Econ 240A: Problem Set 7
Solutions to Selected Problems from Chapter 7

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1.

We consider a random sample $X = (X_1, \ldots, X_n)'$, where $X_i \sim N(\mu, \sigma^2)$. The likelihood function of the sample is therefore given by

$$L(\mu, \sigma^2, X) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right].$$

Let $\theta = (\mu, \sigma^2)'$. A uniformly most powerful (UMP) test of $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ maximizes

$$P_{H_1}[\text{Reject} H_0]$$

subject to a pre-specified upper bound α on the probability of incorrect rejection[1]:

$$P_{H_0}[\text{Reject} H_0] \leq \alpha$$

Our general strategy for constructing tests of the parameters of a normal distribution given an iid sample consists of finding a statistic $T(X)$ that is sufficient for the parameter of interest in the problem at hand and that has a known distribution for some value of the parameter of interest in the null hypothesis. Suppose that we find ourselves in the uncommon (and fortunate) situation where we know the true value of one of the two parameters characterizing the normal distribution. Denote the unknown parameter of interest by θ. In this case the problem of test construction reduces to finding a statistic T that in addition to being sufficient and having a known distribution for some value of θ in H_0, causes the population to have a *monotone likelihood ratio* (MLR) property in T, i.e., for any fixed parameter values $\theta < \theta'$,

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = g[T(x)],$$

[1] Incorrect rejection is often referred to as *type I error* or "error of the first kind". The upper bound α is known as the *significance level* of the test. Note that since a test with level α of significance is also a level-α' test for any $\alpha' > \alpha$ we generally pick a smallest acceptable probability of type I error which we call the *size* of the test.

where g is a nondecreasing function in T. A UMP level-α test for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ will reject if $T(x)$ is larger than some critical value C_α (see Lehmann, 1986, Theorem 3.2). By the sufficiency of T for θ, the conditional probability of rejection for a given $T(X) = t$ will not depend on θ. Therefore the C_α may be derived by solving

$$P_{\theta_0}[\text{Reject}|T(X) = t] = \alpha.$$

Typically, both parameters of the normal population will be unknown and a UMP test will not exist since the distribution of any test statistics will depend on parameters other than the one of interest. In this case, however, there will exist level-α tests that are UMP over a smaller class of tests that satisfy the desirable property of being *unbiased*, i.e.,

$$\text{Power} = P_{H_1}[\text{Reject} H_0] \geq \alpha$$

and

$$P_{H_0}[\text{Reject} H_0] \leq \alpha.$$

Note that a test that is UMP over all tests is trivially an unbiased test, since its power cannot fall below that of a test that always rejects with probability α. Therefore unbiasedness eliminates the possibility of an embarrassing situation where failure to reject H_0 is more likely for certain values of θ in Θ_1 than for certain values of θ in Θ_0. The argument for the existence of UMP unbiased tests of a single parameter of a multiparameter exponential family is given in Section 4.4 of Lehmann (1986).

a. $H_0 : \mu = .0003$ vs. $H_1 : \mu \neq .0003$

We have

$$LR(\mu, \sigma^2, X) = \frac{\sup_{\mu \neq .0003} \exp\left[-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right]}{\exp\left[-\frac{n(\bar{x}-.0003)^2}{2 \cdot .0003}\right]}.$$

Note that LR is increasing in $\frac{n(\bar{x}-.0003)^2}{s^2}$ (where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$), which is distributed as an $F(1, n-1)$ random variable[2] under H_0. Therefore a UMP unbiased level-α test rejects if

$$\frac{n(\bar{x} - .0003)^2}{s^2} > F_{1-\alpha}(1, n-1).$$

This test is equivalent to the two-tailed t-test that rejects when

$$\left|\frac{\sqrt{n}(\bar{x} - .0003)}{s}\right| > t_{n-1, 1-\frac{\alpha}{2}}.$$

[2] See the handout on sufficiency available on the course homepage for an argument as to why $\bar{X}$ and S^2 are complete sufficient statistics for μ and σ^2, respectively and therefore statistically independent. The content of the remainder of this solution sheet depends crucially on the sufficiency and independence of $\bar{x}$ and s^2.

Suppose $H_1 : \mu = .0005$. Then the test statistic $\frac{n(\bar{x}-.0003)^2}{s^2}$ has a noncentral $F(1, n-1)$ distribution with noncentrality parameter $\frac{n(.0002)^2}{\sigma^2}$. σ^2 is unknown and will have to be estimated by the consistent estimate s^2. The approximate power of our test against $H_1 : \mu = .0005$ is therefore

$$P_{H_1}\left[F(1, n-1, \frac{n(.0002)^2}{s^2}\right] > F_{1-\alpha}(1, n-1)).$$

b. $H_0 : \mu \geq .0003$ vs. $H_1 : \mu < .0003$

We have

$$\begin{aligned} LR(\mu, \sigma^2, X) &= \frac{\sup_{\mu<.0003} \exp\left[-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right]}{\sup_{\mu\geq.0003} \exp\left[-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right]} \\ &= \begin{cases} \exp\left[\frac{1}{2\sigma^2}n(\bar{x}-\mu)^2\right] & \text{if} \quad \bar{x} < .0003 \\ \exp\left[-\frac{1}{2\sigma^2}n(\bar{x}-\mu)^2\right] & \text{if} \quad \bar{x} \geq .0003 \end{cases}, \end{aligned}$$

which is a nonincreasing function in $\frac{\sqrt{n}(\bar{x}-.0003)}{s}$, which has a a t_{n-1} distribution when $\mu = .0003$. Therefore a UMP unbiased level-α test rejects if

$$\frac{\sqrt{n}(\bar{x}-.0003)}{s} < t_{n-1,\alpha}.$$

Suppose $H_1 : \mu = .0001$. Therefore under H_1, $\frac{\sqrt{n}(\bar{x}-.0003)}{s}$ is distributed as a noncentral t_{n-1} distribution with noncentrality $-\frac{\sqrt{n}(.0002)}{\sigma}$. Its power may be derived accordingly.

c. $H_0 : \sigma^2 = .0001$ vs. $H_1 : \sigma^2 \neq .0001$

We have

$$LR(\mu, \sigma^2, X) = \frac{\sup_{\sigma^2\neq.0001}(\sigma^2)^{-\frac{n}{2}} \exp[-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2]}{(.0001)^{-\frac{n}{2}} \exp[-\frac{1}{2\cdot.0001}\sum_{i=1}^n (x_i-\mu)^2]}.$$

Since $\sum(x_i-\mu)^2 = (n-1)s^2 + n(\bar{x}-\mu)^2$, a UMP unbiased level-$\alpha$ test will reject if $\frac{(n-1)s^2}{.0001}$ is either large—making the denominator of LR small—or small, by making the numerator large. The test statistic is distributed χ^2_{n-1} under the null, so we would reject if

$$\frac{(n-1)s^2}{.0001} > \chi^2_{n-1,1-\frac{\alpha}{2}},$$

or

$$\frac{(n-1)s^2}{.0001} < \chi^2_{n-1,\frac{\alpha}{2}}.$$

Now suppose $H_1 : \sigma^2 = .000095$. Let $T = \frac{(n-1)s^2}{.0001}$. Then under H_1, $\frac{(n-1)s^2}{.000095} = \frac{20T}{19} \sim \chi^2_{n-1}$. The power of the test against H_1 is given by

$$\begin{aligned} P_{H_1}\left(T > \chi^2_{n-1,1-\alpha}\right) &= P_{H_1}\left(\frac{20T}{19} > \frac{20\chi^2_{n-1,1-\alpha}}{19}\right) \\ &= P\left(\chi^2_{n-1} > \frac{20}{19}\chi^2_{n-1,1-\alpha}\right). \end{aligned}$$

d. Testing for equality of two normal means assuming a common variance

Let index $i = 1$ denote the subsample of n_1 observations occurring before 1 January 1985. Let $i = 2$ denote the subsample containing n_2 observations occurring on or after 1 January 1985. Note the following:

(a) $\bar{x}_1 - \bar{x}_2 \sim N\left\{\mu_1 - \mu_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\sigma^2\right\}$,

(b) $\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}$,

and $s_1^2, s_2^2, \bar{x}_1, \bar{x}_2$ are all independent. Let x_{ji} denote the jth observation in subsample i.

Consider the hypotheses $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$. Note that

$$-\frac{1}{2\sigma^2}\sum_{j,i}(x_{ji} - \mu_i)^2 = -\frac{1}{2\sigma^2}\left(\sum x_{j1}^2 + \sum x_{j2}^2\right) + \frac{n_1\mu_1\bar{x}_1}{\sigma^2} + \frac{n_2\mu_2\bar{x}_2}{\sigma^2}.$$

But

$$n_1\mu_1\bar{x}_1 + n_2\mu_2\bar{x}_2 = \frac{(\bar{x}_2 - \bar{x}_1)(\mu_2 - \mu_1)}{\frac{1}{n_1} + \frac{1}{n_2}} + \frac{(n_1\bar{x}_1 + n_2\bar{x}_2)(n_1\mu_1 + n_2\mu_2)}{n_1 + n_2}.$$

It follows that $\frac{\bar{x}_2-\bar{x}_1}{\frac{1}{n_1}+\frac{1}{n_2}}$ and $(n_1-1)s_1^2+(n_2-1)s_2^2$ are complete sufficient statistics for the parameters $\frac{\mu_2-\mu_1}{\sigma^2}$ and $\frac{1}{\sigma^2}$, respectively. A glance at the numerator of the likelihood ratio statistic for H_0 and H_1 reveals that a UMP unbiased level-α test rejects when

$$T(x) = \left|\frac{\frac{\bar{x}_2-\bar{x}_1}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}{\sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}}\right|$$

is large. Moreover, the test statistic is distributed as a $t_{n_1+n_2-2}$ random variable, so rejection occurs whenever $T(x) < t_{n_1+n_2-2,\frac{\alpha}{2}}$ or $T(x) > t_{n_1+n_2-2,\frac{\alpha}{2}}$.

e. $H_0 : \sigma_1^2 \leq \sigma_2^2$ **vs.** $H_1 : \sigma_1^2 > \sigma_2^2$

Using the notation adopted in part d above, note that the likelihood function of all the observations is given by

$$\begin{aligned} f(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2; x_1, x_2) &= (2\pi)^{-\frac{n_1+n_2}{2}} (\sigma_1)^{-n_1} (\sigma_2)^{-n_2} \exp\left\{ -\frac{1}{2\sigma_1^2} \sum x_{j1}^2 \right. \\ &\quad \left. -\frac{1}{2\sigma_2^2} \sum x_{j2}^2 + \frac{n_1\mu_1\bar{x}_1}{\sigma_1^2} + \frac{n_2\mu_2\bar{x}_2}{\sigma_2^2} \right\} \\ &\equiv K(n_1, n_2, \sigma_1^2, \sigma_2^2) \exp(J(\theta, x_1, x_2)) \end{aligned}$$

But

$$\begin{aligned} J(\theta, x_1, x_2) &\equiv -\frac{(n_1-1)s_1^2}{2\sigma_1^2} - \frac{(n_2-1)s_2^2}{2\sigma_2^2} - \frac{n_1(\bar{x}_1-\mu_1)^2}{2\sigma_1^2} - \frac{n_2(\bar{x}_2-\mu_2)^2}{2\sigma_2^2} \\ &= \left(-\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} \right)(n_1-1)s_1^2 - \frac{1}{2\sigma_2^2}\left[(n_2-1)s_2^2 - (n_1-1)s_1^2\right] \\ &\quad -\frac{n_1(\bar{x}_1-\mu_1)^2}{2\sigma_1^2} - \frac{n_2(\bar{x}_2-\mu_2)^2}{2\sigma_2^2} \\ &\equiv \theta_1(n_1-1)s_1^2 + \theta_2\left[(n_2-1)s_2^2 - (n_1-1)s_1^2\right] + \theta_3 n_1(\bar{x}_1-\mu_1)^2 \\ &\quad +\theta_4 n_2(\bar{x}_2-\mu_2)^2 \end{aligned}$$

Under H_0, $\theta_1 \equiv -\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} \leq 0$ while under H_1, $\theta_1 > 0$. Note that $(n_1-1)s_1^2$ is a complete sufficient statistic for θ_1 and is therefore independent of $(n_2 - 1)s_2^2 - (n_1-1)s_1^2$, $n_1(\bar{x}_1-\mu_1)^2$, and $n_2(\bar{x}_2-\mu_2)^2$. These facts imply that a UMP unbiased level-α test for H_0 against H_1 will reject when $\frac{s_1^2}{s_2^2}$ is large. Under H_0 the test statistic has an $F(n_1-1, n_2-1)$ distribution. Therefore the UMP unbiased test rejects when

$$\frac{s_1^2}{s_2^2} > F_{1-\alpha}(n_1-1, n_2-1).$$

4.

We have $X_1, \ldots, X_n$ iid $N(\mu, 25)$. A UMP level-α test for $H_0 : \mu \geq 2$ against $H_1 : \mu < 2$ rejects when

$$\frac{\sqrt{n}(\bar{x}-2)}{5} < z_\alpha.$$

Note that under $H_1 : \mu = 1$ the test statistic has a $N(-\frac{\sqrt{n}}{5}, 1)$ distribution. For $\alpha = .01$ the power of the UMP test against $H_1 : \mu = 1$ is given by

$$\text{Power} = P_{H_1}\left[\frac{\sqrt{n}(\bar{x}-2)}{5} < -2.3263\right]$$

$$= \quad P\left[N(0,1) < -2.3263 + \frac{\sqrt{n}}{5}\right].$$

So if we want the test to have power $\geq .99$, the sample size n must satisfy

$$-2.3263 + \frac{\sqrt{n}}{5} \geq 2.3263.$$

Therefore at least 542 observations will be needed.

References

[1] Lehmann, E. L. (1986): *Testing Statistical Hypotheses*, Second Edition. New York: Springer-Verlag.