Econ. 240A, Spring 2000 D. McFadden

PROBLEM SET 3 (Stochastic Limits)

(Due Monday, Feb. 14, with discussion in sections on Feb. 15-17)

1. The sequence of random variables X_n satisfy $X_n(s) = s^n$, where s is a state of Nature in the sample space S = [0,1] with uniform probability on S. Does X_n have a stochastic limit, and if so in what sense (weak, strong, quadratic mean, distribution)? What about $Y_n = n\cdot X_n$ or $Z_n = \log(X_n)$?

2. A sequence of random variables Z_n are multivariate normal with mean zero, variance $\sigma^2 n$, and covariances $\mathbf{E}\, Z_n Z_{n+m} = \sigma^2 n$ for m > n. (For an infinite sequence, this means that every finite subsequence is multivariate normal.) Let $S_n = \sum_{k=1}^{n} Z_k$. Does S_n/n converge in probability? Is there a scale factor $\alpha(n)$ such that $S_n/\alpha(n)$ converges in probability? Is there a scale factor $\beta(n)$ such that $S_n/\beta(n)$ is asymptotically normal?

3. Ignoring adjustments for family composition and location, an American family is said to be below the poverty line if its annual income is less than \$14,800 per year. Let Y_i be the income level of family i, drawn randomly and independently from the American population, and let Q_i be one if Y_i is less than \$14,800, zero otherwise. Family income can obviously never be larger than GDP, so that it is bounded above by a (very big) constant G, and cannot be negative. Let μ denote the population mean annual income and π denote the population proportion below the poverty line. Let m_n and p_n denote the corresponding sample means in a simple random sample of size n. Prove that sample mean annual income m_n converges in probability to population mean annual income; i.e., show the requirements for a WLLN are met. Prove that $n^{1/2}(m_n - \mu)$ converges in distribution to a normal; i.e., show the requirements for a CLT are met. Similarly, prove that p_n converges in probability to π and $n^{1/2}(p_n - \pi)$ converges in distribution to a normal with mean 0 and variance $\pi(1-\pi)$.

4. Empirical illustration of stochastic limits: On the computer, construct a simple random sample of observations X_k by drawing independent uniform random variables U_k and V_k from (0,1) and defining $X_k = 1$ if $U_k > 1/2$ and $X_k = \log(V_k)$ if $U_k \le 1/2$. Let m_n be the sample mean of the X_k from a sample of size n for n = 10, 100, 1000. (a) Does m_n appear to converge in probability? To what limit? (b) Draw 100 samples of size 10 by the procedure described above, and keep the sample means from each of the 100 samples. Calculate what are called "studentized residuals" by subtracting the mean of the 100 sample means, and dividing these differences by their sample standard deviation (i.e., the square root of the average of the squared deviations). Sort these studentized residuals from low to high and plot them against quantiles of the standard normal, $Q_k = \Phi^{-1}((k-0.5)/n)$. This is called a normal probability plot, and deviations from the 45-degree line reflect differences in the exact distribution of the sample means from normal, plus random noise. Are there systematic deviations that suggest the normal approximation is not very good? (c) Repeat part (b) with 100 samples of size 100. Has the normal approximation become more accurate?