Econ 240A: Problem Set 3
Solutions to Selected Problems from Chapter 3

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16.

Note that c is a *median* of a random variable X iff $P(X \leq c) \geq \frac{1}{2}, P(X \geq c) \geq \frac{1}{2}$. Note the following two facts:

- The definition of a median is equivalent to the statement that c is a median of X iff $P(X < c) \leq \frac{1}{2}$ and $P(X > c) \leq \frac{1}{2}$.
- It can be shown that the set of all medians $med(X)$ of X is a closed interval (or generalized rectangle): that is, there exist medians c_0, c_1 such that for every median c, $c_0 \leq c \leq c_1$.[1]

We use the above two facts to show that $E|X - c|$ is minimized by requiring $c \in med(X)$:
Let $d > c_1$. Let c be a median of X. Therefore $c_0 \leq c \leq c_1$. We have

$$\begin{aligned} E|X-d| - E|X-c| &= \int_{x>d} (x-d)dP(x) - \int_{x<d} (x-d)dP(x) \\ &\quad - \int_{x>c} (x-c)dP(x) + \int_{x<c} (x-c)dP(x) \\ &= \int_{x<d} (d-x)dP(x) + \int_{x\leq c} (x-d+d-c)dP(x) \\ &\quad + \int_{x>d} (x-d)dP(x) - \int_{x>c} (x-d+d-c)dP(x) \\ &= \int_{c<x<d} (d-x)dP(x) + (d-c)P(X \leq c) \end{aligned}$$

[1] It is clear that $med(X)$ contains a maximal and a minimal element (it's easy to construct a proof by contradiction). It remains to show that $med(X)$ doesn't contain any "holes" or gaps. Let $c_0 \leq c_1$ be medians. Then $P(X \leq c_0) \geq \frac{1}{2}, P(X \geq c_1) \geq \frac{1}{2}$. It follows that for any c such that $c_0 \leq c \leq c_1$, $P(X \leq c) \geq \frac{1}{2}, P(X \geq c) \geq \frac{1}{2}$. Therefore c is also a median, which shows that $med(X)$ is a closed interval (if X is scalar-valued) or a closed rectangle if X is multidimensional.

$$
\begin{aligned}
& -\int_{x\geq d}(d-x)dP(x)+\int_{x>c}(d-x)dP(x) \\
& -(d-c)P(X>c) \\
= & \int_{c<x<d}(d-x)dP(x)+(d-c)[P(X\leq c)-P(X>c)] \\
& +\int_{c<x<d}(d-x)dP(x) \\
= & (d-c)[P(X\leq c)-P(X>c)]+2\int_{c<x<d}(d-x)dP(x) \\
\geq & \ 0.
\end{aligned}
$$

Similarly, we can show

$$E|X-f|-E|X-c|=2\int_{f<x<c}(x-f)dP(x)+(c-f)[P(X\geq c)-P(X<c)]\geq 0$$

for any $f<c_0$.

22.

We have $E(X+aZ)^2=EX^2+2aE[XZ]+a^2EZ^2\geq 0\forall a\in\Re$. If $E[Z^2]=0$ then $E[X^2]=E[XZ]=0$ and the Cauchy-Schwarz inequality holds with equality. Now assume $E[Z^2]>0$. Note that if $E(X+aZ)^2>0$, the roots of $E(X+aZ)^2$ are nonreal[2] and are given by

$$a=\frac{-2E[XZ]\pm\sqrt{4(E[XZ])^2-4E[Z^2]E[X^2]}}{2E[Z^2]}.$$

Complexity requires $(E[XZ])^2<E[Z^2]E[X^2]$.

Now $\forall\epsilon>0$ we have by Markov's inequality that

$$P[|X+aZ|>\epsilon]\leq\frac{E(X+aZ)^2}{\epsilon^2}.$$

If $E(X+aZ)^2=0$ it follows that for any scalar a, $P(X=-aZ)=1$, so $(E[XZ])^2=E[Z^2]E[X^2]$.

25.

We assume that $Cov(X,Z)=2\rho$ for $-1\leq\rho\leq 1$. Let

$$\Sigma\equiv\begin{bmatrix} 1 & 2\rho \\ 2\rho & 4 \end{bmatrix},$$

[2]Recall that if $a>0$ and $b^2-4ac\geq 0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ will solve $ax^2+bx+c=0$; otherwise $ax^2+bx+c>0\Leftrightarrow(\sqrt{a}x+\frac{b}{2\sqrt{a}})^2+\frac{4ac-b^2}{4a}>0\Rightarrow 4ac-b^2>0$.

which implies that

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{1-\rho^2} & -\frac{\rho}{2-2\rho^2} \\ -\frac{\rho}{2-2\rho^2} & \frac{1}{4-4\rho^2} \end{bmatrix}.$$

Also, let $Y \equiv \begin{bmatrix} X \\ Z \end{bmatrix}$, $\mu \equiv \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Note that

$$\begin{aligned} (y-\mu)'\Sigma^{-1}(y-\mu) &= \frac{(x-1)^2}{1-\rho^2} + 2(x-1)(z-2)(-\frac{2\rho}{4-4\rho^2}) + \frac{(z-2)^2}{4-4\rho^2} \\ &= \frac{1}{1-\rho^2}((x-1) - \frac{\rho}{2}(z-2))^2 - \frac{\rho^2(z-2)^2}{4(1-\rho^2)} + \frac{(z-2)^2}{4-4\rho^2} \\ &= \frac{1}{1-\rho^2}(x-1-\frac{\rho}{2}(z-2))^2 + \frac{(z-2)^2}{4}. \end{aligned}$$

Therefore $p(x,z) = p(x|z)p(z) \propto \frac{1}{\sqrt{1-\rho^2}} \exp[-\frac{1}{2}\frac{(x-1-\frac{\rho}{2}(z-2))^2}{1-\rho^2}] \cdot \frac{1}{2}\exp[-\frac{1}{2}\frac{(z-2)^2}{4}]$. It follows that $X|Z = z \sim N(1+\frac{\rho}{2}(z-2), 1-\rho^2)$.

Bayes' theorem states that $p(Z|X) = \frac{P(X|Z)P(Z)}{\int p(X|z)p(z)dz}$. We have

$$\begin{aligned} (y-\mu)'\Sigma^{-1}(y-\mu) &= \frac{(x-1)^2}{1-\rho^2} + 2(x-1)(z-2)(-\frac{2\rho}{4-4\rho^2}) + \frac{(z-2)^2}{4-4\rho^2} \\ &= \frac{(x-1)^2}{1-\rho^2}(1-\rho^2) + \frac{1}{4-4\rho^2}(z-2-2\rho(x-1))^2 \\ &= (x-1)^2 + \frac{1}{4-4\rho^2}(z-2-2\rho(x-1))^2, \end{aligned}$$

so $p(x,z) \propto \exp[-\frac{1}{2}(x-1)^2] \cdot \frac{1}{2\sqrt{1-\rho^2}} \exp[-\frac{1}{2}\frac{(z-2-2\rho(x-1))^2}{4-4\rho^2}]$.

We have $\int p(x|z)p(z)dz = \int p(z|x)p(x)dz = p(x) \propto \exp[-\frac{1}{2}(x-1)^2]$. Therefore $p(z|x) \propto \frac{1}{2\sqrt{1-\rho^2}} \exp[-\frac{1}{2}\frac{z-2-2\rho(x-1))^2}{4-4\rho^2}]$, so $Z|X = x \sim N(2+2\rho(x-1), 4-4\rho^2)$.

34.

Note that the moment-generating function of $X - Y$ is given by $M_{X-Y}(t) = M_X(t)M_Y(-t)$. We show that $M_{X-Y}(t)$ is in fact the mgf of a member of the class of logistic distributions.

If X is distributed as an extreme value (or *Gumbel*) random variable with parameters a and b we have that

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{b}\exp[-\frac{x-a}{b} - e^{-\frac{x-a}{b}}]dx.$$

Let $y = \exp[-\frac{x-a}{b}]$. Changing the variable of integration gives us

$$M_X(t) = e^{at} \int_0^\infty y^{-bt} e^{-y} dy = e^{at}\Gamma(1-bt),$$

assuming that $t < \frac{1}{b}$.

Now if Z is a logistic random variable with parameters a and b we have

$$M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{b} \frac{\exp(\frac{a-z}{b})}{(1+\exp(\frac{a-z}{b}))^2} dz.$$

Let $y = \frac{1}{1+\exp[\frac{a-z}{b}]}$. Changing variables allows us to write

$$M_Z(t) = e^{at} \int_0^1 y^{bt}(1-y)^{-bt} dy = e^{at} B(1+bt, 1-bt) = e^{at}\Gamma(1+bt)\Gamma(1-bt).$$

Therefore $L = X - Y$ is distributed as a logistic random variable with density $\frac{1}{b} \cdot \frac{\exp[\frac{-l}{b}]}{(1+\exp[\frac{-l}{b}])^2}$. As expected, L has mean 0 and variance equal to twice the variance of X and Y, namely $\frac{(\pi b)^2}{6}$.

37.

Let $Z = \log Y$. We have that $Z \sim N(\mu, \sigma^2)$. One should check that Y does not in fact have a mgf. However one can in fact derive an expression for $E[Y^t]$:

Proceed by noting that the density of Y is given by transforming from the density of a normal (μ, σ^2) distribution:

$$p(y|\mu,\sigma^2) \propto \frac{1}{\sigma} \exp[-\frac{1}{2\sigma^2}(\log y - \mu)^2]\frac{1}{y}.$$

Therefore

$$\begin{aligned} E[Y^t] &= \int_0^\infty \frac{1}{\sigma} \exp[-\frac{1}{2\sigma^2}(\log y - \mu)^2] y^{t-1} dy \\ &= \int_{-\infty}^{\infty} e^{z(t-1)+z} \frac{1}{\sigma}\phi(\frac{z-\mu}{\sigma}) dz, \end{aligned}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$.[3] It follows that $E[Y^t] = M_Z(t) = \exp[\mu t + \frac{1}{2}\sigma^2 t^2]$ (i.e., the mgf for a $N(\mu, \sigma^2)$ random variable). Therefore

$$\begin{aligned} E[Y] &= e^{\mu + \frac{1}{2}\sigma^2} \\ Var[Y] &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1). \end{aligned}$$

[3] This is standard notation for the pdf of a $N(0,1)$ random variable.

40.

Suppose

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix}\right),$$

where $-1 < \rho < 1$. Proceeding as in Problem 25 above, one can show that

$$\begin{aligned} E[X|Y=y] &= \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), \\ Var[X|Y=y] &= \sigma_X^2(1-\rho^2), \\ E[Y|X=x] &= \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \\ Var[Y|X=x] &= \sigma_Y^2(1-\rho^2). \end{aligned}$$

So if $\mu_X = 1, \mu_Y = 3$ and $\sigma_X^2 = 4, \sigma_Y^2 = 9, \rho = \frac{5}{6}$, we have

$$X|Y=y \sim N(1 + \frac{5}{9}(y-3), \frac{11}{9})$$

and

$$Y|X=x \sim N(3 + \frac{25}{4}(x-1), \frac{11}{4}).$$

Therefore

(a)

$$\begin{aligned} E[2X-Y] &= E(E[2X-Y]|Y) \\ &= E(2E[X|Y]-Y) \\ &= E(2[1+\frac{5}{9}Y-3]-Y) \\ &= -\frac{11}{3}; \end{aligned}$$

(b)

$$\begin{aligned} Var[2X-Y] &= Var(E[2X-Y|Y]) + E(Var[2X-Y|Y]) \\ &= Var(2+\frac{10}{9}Y-6-Y) + E(4\cdot\frac{11}{9}-Y) \\ &= 2; \end{aligned}$$

(c)

$$\begin{aligned} E[2X-Y|X=5] &= 10 - E[Y|X=5] \\ &= 10 - 3 - \frac{25}{4}\cdot 4 \\ &= -18; \end{aligned}$$

(d)

$$\begin{aligned} Var[2X - Y|X = 5] &= Var[Y|X = 5] \\ &= \frac{11}{4}. \end{aligned}$$