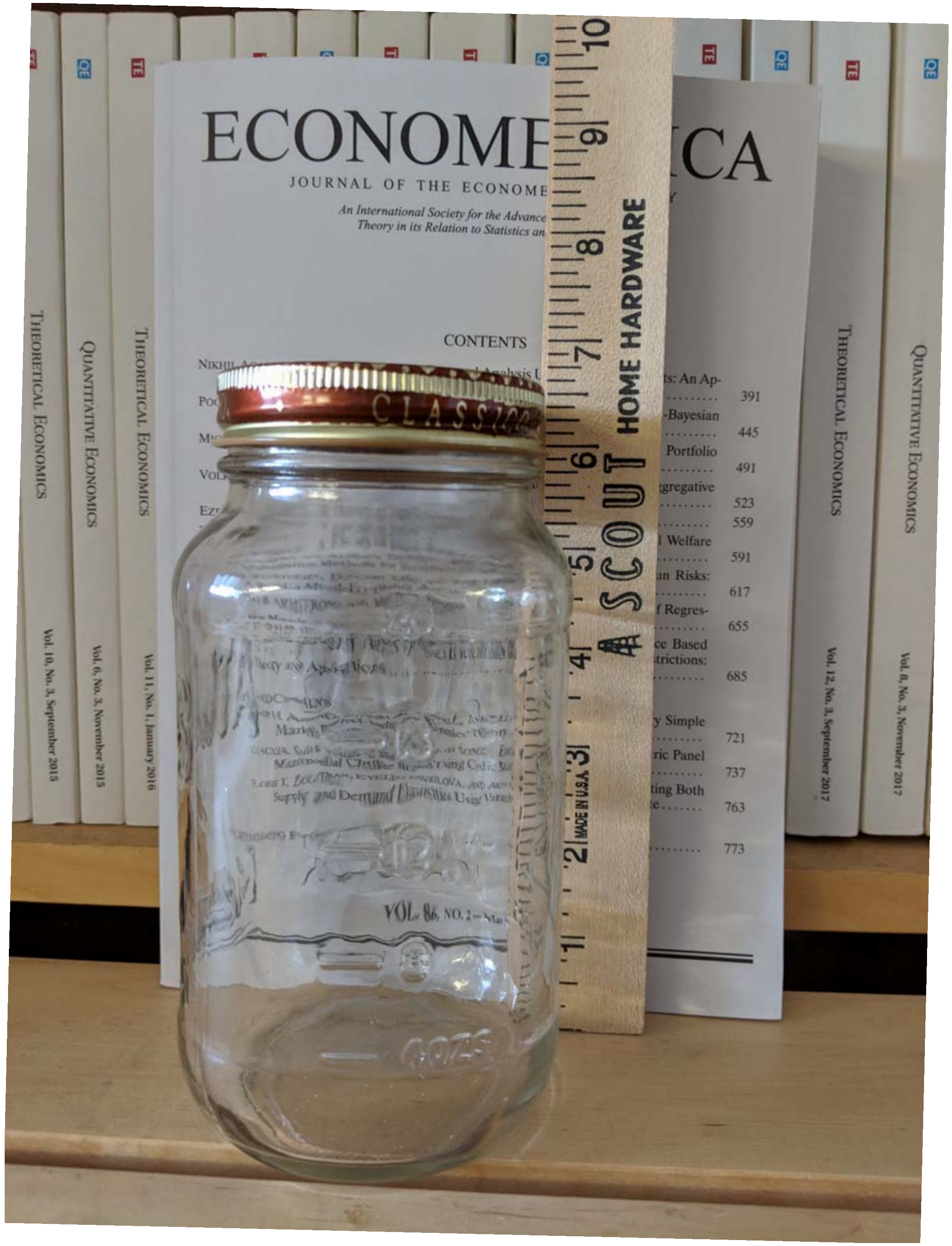
More on strategic games and extensive games (w/ perfect information)

Block II

Jan 29 and Feb 1-2, 2025

Game plan

1. Auction results, LUPI and Morra
2. Strategic games — review
3. Extensive games (w/ perfect information)
4. Oligopolistic competition
5. The tragedy of the commons



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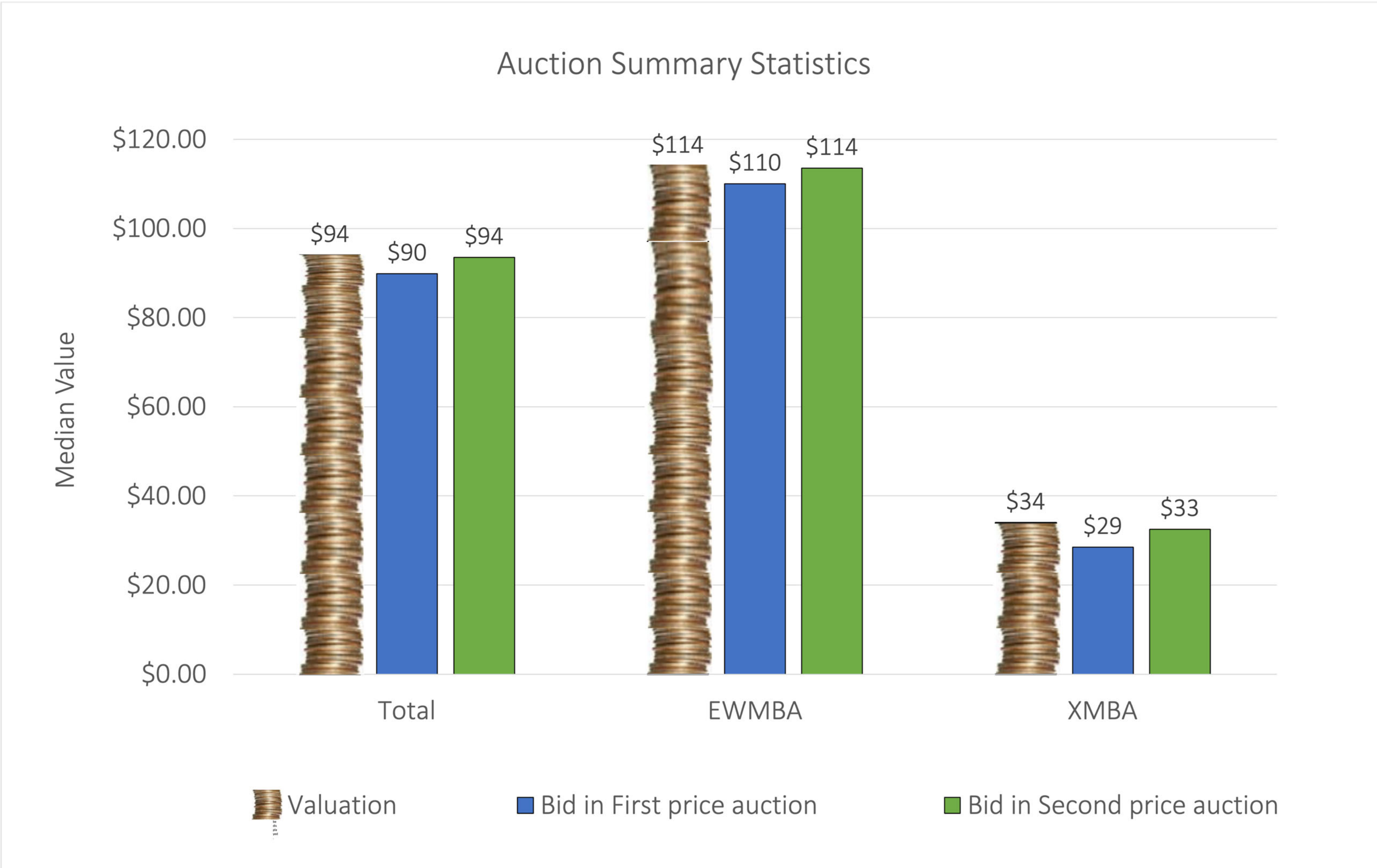
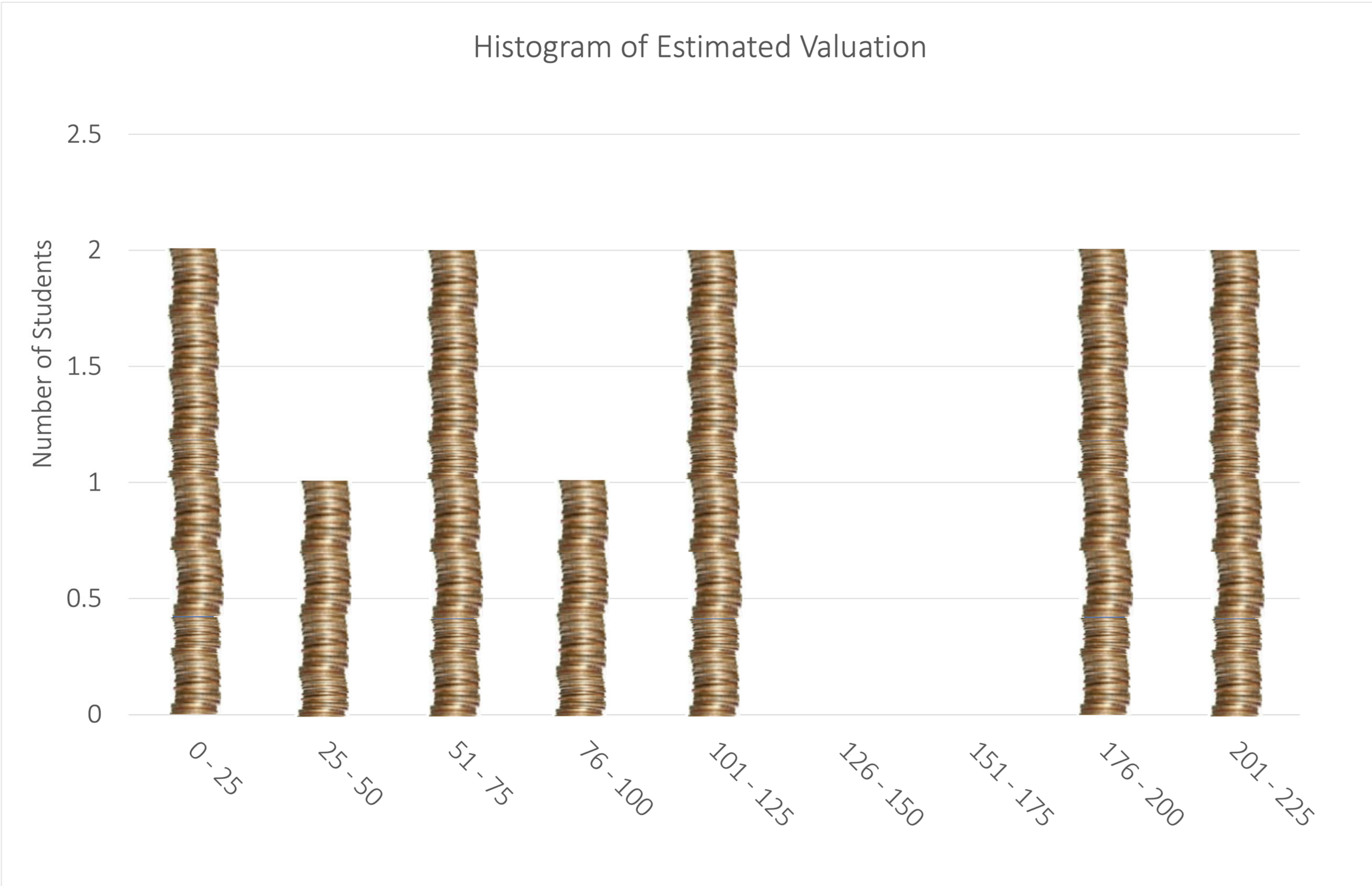
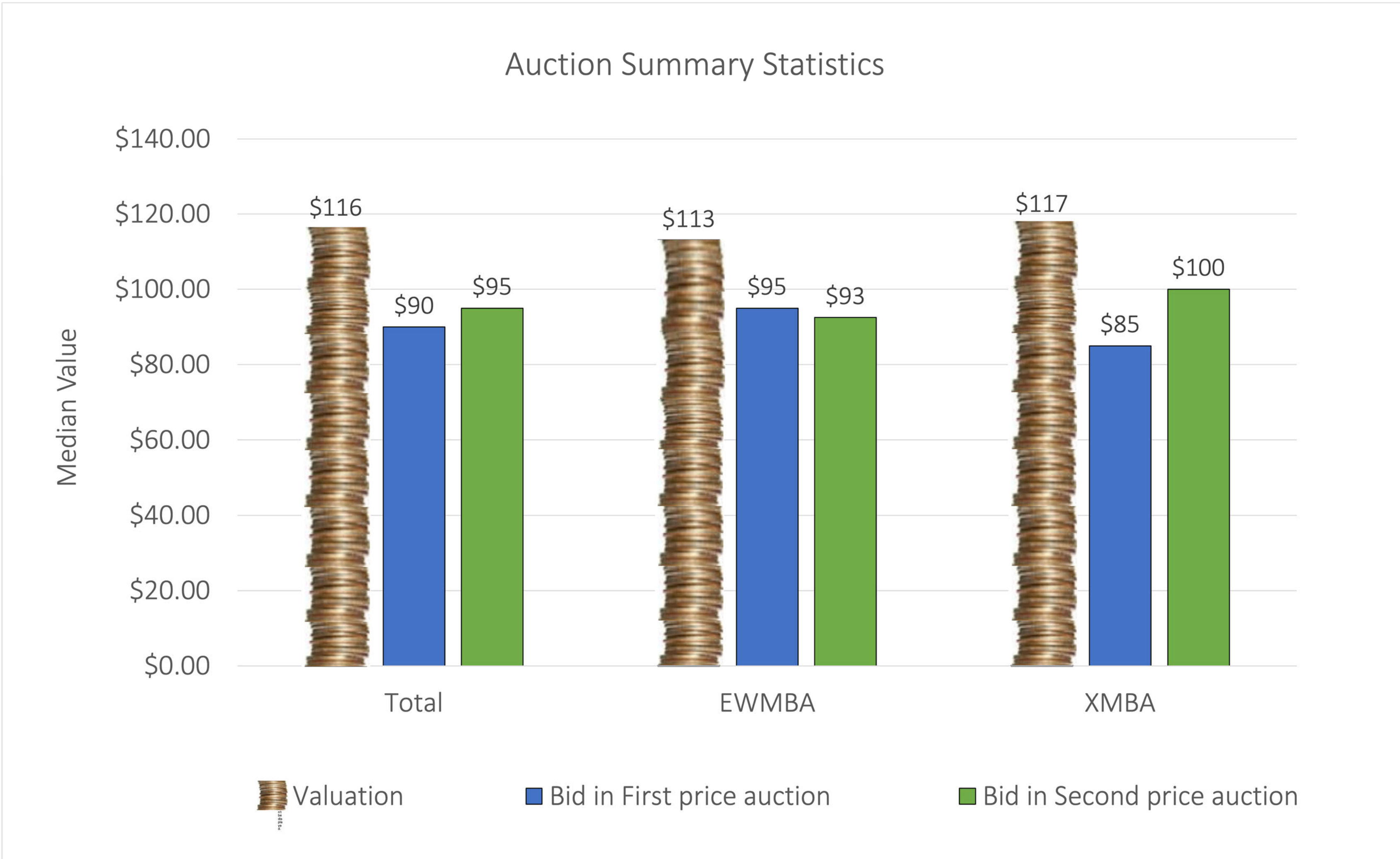
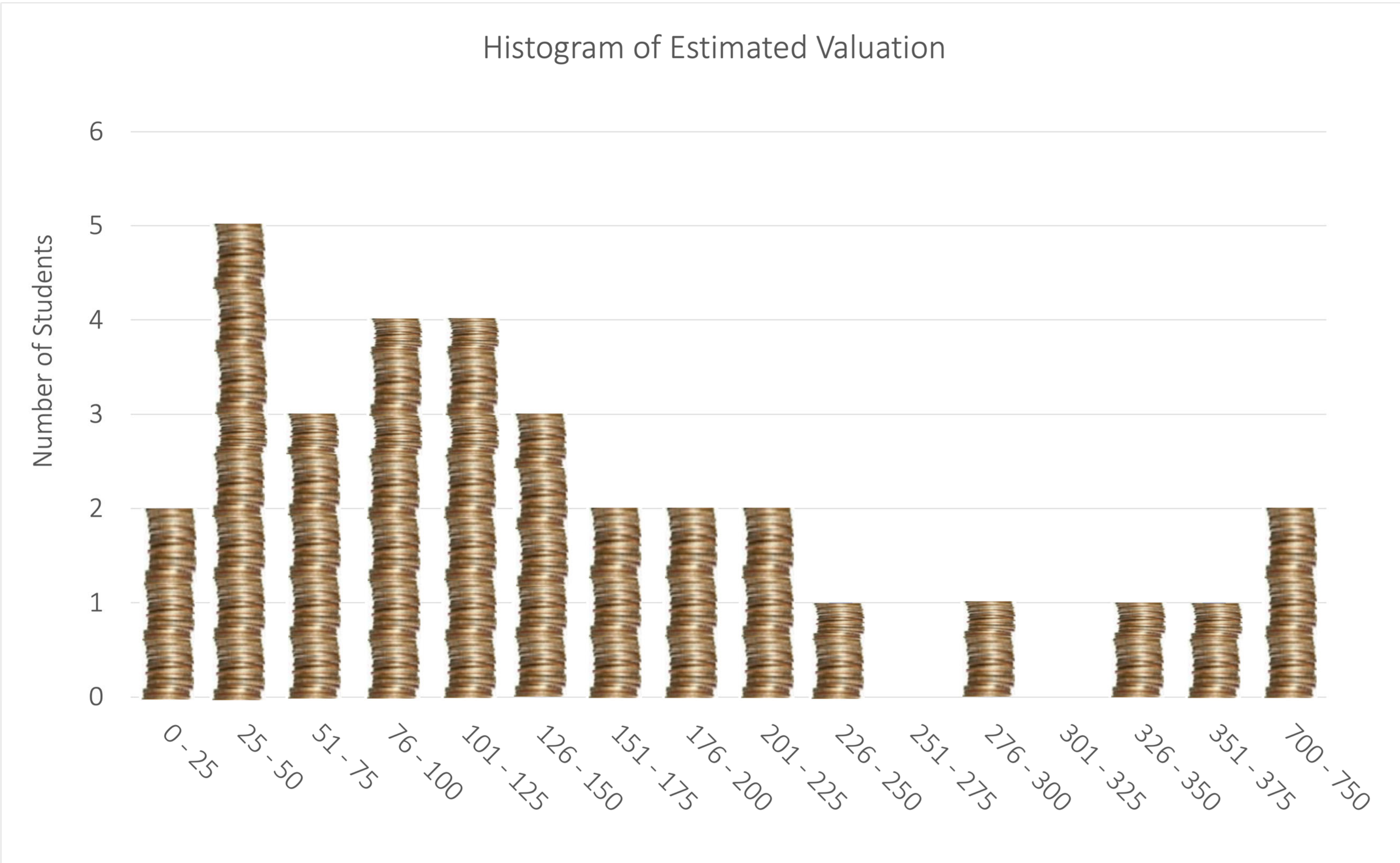


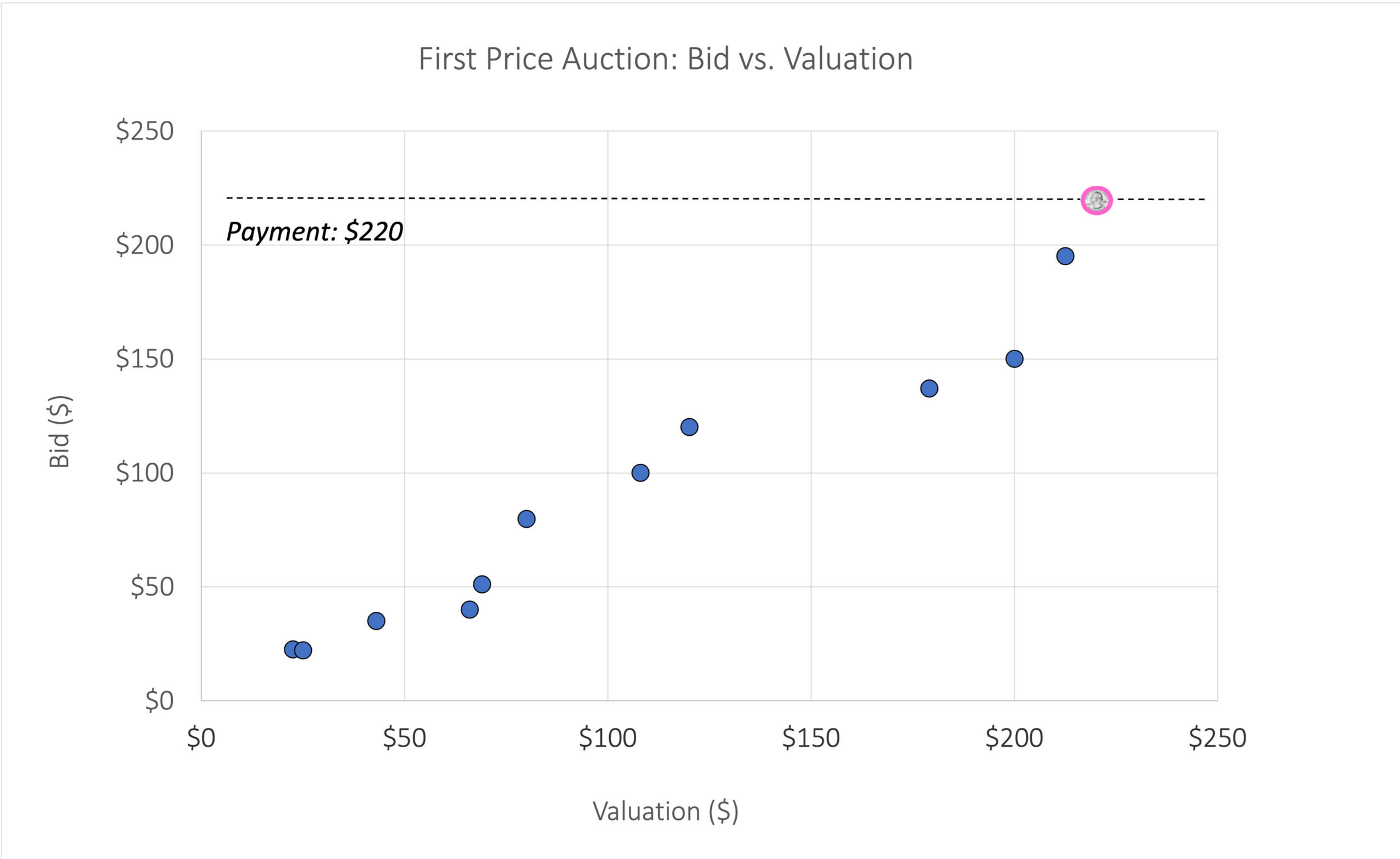
Auction Summary Statistics
Median Value
$120.00
$100.00
$80.00
$60.00
$40.00
$20.00
$0.00
$94
$90
$94
$114
$110
$114
$34
$29
$33
Total
EWMBA
XMBA
Valuation
Bid in First price auction
Bid in Second price auction

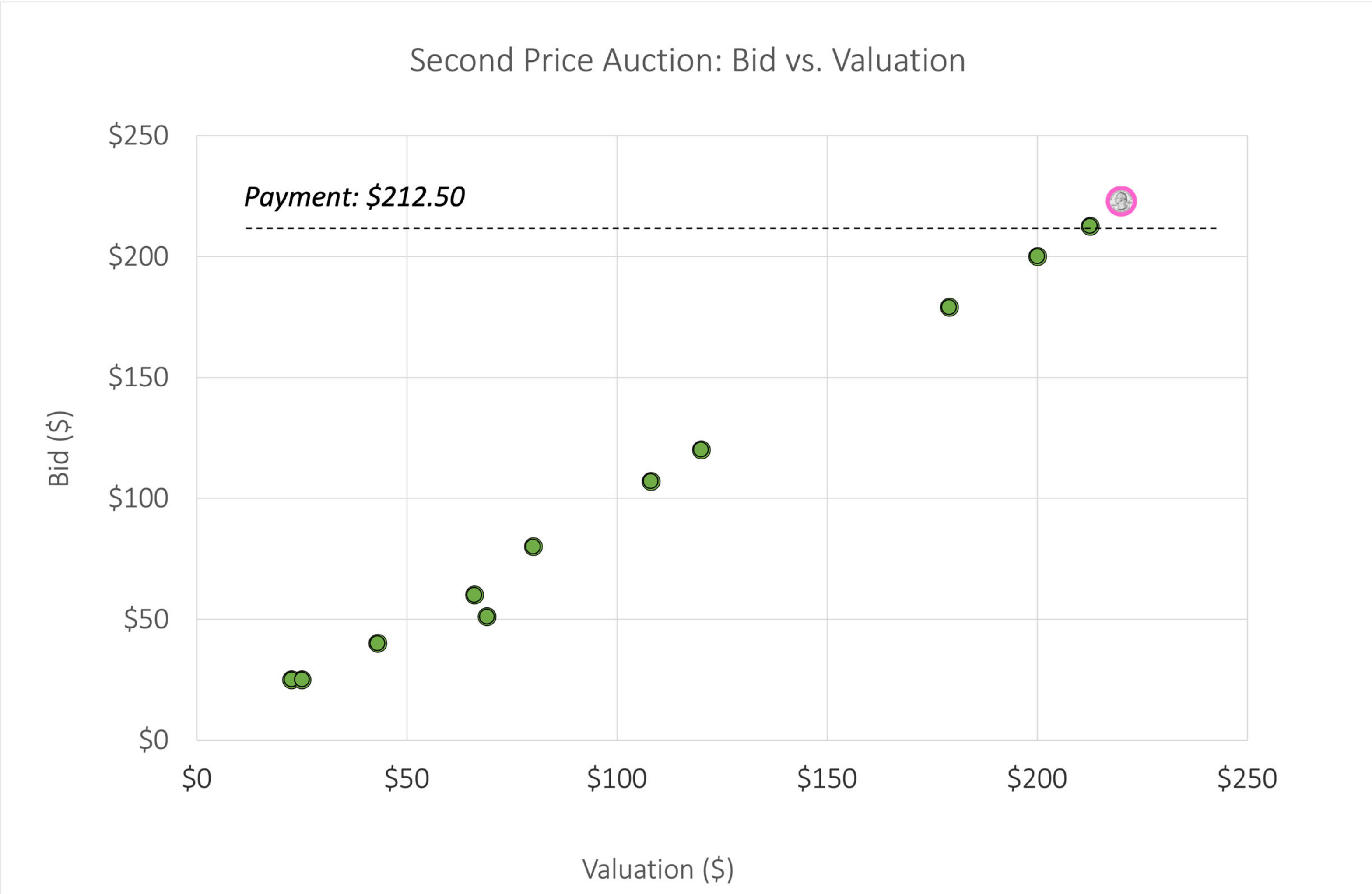

Histogram of Estimated Valuation
Number of Students
2.5
2
1.5
1
0.5
0
0 - 25
25 - 50
51 - 75
76 - 100
101 - 125
126 - 150
151 - 175
176 - 200
201 - 225

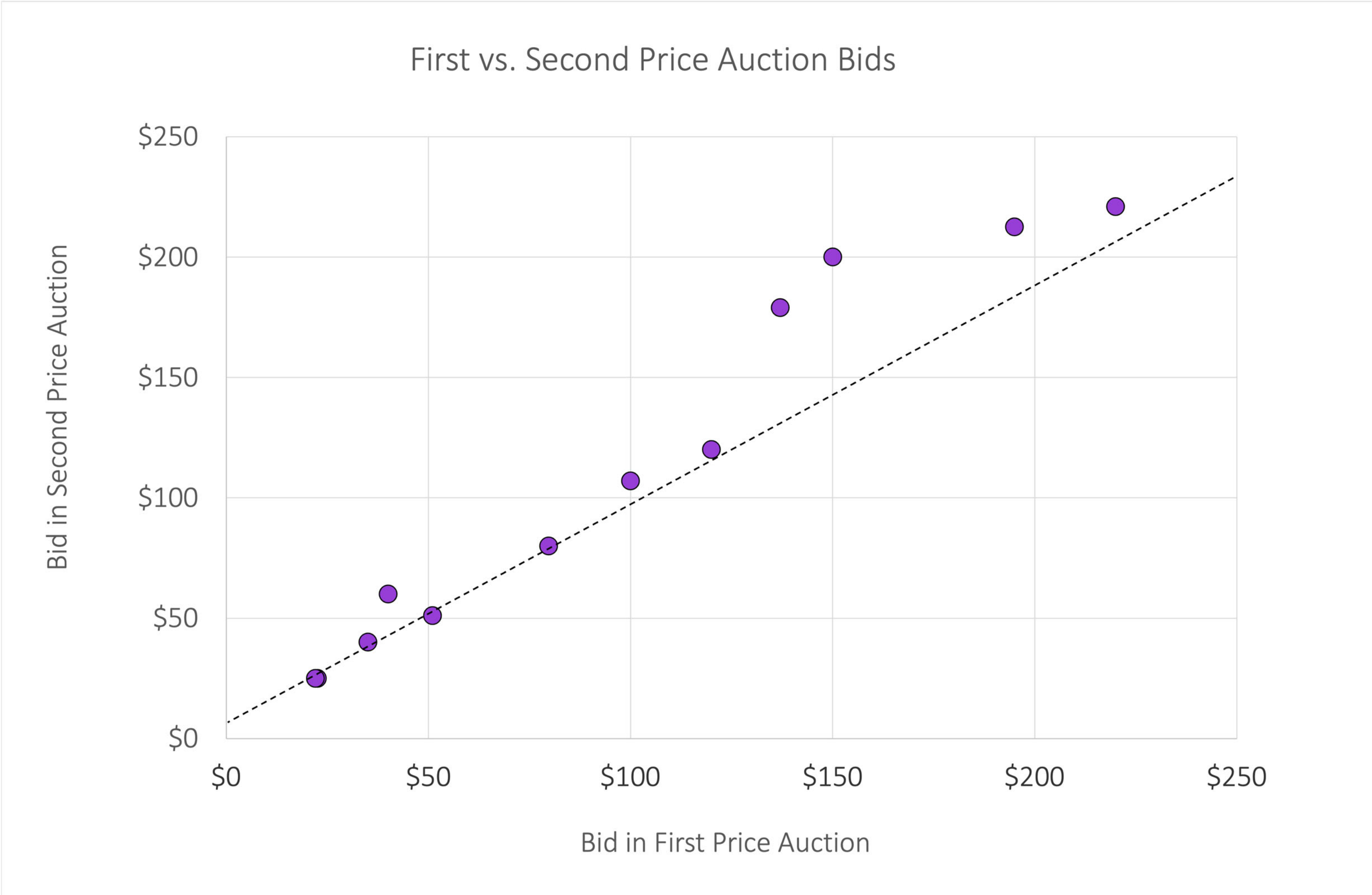

Auction Summary Statistics
Median Value
$140.00
$120.00
$100.00
$80.00
$60.00
$40.00
$20.00
$0.00
$116
$90
$95
$113
$95
$93
$117
$85
$100
Total
EWMBA
XMBA
Valuation
Bid in First price auction
Bid in Second price auction



Histogram of Estimated Valuation
Number of Students
0
1
2
3
4
5
6
0 - 25
25 - 50
51 - 75
76 - 100
101 - 125
126 - 150
151 - 175
176 - 200
201 - 225
226 - 250
251 - 275
276 - 300
301 - 325
326 - 350
351 - 375
700 - 750


First Price Auction: Bid vs. Valuation
Payment: $220
Bid ($)
Valuation ($)
$0
$50
$100
$150
$200
$250


Second Price Auction: Bid vs. Valuation
Payment: $212.50
$250
$200
$150
$100
$50
$0
Bid ($)
$0
$50
$100
$150
$200
$250
Valuation ($)

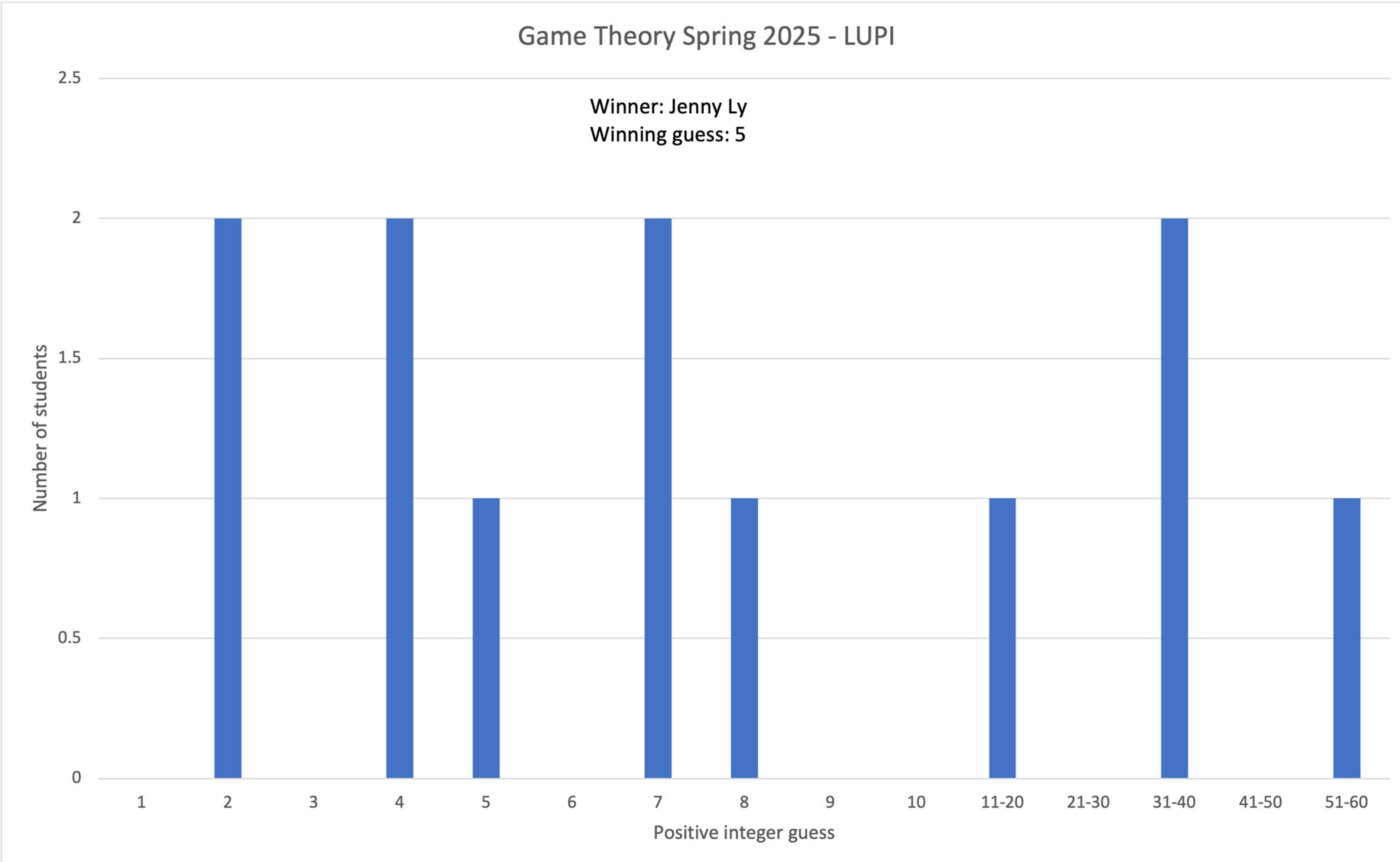

First vs. Second Price Auction Bids
$250
$200
$150
$100
$50
$0
Bid in Second Price Auction
$0
$50
$100
$150
$200
$250
Bid in First Price Auction

LUPI

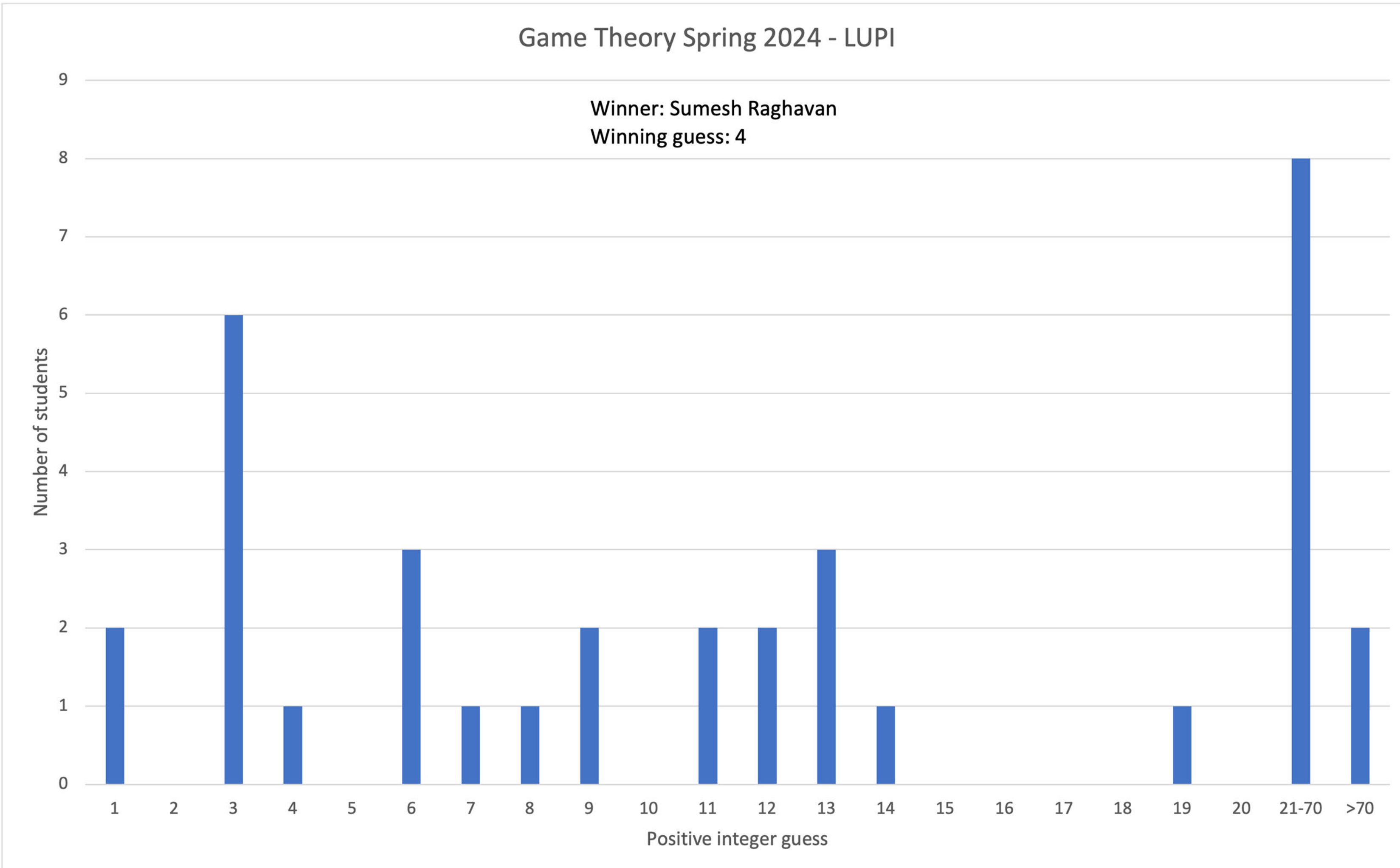
Many players simultaneously chose an integer between 1 and 99,999. Whoever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.



Game Theory Spring 2025 - LUPI
Winner: Jenny Ly
Winning guess: 5
Number of students
2.5
2
1.5
1
0.5
0
1
2
3
4
5
6
7
8
9
10
11-20
21-30
31-40
41-50
51-60
Positive integer guess

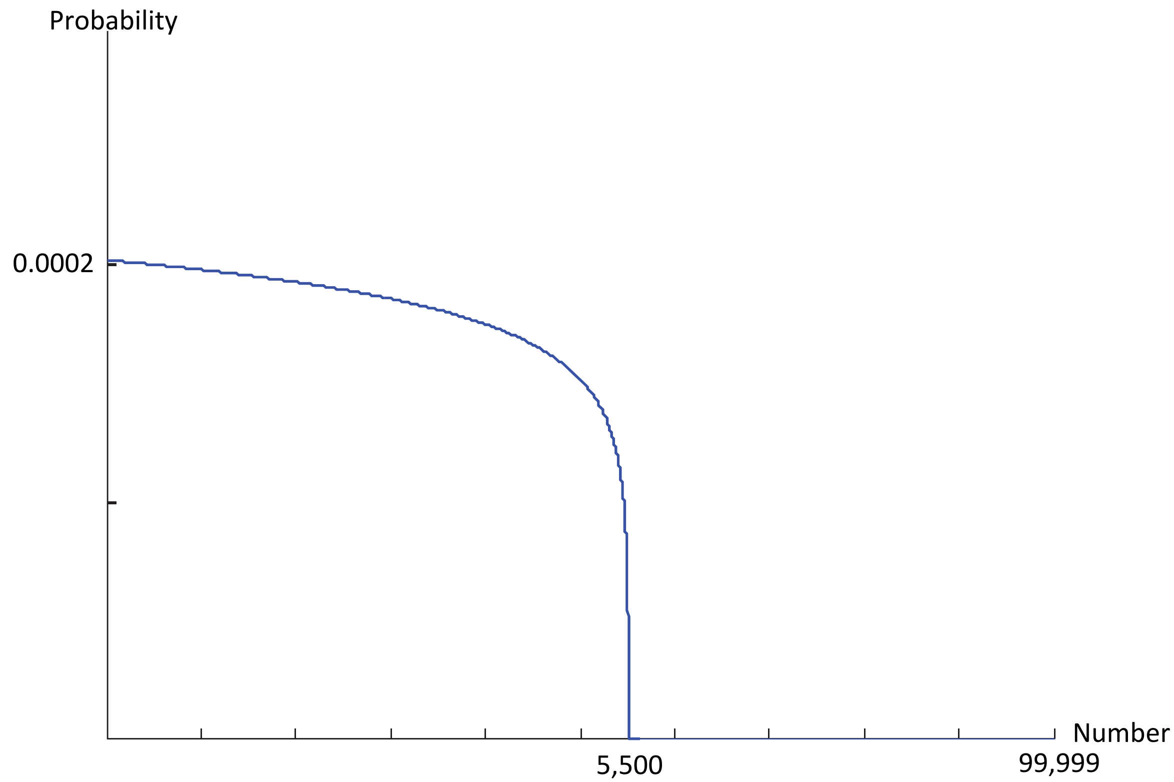

Game Theory Spring 2024 - LUPI
Winner: Sumesh Raghavan
Winning guess: 4
Number of students
0
1
2
3
4
5
6
7
8
9
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21-70
>70
Positive integer guess

Games with population uncertainty relax the assumption that the exact number of players is common knowledge.

In particular, in a Poisson game (Myerson; 1998, 2000) the number of players N is a random variable that follows a Poisson distribution with mean n so the probability that $N = k$ is given by

$$\frac{e^{-n} n^k}{k!}$$

In the Swedish game the average number of players was $n = 53,783$ and number choices were positive integers up to $99,999$.


Probability
0.0002
Number
5,500
99,999

Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

In Morra there are two players, each of whom has four (relevant) actions, S_1G_2, S_1G_3, S_2G_3, and S_2G_4, where S_iG_j denotes the strategy (Show i, Guess j).

The payoffs in the game are as follows

	S_1G_2	S_1G_3	S_2G_3	S_2G_4
S_1G_2	$0, 0$	$2, -2$	$-3, 3$	$0, 0$
S_1G_3	$-2, 2$	$0, 0$	$0, 0$	$3, -3$
S_2G_3	$3, -3$	$0, 0$	$0, 0$	$-4, 4$
S_2G_4	$0, 0$	$-3, 3$	$4, -4$	$0, 0$

Strategic games
(review)

A two-player (finite) strategic game

The game can be described conveniently in a so-called bi-matrix. For example, a generic 2×2 (two players and two possible actions for each player) game

	L	R
T	a_1, a_2	b_1, b_2
B	c_1, c_2	d_1, d_2

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2). The two numbers in a box formed by a specific row and column are the players' payoffs given that these actions were chosen.

In this game above a_1 and a_2 are the payoffs of player 1 and player 2 respectively when player 1 is choosing strategy T and player 2 strategy L.

Classical 2×2 games

- The following simple 2×2 games represent a variety of strategic situations.

- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.

- These classical games “span” the set of almost *all* games (strategic equivalence).

Game I: Prisoner's Dilemma

	$Work$	$Goof$
$Work$	$3, 3$	$0, 4$
$Goof$	$4, 0$	$1, 1$

A situation where there are gains from cooperation but each player has an incentive to "free ride."

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.

Game II: Battle of the Sexes (BoS)

	Ball	*Show*
Ball	2, 1	0, 0
Show	0, 0	1, 2

Like the Prisoner's Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.

Game III-V: Coordination, Hawk-Dove, and Matching Pennies

	Ball	*Show*
Ball	$2, 2$	$0, 0$
Show	$0, 0$	$1, 1$

	Dove	*Hawk*
Dove	$3, 3$	$1, 4$
Hawk	$4, 1$	$0, 0$

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Best response and dominated actions

Action T is player 1's *best response* to action L player 2 if T is the optimal choice when 1 *conjectures* that 2 will play L.

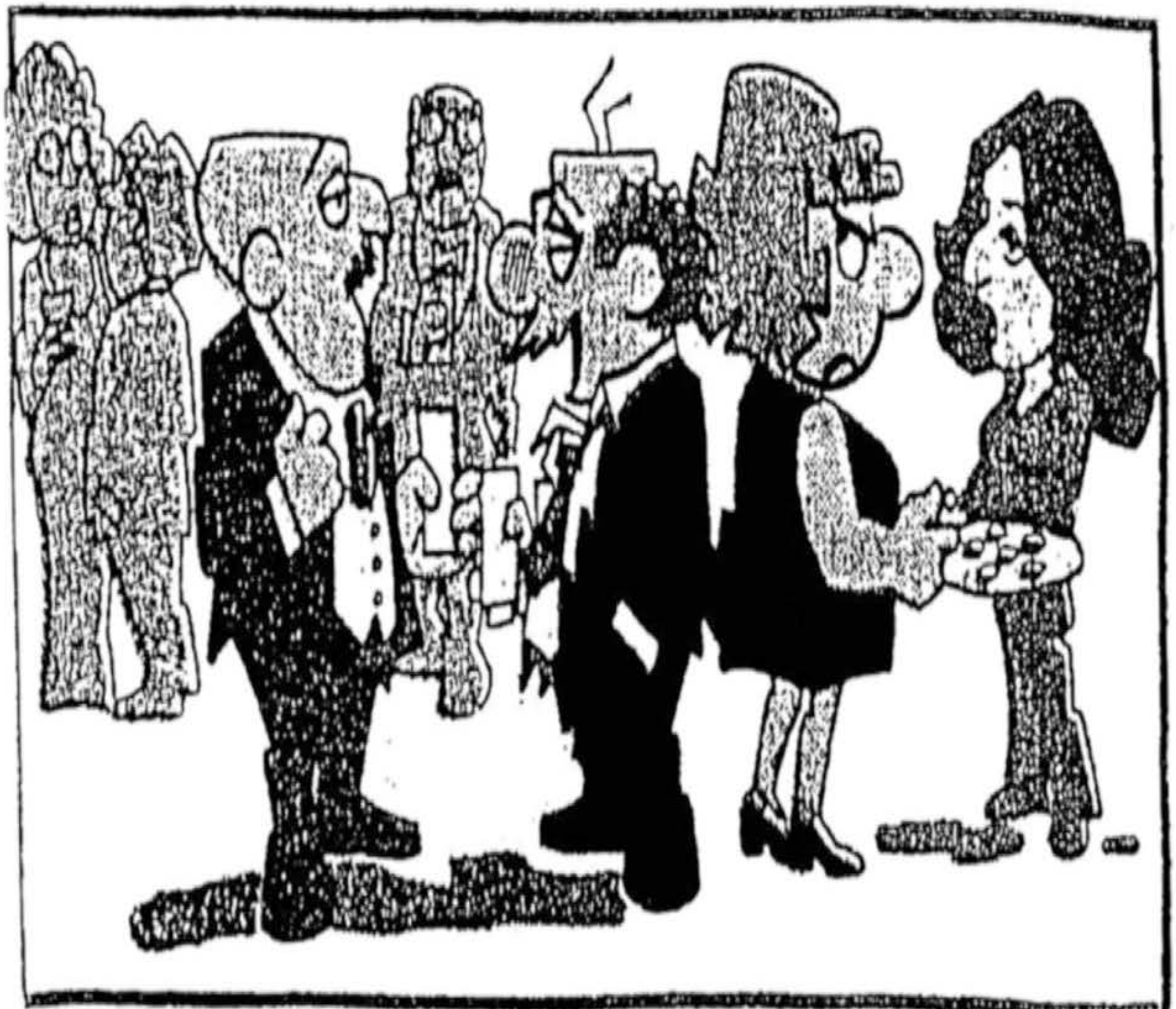
Player 1's action T is *strictly* dominated if it is never a best response (inferior to B no matter what the other players do).

In the Prisoner's Dilemma, for example, action $Work$ is strictly dominated by action $Goof$. As we will see, a strictly dominated action is not used in any Nash equilibrium.

Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a NE is a set of actions such that all players are doing their best given the actions of the other players.



"LORETTA'S DRIVING BECAUSE I'M DRINKING, AND I'M DRINKING BECAUSE SHE'S DRIVING."

Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

		q L	$1-q$ R
p	T	pq	$p(1-q)$
$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

Let p and q be the probabilities that player 1 and 2 respectively assign to the strategy *Ball*.

Player 2 will be indifferent between using her strategy B and S when player 1 assigns a probability p such that her expected payoffs from playing B and S are the same. That is,

$$1p + 0(1-p) = 0p + 2(1-p)$$
$$p = 2 - 2p$$
$$p^* = 2/3$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy B and probability $1 - p^* = 1/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

Similarly, player 1 will be indifferent between using her strategy B and S when player 2 assigns a probability q such that her expected payoffs from playing B and S are the same. That is,

$$2q + 0(1-q) = 0q + 1(1-q)$$
$$2q = 1 - q$$
$$q^* = 1/3$$

Hence, when player 2 assigns probability $q^* = 1/3$ to her strategy B and probability $1 - q^* = 2/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

In terms of best responses:

$$B_1(q) = \begin{cases} p = 1 & if \quad p > 1/3 \\ p \in [0,1] & if \quad p = 1/3 \\ p = 0 & if \quad p < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} q = 1 & if \quad p > 2/3 \\ q \in [0,1] & if \quad p = 2/3 \\ q = 0 & if \quad p < 2/3 \end{cases}$$

The BoS has two Nash equilibria in pure strategies $\{(B,B),(S,S)\}$ and one in mixed strategies $\{(2/3,1/3)\}$. In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).

Three Matching Pennies games in the laboratory

		.48	.52
		a_2	b_2
.48	a_1	80, 40	40, 80
.52	b_1	40, 80	80, 40

		.16	.84
		a_2	b_2
.96	a_1	320, 40	40, 80
.04	b_1	40, 80	80, 40

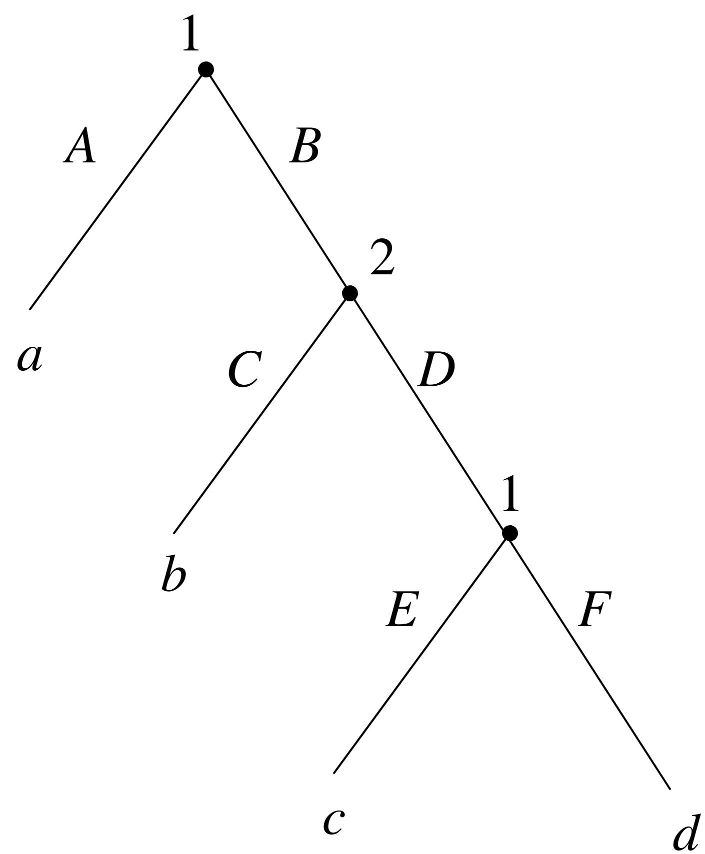
		.80	.20
		a_2	b_2
.08	a_1	44, 40	40, 80
.92	b_1	40, 80	80, 40

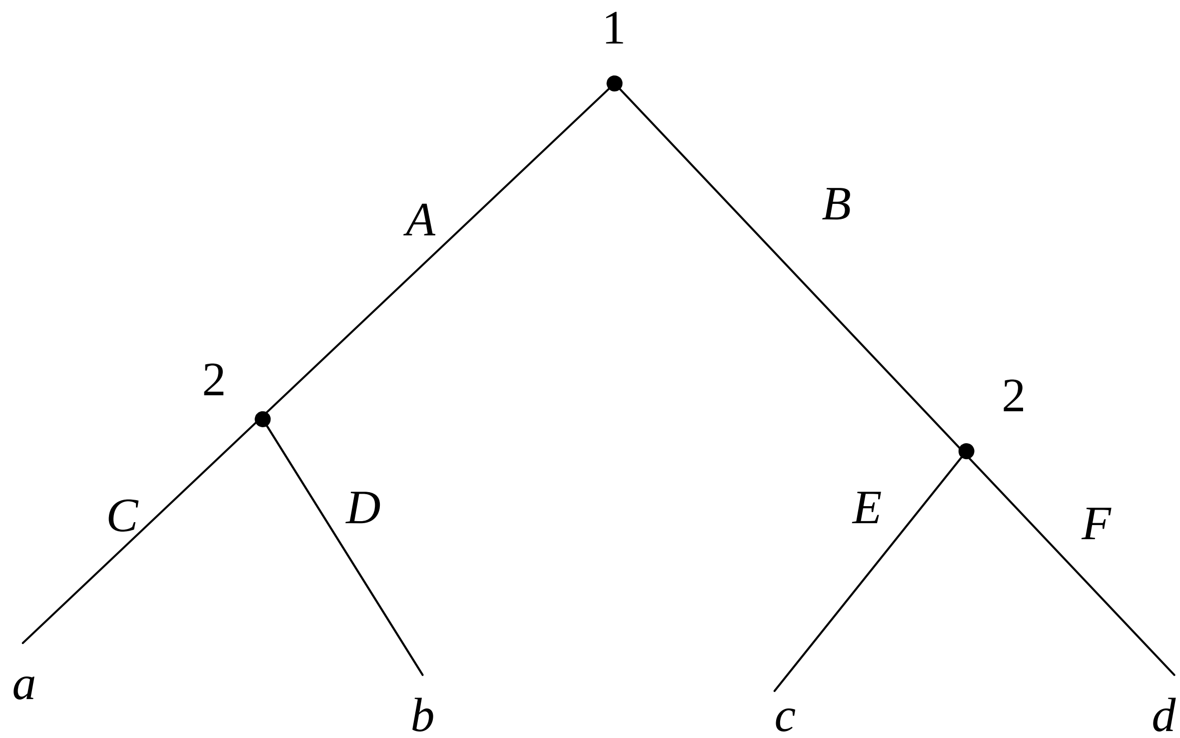
Extensive games with perfect information

Extensive games with perfect information

- The model of a strategic suppresses the sequential structure of decision making.
  - All players simultaneously choose their plan of action once and for all.

- The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly.
  - In an extensive game of perfect information all players are fully informed about all previous actions.


1
A
B
a
2
C
D
b
1
E
F
c
d


1
A
B
2
2
C
D
E
F
a
b
c
d

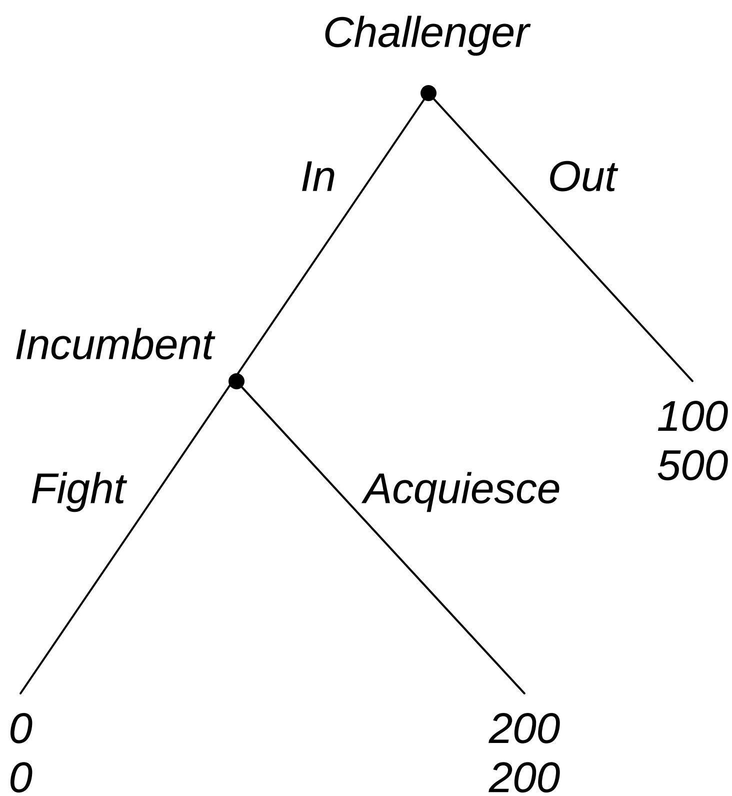
Subgame perfect equilibrium

- The notion of Nash equilibrium ignores the sequential structure of the game.

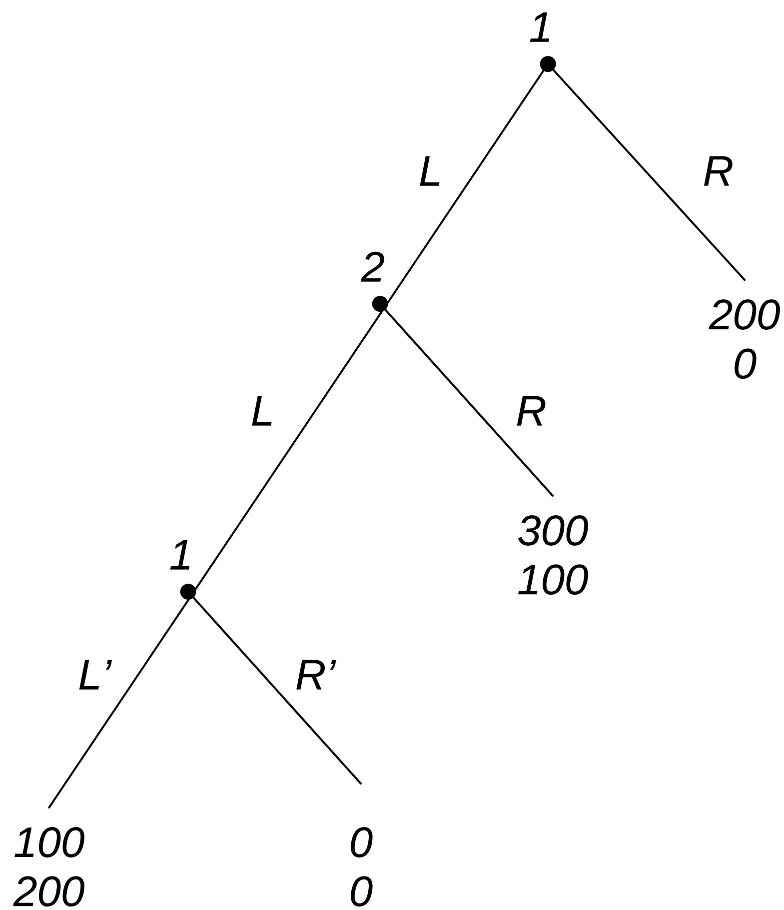
- Consequently, the steady state to which a Nash Equilibrium corresponds may not be robust.

- A *subgame perfect equilibrium* is an action profile that induces a Nash equilibrium in every *subgame* (so every subgame perfect equilibrium is also a Nash equilibrium).

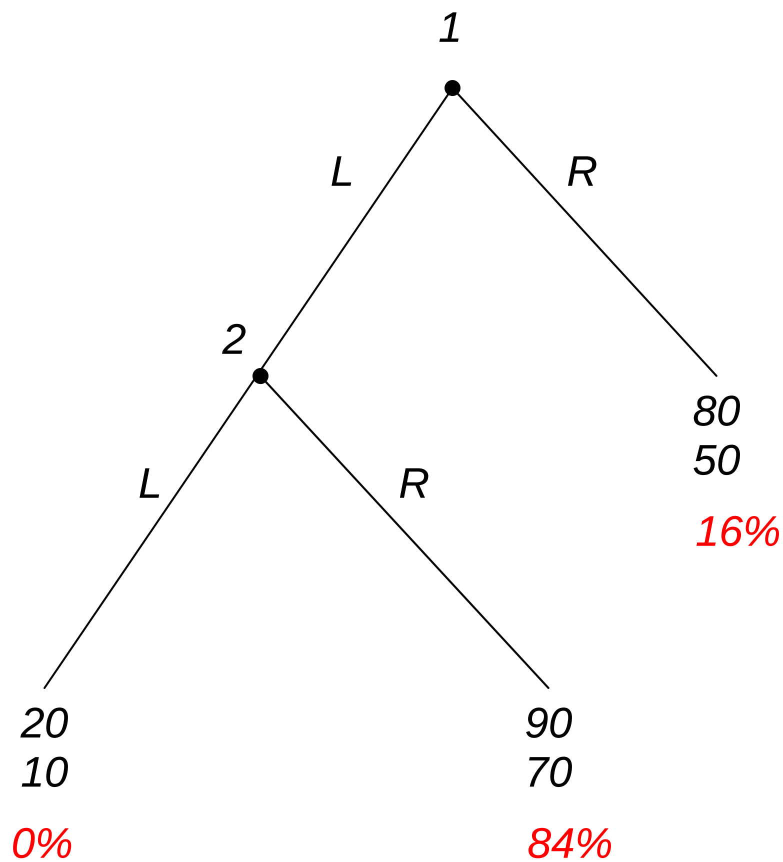
An example: entry game

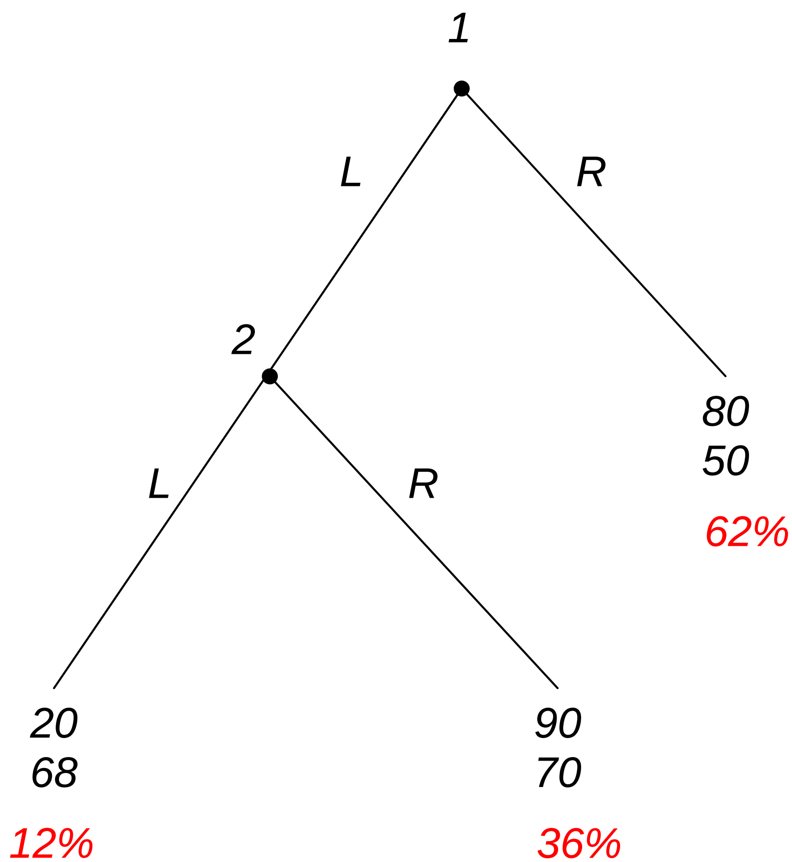


Subgame perfect and backward induction



Two entry games in the laboratory




1
L
R
2
L
R
80
50
62%
20
68
12%
90
70
36%

A review of the main ideas

We study two (out of four) groups of game theoretic models:

[1] Strategic games – all players simultaneously choose their plan of action once and for all.

[2] Extensive games (with perfect information) – players choose sequentially (and fully informed about all previous actions).

A solution (equilibrium) is a systematic description of the outcomes that may emerge in a family of games. We study two solution concepts:

[1] Nash equilibrium – a steady state of the play of a <u>strategic</u> game (no player has a profitable deviation given the actions of the other players).

[1] Subgame equilibrium – a steady state of the play of an <u>extensive</u> game (a Nash equilibrium in every <u>subgame</u> of the extensive game).

$\Longrightarrow$ Every subgame perfect equilibrium is also a Nash equilibrium.

Oligopoly

Cournot's oligopoly model (1838)

- A single good is produced by two firms (the industry is a "duopoly").

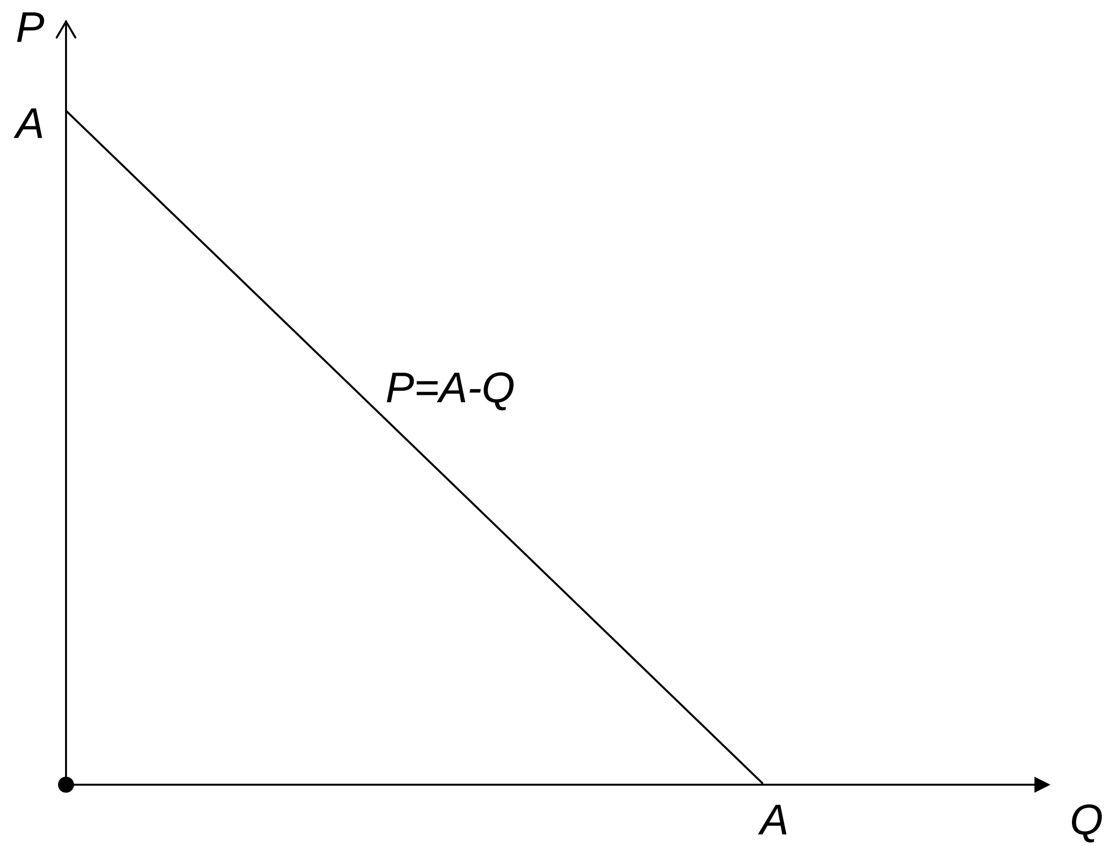
- The cost for firm $i = 1, 2$ for producing q_i units of the good is given by $c_i q_i$ ("unit cost" is constant equal to $c_i > 0$).

- If the firms' total output is $Q = q_1 + q_2$ then the market price is

$$P = A - Q$$

  if $A \geq Q$ and zero otherwise (linear inverse demand function). We also assume that $A > c$.

The inverse demand function



To find the Nash equilibria of the Cournot's game, we can use the procedures based on the firms' best response functions.

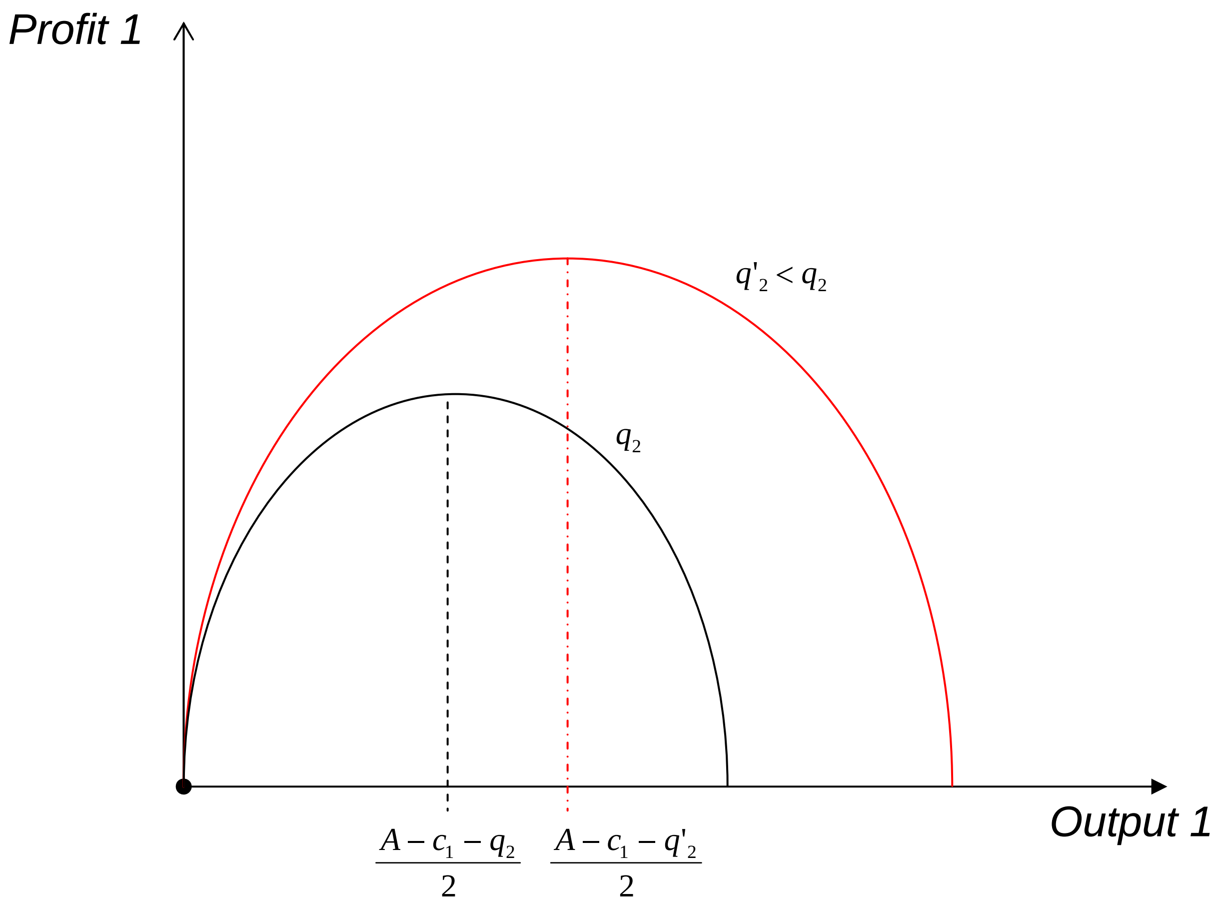
But first we need the firms payoffs (profits):

$$\begin{aligned}\pi_1 &= Pq_1 - c_1q_1 \\ &= (A - Q)q_1 - c_1q_1 \\ &= (A - q_1 - q_2)q_1 - c_1q_1 \\ &= (A - q_1 - q_2 - c_1)q_1\end{aligned}$$

and similarly,

$$\pi_2 = (A - q_1 - q_2 - c_2)q_2$$

Firm 1's profit as a function of its output (given firm 2's output)



To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2.

Using calculus, we set the derivative of firm 1's profit with respect to q_1 equal to zero and solve for q_1:

$$q_1 = \frac{1}{2}(A - q_2 - c_1).$$

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the values of q_2 and c_1.

Because firm 2's cost function is $c_2 \neq c_1$, its best response function is given by

$$q_2 = \frac{1}{2}(A - q_1 - c_2).$$

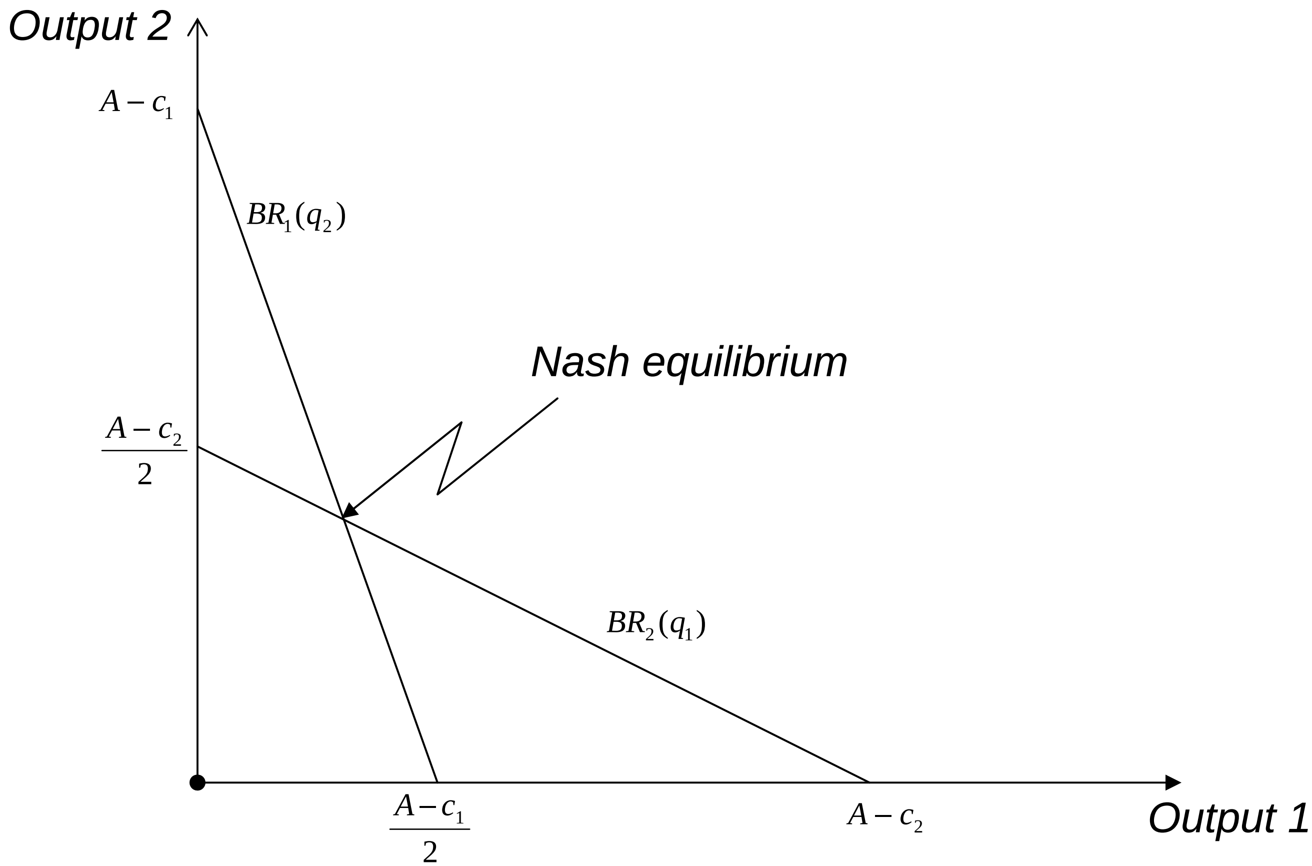
A Nash equilibrium of the Cournot's game is a pair (q_1^*, q_2^*) of outputs such that q_1^* is a best response to q_2^* and q_2^* is a best response to q_1^*.

From the figure below, we see that there is exactly one such pair of outputs

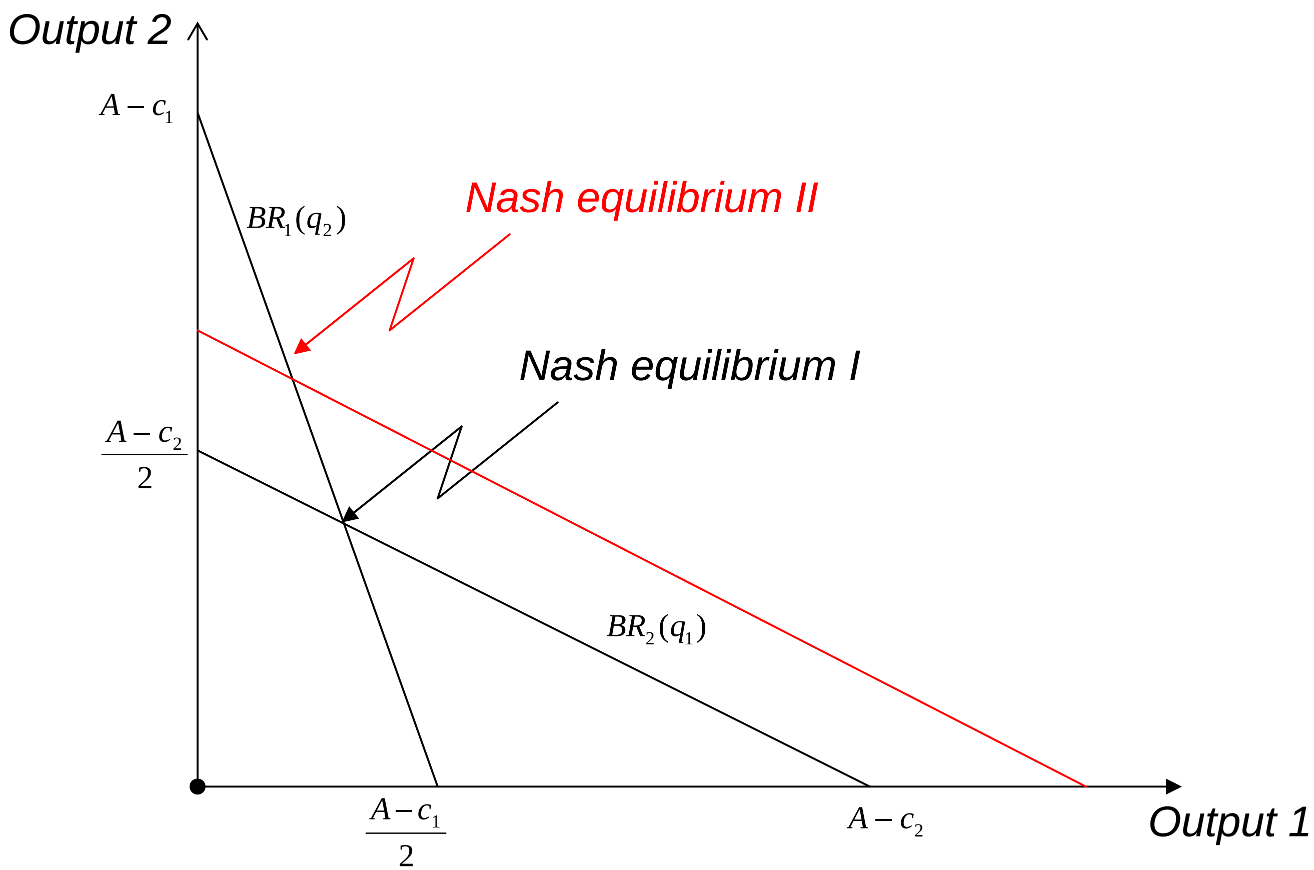
$$q_1^* = \frac{A+c_2-2c_1}{3} \quad \text{and} \quad q_2^* = \frac{A+c_1-2c_2}{3}$$

which is the solution to the two equations above.

The best response functions in the Cournot's duopoly game



Nash equilibrium comparative statics
(a decrease in the cost of firm 2)



A question: what happens when consumers are willing to pay more (*A* increases)?

In summary, this simple Cournot's duopoly game has a unique Nash equilibrium.

Two economically important properties of the Nash equilibrium are (to economic regulatory agencies):

[1] The relation between the firms' equilibrium profits and the profit they could make if they act collusively.

[2] The relation between the equilibrium profits and the number of firms.

[1] Collusive outcomes: in the Cournot's duopoly game, there is a pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium.

[2] Competition: The price at the Nash equilibrium if the two firms have the *same* unit cost $c_1 = c_2 = c$ is given by

$$\begin{aligned} P^* &= A - q_1^* - q_2^* \\ &= \frac{1}{3}(A + 2c) \end{aligned}$$

which is above the unit cost c. But as the number of firm increases, the equilibrium price deceases, approaching c (zero profits!).

Stackelberg's duopoly model (1934)

How do the conclusions of the Cournot's duopoly game change when the firms move sequentially? Is a firm better off moving before or after the other firm?

Suppose that $c_1 = c_2 = c$ and that firm 1 moves at the start of the game. We may use backward induction to find the subgame perfect equilibrium.

- First, for *any* output q_1 of firm 1, we find the output q_2 of firm 2 that maximizes its profit. Next, we find the output q_1 of firm 1 that maximizes its profit, *given the strategy* of firm 2.

Firm 2

Since firm 2 moves after firm 1, a strategy of firm 2 is a *function* that associate an output q_2 for firm 2 for each possible output q_1 of firm 1.

We found that under the assumptions of the Cournot's duopoly game Firm 2 has a unique best response to each output q_1 of firm 1, given by

$$q_2 = \frac{1}{2}(A - q_1 - c)$$

(Recall that $c_1 = c_2 = c$).

Firm 1

Firm 1's strategy is the output q_1 the maximizes

$$\pi_1 = (A - q_1 - q_2 - c)q_1 \quad \text{subject to} \quad q_2 = \tfrac{1}{2}(A - q_1 - c)$$

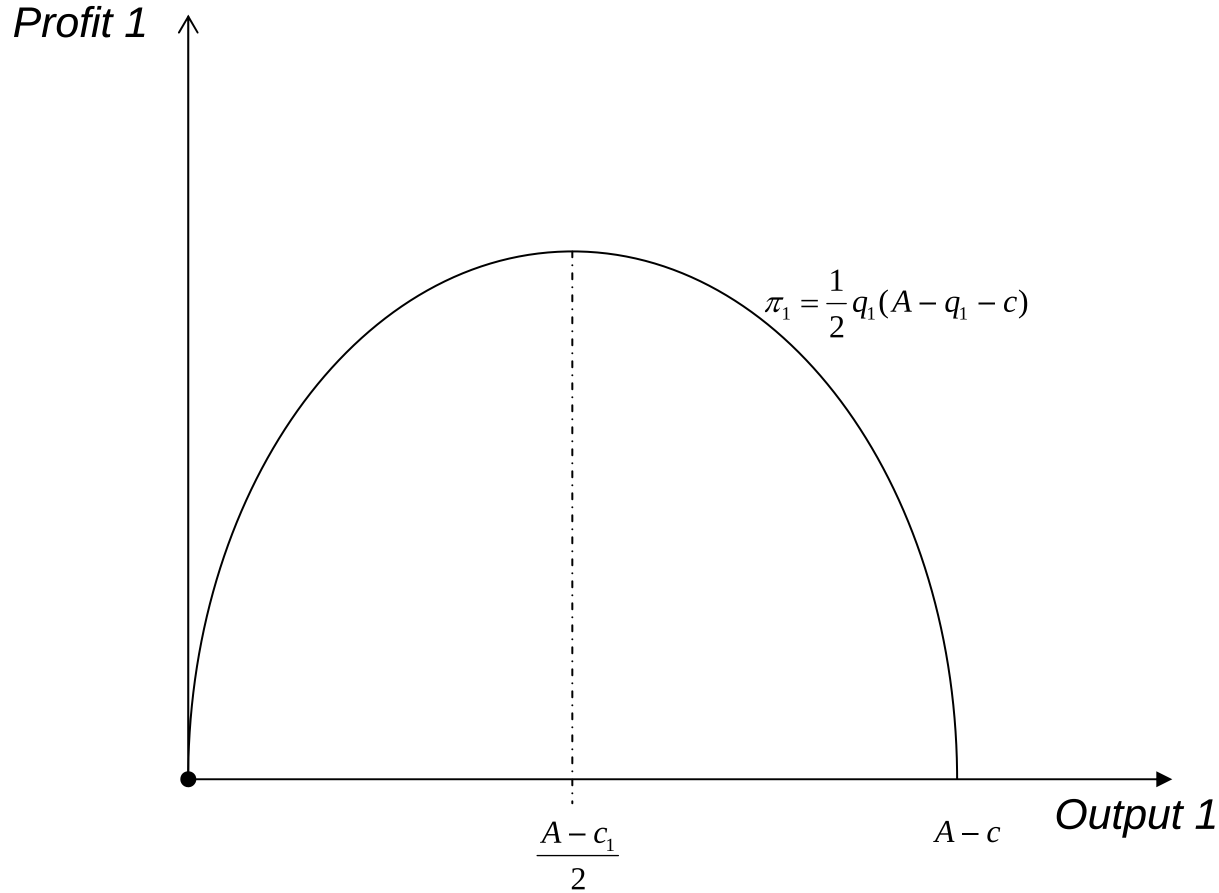
Thus, firm 1 maximizes

$$\pi_1 = (A - q_1 - (\frac{1}{2}(A - q_1 - c)) - c)q_1 = \frac{1}{2}q_1(A - q_1 - c).$$

This function is quadratic in q_1 that is zero when $q_1 = 0$ and when $q_1 = A - c$. Thus its maximizer is

$$q_1^* = \frac{1}{2}(A - c).$$

Firm 1’s (first-mover) profit in Stackelberg's duopoly game



We conclude that Stackelberg's duopoly game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output

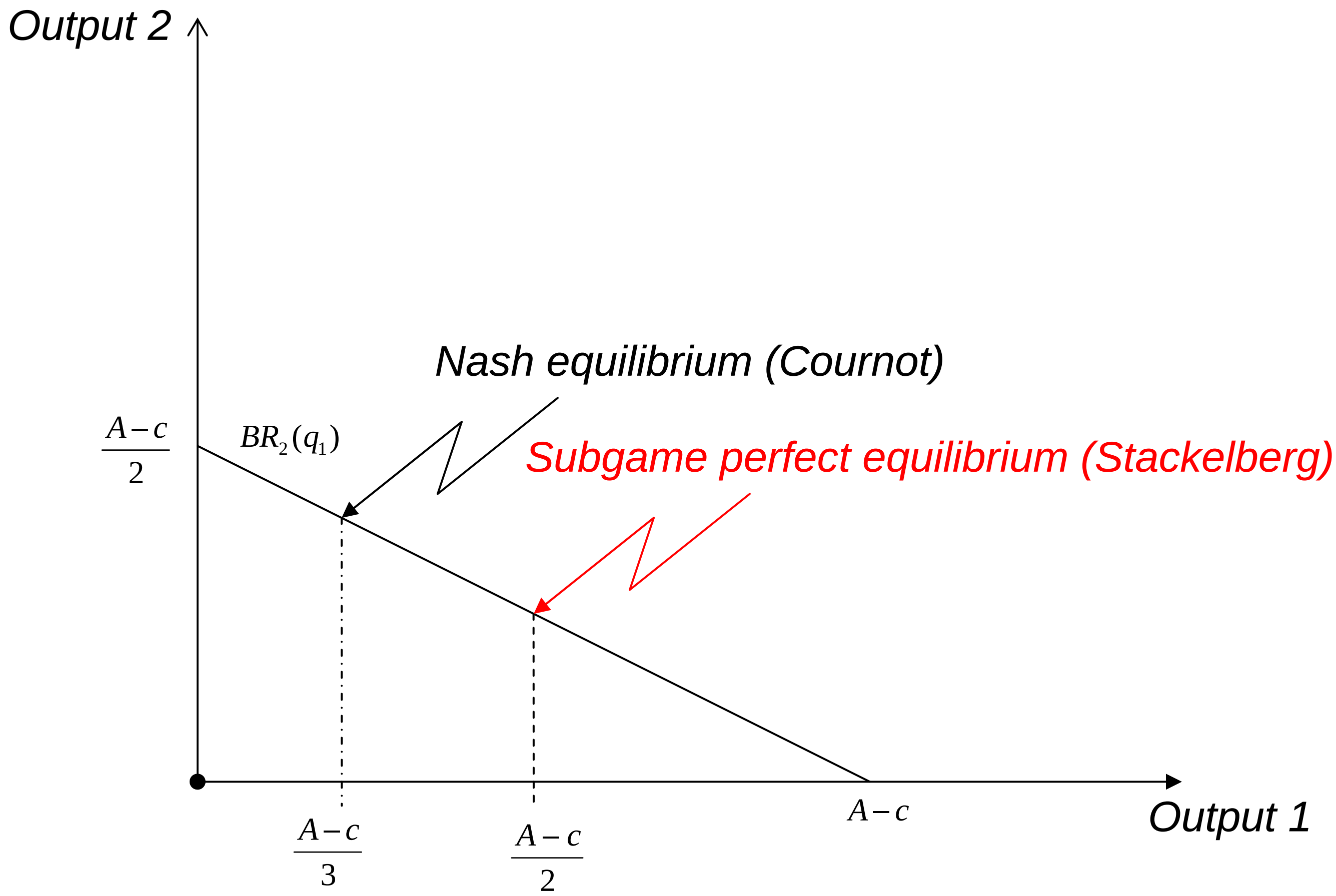
$$q_1^* = \frac{1}{2}(A - c)$$

and firm 2's output is

$$\begin{aligned} q_2^* &= \frac{1}{2}(A - q_1^* - c) \\ &= \frac{1}{2}(A - \frac{1}{2}(A - c) - c) \\ &= \frac{1}{4}(A - c). \end{aligned}$$

By contrast, in the unique Nash equilibrium of the Cournot's duopoly game under the same assumptions ($c_1 = c_2 = c$), each firm produces $\frac{1}{3}(A - c)$.

The subgame perfect equilibrium of Stackelberg's duopoly game



The tragedy of the commons

William Forster Lloyd (1833)

– Cattle herders sharing a common parcel of land (the commons) on which they are each entitled to let their cows graze. If a herder put more than his allotted number of cattle on the common, overgrazing could result.

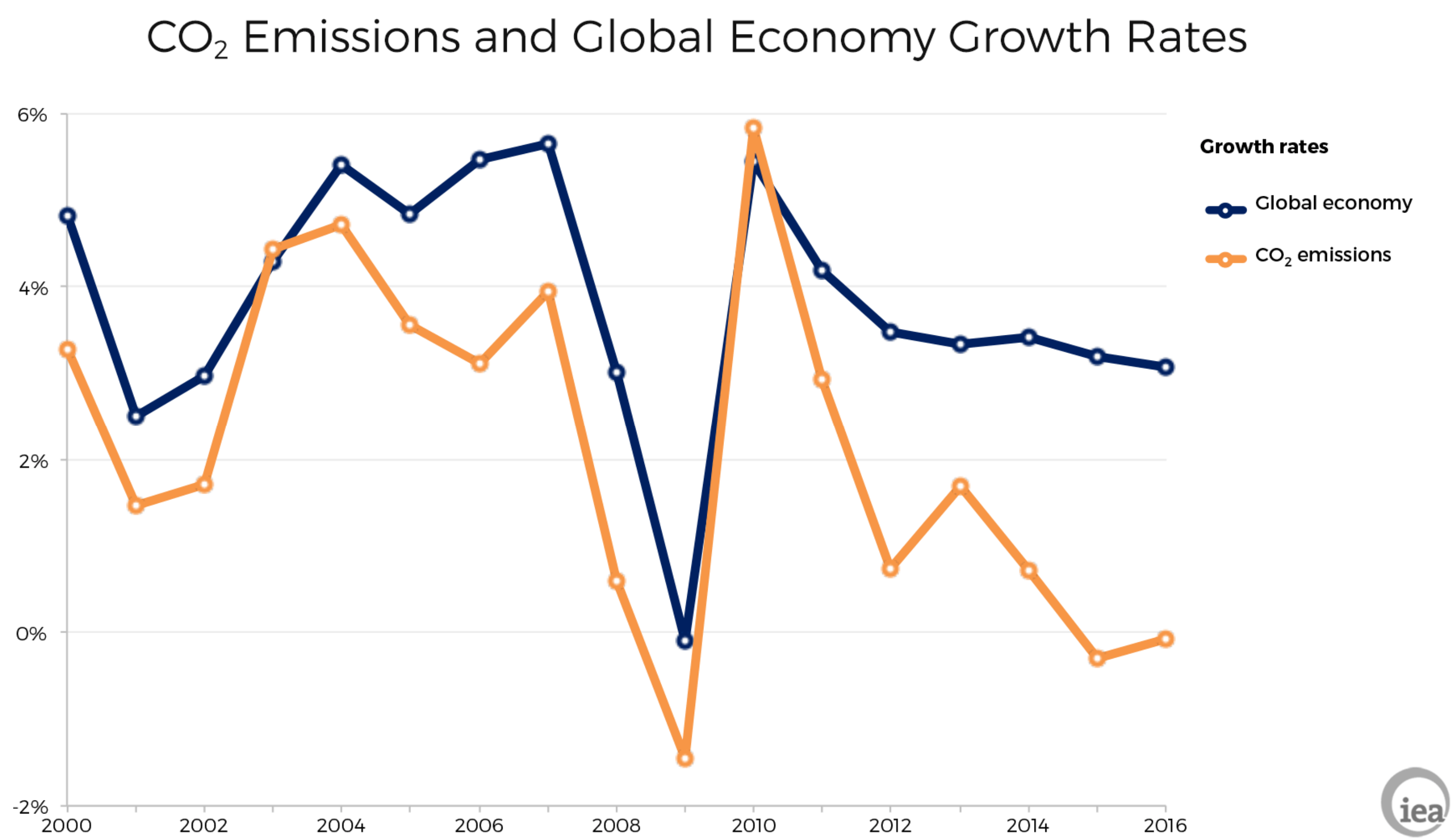
– Each additional animal has a positive effect for its herder, but the cost of the extra animal is shared by all other herders, causing a so-called “free-rider” problem. Today’s commons include fish stocks, rivers, oceans, and the atmosphere.

Science AAAS

The Tragedy of the Commons

The population problem has no technical solution;
it requires a fundamental extension in morality.

Garrett Hardin



CO2 Emissions and Global Economy Growth Rates
6%
4%
2%
0%
-2%
2000
2002
2004
2006
2008
2010
2012
2014
2016
Growth rates
Global economy
CO2 emissions
iea

Garrett Hardin (1968)

- This social dilemma was populated by Hardin in his article "The Tragedy of the Commons," published in the journal *Science*. The essay derived its title from Lloyd (1833) on the over-grazing of common land.

- Hardin concluded that "...the commons, if justifiable at all, is justifiable only under conditions of low-population density. As the human population has increased, the commons has had to be abandoned in one aspect after another."

– “The only way we can preserve and nurture other and more precious freedoms is by relinquishing the freedom to breed, and that very soon. “Freedom is the recognition of necessity” – and it is the role of education to reveal to all the necessity of abandoning the freedom to breed. Only so, can we put an end to this aspect of the tragedy of the commons.”

“Freedom to breed will bring ruin to all.”

Let's put some game theoretic analysis (rigorous sense) behind this story:

- There are n players, each choosing how much to produce in a production activity that 'consumes' some of the clean air that surrounds our planet.

- There is K amount of clean air, and any consumption of clean air comes out of this common resource. Each player $i = 1, ..., n$ chooses his consumption of clean air for production $k_i \geq 0$ and the amount of clean air left is therefore

$$K - \sum_{i=1}^{n} k_i.$$

- The benefit of consuming an amount $k_i \geq 0$ of clean air gives player i a benefit equal to $\ln(k_i)$. Each player also enjoys consuming the reminder of the clean air, giving each a benefit

$$\ln\left(K - \sum_{i=1}^{n} k_i\right).$$

- Hence, the value for each player i from the action profile (outcome) $k = (k_1, ..., k_n)$ is give by

$$v_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right).$$

– To get player i's best-response function, we write down the first-order condition of his payoff function:

$$\frac{\partial v_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0$$

and thus

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}.$$

The two-player Tragedy of the Commons

– To find the Nash equilibrium, there are n equations with n unknown that need to be solved. We first solve the equilibrium for two players. Letting $k_i(k_j)$ be the best response of player i, we have two best-response functions:

$$k_1(k_2) = \frac{K - k_2}{2} \quad \text{and} \quad k_2(k_1) = \frac{K - k_1}{2}.$$

– If we solve the two best-response functions simultaneously, we find the unique (pure-strategy) Nash equilibrium

$$k_1^{NE} = k_2^{NE} = \frac{K}{3}.$$

Can this two-player society do better? More specifically, is consuming $\frac{K}{3}$ clean air for each player too much (or too little)?

– The 'right way' to answer this question is using the Pareto principle (Vilfredo Pareto, 1848-1923) – can we find another action profile $k = (k_1, k_2)$ that will make both players better off than in the Nash equilibrium?

– To this end, the function we seek to maximize is the social welfare function w given by

$$w(v_1, v_2) = v_1 + v_2 = \sum_{i=1}^{2} \ln(k_i) + 2 \ln \left(K - \sum_{i=1}^{2} k_i \right).$$

– The first-order conditions for this problem are

$$\frac{\partial w(k_1, k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

and

$$\frac{\partial w(k_1, k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0.$$

– Solving these two equations simultaneously result the unique Pareto optimal outcome

$$k_1^{PO} = k_2^{PO} = \frac{K}{4}.$$

The n-player Tragedy of the Commons

- In the n-player Tragedy of the Commons, the best response of each player $i = 1, ..., n$, $k_i(k_{-i})$, is given by

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}.$$

- We consider a symmetric Nash equilibrium where each player i chooses the same level of consumption of clean air k^* (it is subtle to show that there cannot be asymmetric Nash equilibria).

– Because the best response must hold for each player i and they all choose the same level k^{SNE} then in the symmetric Nash equilibrium all best-response functions reduce to

$$k^{SNE} = \frac{K - \sum_{j \neq i} k^{SNE}}{2} = \frac{K - (n-1)k^{SNE}}{2}$$

or

$$k^{SNE} = \frac{K}{n+1}.$$

Hence, the sum of clean air consumed by the firms is $\frac{n}{n+1}K$, which increases with n as Hardin conjectured.

What is the socially optimal outcome with n players? And how does society size affect this outcome?

– With n players, the social welfare function w given by

$$\begin{aligned} w(v_1, ..., v_n) &= \textstyle\sum_{i=1}^n v_i \\ &= \textstyle\sum_{i=1}^n \ln(k_i) + n \ln \left(K - \sum_{i=1}^n k_i \right). \end{aligned}$$

And the n first-order conditions for the problem of maximizing this function are

$$\frac{\partial w(k_1, ..., k_n)}{\partial k_i} = \frac{1}{k_i} - \frac{n}{K - \sum_{j=1}^n k_j} = 0$$

for $i = 1, ..., n$.

– Just as for the analysis of the Nash equilibrium with n players, the solution here is also symmetric. Therefore, the Pareto optimal consumption of each player k^{PO} can be found using the following equation:

$$\frac{1}{k^{PO}} - \frac{n}{K - nk^{PO}} = 0$$

or

$$k^{PO} = \frac{K}{2n}$$

and thus the Pareto optimal consumption of air is equal $\frac{K}{2}$, for any society size n. for $i = 1, ..., n$.

Finally, we show there is no asymmetric equilibrium.

- To this end, assume there are two players, i and j, choosing two different $k_i \neq k_j$ in equilibrium.

- Because we assume a Nash equilibrium the best-response functions of i and j must hold simultaneously, that is

$$k_i = \frac{K - \bar{k} - k_j}{2} \quad \text{and} \quad k_j = \frac{K - \bar{k} - k_i}{2}$$

  where $\bar{k}$ be the sum of equilibrium choices of all other players except i and j.

– However, if we solve the best-response functions of players i and j simultaneously, we find that

$$k_i = k_j = \frac{K - \bar{k}}{3}$$

contracting the assumption we started with that $k_i \neq k_j$.