Econ. 240A, Spring 2000 D. McFadden

PROBLEM SET 1: Review of Elementary Probability theory

(Solve 1-6 for section discussion this week; turn in 7-12 on Monday, Jan. 31)

1. Prove that a σ-field of events contains *countable* intersections of its members.

2. It is known that 0.2 percent of the population is HIV-positive. It is known that a screening test for HIV has a 10 percent chance of incorrectly showing positive when the subject is negative, and a 2 percent chance of incorrectly showing negative when the subject is positive. What proportion of the population that tests positive has HIV?

3. John and Kate are 80 years old. The probability that John will die in the next year is 0.08, and the probability that Kate will die in the next year is 0.05. The probability that John will die, given that Kate dies, is 0.2. What is the probability that both will die? That at least one will die? That Kate will die, given that John dies?

4. The probability that a driver will have an accident next year if she has a Ph.D. is 0.2. The probability she will have an accident if she does not have a Ph.D. is 0.25. The probability the driver has a Ph.D. <u>and</u> an accident is 0.01. What is the probability the driver has a Ph.D.? What is the probability of a Ph.D. given an accident?

5. A quiz show offers you the opportunity to become a millionaire if you answer nine questions correctly. Questions can be easy (E), moderate (M), or hard (H). The respective probabilities that you will answer an E, M, or H question correctly are 2/3, 1/2, and 1/3. If you get an E question, your next question will be E, M, or H with probabilities 1/4, 1/2, and 1/4 respectively. If you get a M question, your next question will be E, M, or H with probabilities 1/3, 1/3, and 1/3 respectively. If you get a H question, your next question will be E, M, or H with probabilities 1/2, 0, and 1/2 respectively. The first question is always an E question. What is the probability that you will become a millionaire? [Hint: Show that the probability of winning if you reach question 9 is independent of whether this question is E, M, or H. Then use backward recursion.]

6. Show that if $\mathbf{A} \subseteq \mathbf{B}$ and $P(\mathbf{A}) > 0$, then $P(\mathbf{C}|\mathbf{A})$ can be either larger or smaller than $P(\mathbf{C}|\mathbf{B})$.

7. An airplane has 40 seats. The probability that a ticketed passenger shows up for the flight is 0.95, and the events that any two different passengers show up is statistically independent. If the airline sells 45 seats, what is the probability that the plane will be overbooked? How many seats can the airline sell, and keep the probability of overbooking to 5 percent or less?

8. Prove that the expectation $\mathbf{E}(X - c)^2$ is minimized when $c = \mathbf{E}X$.

9. Prove that the expectation $\mathbf{E}|X - c|$ is minimized when $c = \text{median}(X)$.

10. A sealed bid auction has a tract of land for sale to the highest of n bidders. You are bidder 1. Your experience is that the bids of each other bidder is distributed with a Power distribution $F(X) = X^\alpha$ for $0 \le X \le 1$. Your profit if you are successful in buying tract at price y is $1 - y$. What should you bid to maximize your expected profit? What is your probability of winning the auction?

11. A random variable X has a normal distribution if its density is $f(x) = (2\pi\sigma^2)^{-1/2}\cdot\exp(-(x-\mu)^2/2\sigma^2)$, where μ and σ^2 are parameters. Prove that X has mean μ and variance σ^2. Prove that $\mathbf{E}(X-\mu)^3 = 0$ and $\mathbf{E}(X-\mu)^4 = 3\sigma^4$. [Hint: First show that $\int x\cdot\exp(-x^2/2)dx = -\exp(-x^2/2)$ and for $k > 1$, the integration by parts formula $\int x^k\cdot\exp(-x^2/2)dx = -x^{k-1}\cdot\exp(-x^2/2) + (k-1)\int x^{k-2}\cdot\exp(-x^2/2)dx$.]

12. Suppose the stock market has two regimes, Up and Down. In an Up regime, the probability that the market index will rise on any given day is P. In a Down regime, the probability that the market index will rise on any given day is Q, with $Q < P$. Within a regime, the probability that the market rises on a given day is independent of its history. The probability of being in a Up regime is 1/2, so that if you do not know which regime you are in, then all you can say is that the probability that the market will rise on any given day is $R = (P+Q)/2$. Assume that regimes persist far longer than runs of rises, so that when analyzing runs the regime can be treated as persisting indefinitely. Show that when you are in the Up regime, the probability of a run of k or more successive days in which the market rises is P^{k-1}, and that the probability of a run of exactly k days in which the market rises is $P^{k-1}(1-P)$. A similar formula with Q instead of P holds when you are in a Down regime. Show that expected length in an Up regime of a run of rises is $1/(1-P)$. Show that $\frac{1}{2}/(1-P) + \frac{1}{2}/(1-Q) \ge 1/(1-R)$.