Econ 240A: Problem Set 2
Solutions to Selected Problems from Chapter 3

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31 January 2001

1.

We have $S = \{HH, HT, TH, TT\}$. The class of potentially observable events satisfying the properties of a σ-field consists of the set of all subsets of S (i.e., the *power set* of S):

$$\begin{aligned}\mathcal{P}(S) &= \{\emptyset, S, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \\ &\quad \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \\ &\quad \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}\}.\end{aligned}$$

(Note that $\emptyset \neq \{\emptyset\}$.)

4.

Here we have $S = \Re^2$, which is interpreted as the set of all ordered pairs of possible values of the index on two successive days. We assume that the index is normalized to one, so the event E that the change in the index on two successive days is the same is given as a diagonal line in $\Re^2$: $E = \{(y_1, y_2) : y_2 = 2y_1 - 1\}$. Since the Cartesian product $\mathcal{B} \times \mathcal{B}$ of the Borel σ-field $\mathcal{B}$ on $\Re$ with itself consists strictly of unions and intersections of open rectangles and their complements, it follows that the only lines of infinite length contained in $\mathcal{B} \times \mathcal{B}$ run parallel to one of the axes, a criterion that clearly excludes E.

However E can be represented as the intersection of a monotonically decreasing sequence of sets in $\mathcal{B} \times \mathcal{B}$:

$$E = \bigcap_{n \in \Re_+} \bigcup_{a \in \mathcal{Z}} \{(y_1, y_2) : \frac{a}{n} \leq y_1 \leq \frac{a+1}{n}, \ \frac{2a}{n} \leq y_2 + 1 \leq \frac{2a+2}{n}\},$$

where $\Re_+ = \{x : x \in \Re, x \geq 0\}$ and $\mathcal{Z}$ is the set of all integers.

7.

We have consider convergence of the sequence of functions in $L_2(\mu)$ (i.e., the set of square-integrable functions with respect to measure μ) given by $f_n(x) = x^{\frac{1}{n}}$ to 1 as $n \to \infty$ for $x \in (0, 1)$. Since we have $(x^{\frac{1}{n}} - 1)^2 \to 0$, strong convergence can be shown via the Lebesgue dominated convergence theorem (Thm. 3.1, p. 50). It is not difficult to find functions $g \in L_2(\mu)$ on $(0, 1)$ such that $(x^{\frac{1}{n}} - 1)^2 \leq g(x)$ for each n (e.g., $g(x) = -x$). It follows that

$$\lim_{n\to\infty} \int_0^1 (x^{\frac{1}{n}} - 1)^2 \, d\mu(x) = \int_0^1 0 \, d\mu(x) = 0.$$

By continuity of exponentiation we have

$$(\int_0^1 (x^{\frac{1}{n}} - 1)^2 \, d\mu(x))^{\frac{1}{2}} = \| f_n - 1 \|_2 \to 0,$$

which shows strong convergence (and by implication, weak and μ-measure convergence).
(Note that invoking the fundamental theorem of calculus is only legitimate for μ = Lebesgue measure and not for general μ. Of course when μ is taken as Lebesgue measure (i.e., the length of an interval in $\Re$) strong convergence can be shown in this question by direct evaluation of the integral.)