Econ. 241B Dan McFadden, 2000

Exercise 2

1. Facts on Circular Functions: Consider the trigonometric functions $\cos(\omega)$ and $\sin(\omega)$, where ω is a real number giving the angle in radians. These functions are periodic, with $\cos(\omega+2\pi k) = \cos(\omega)$, $\cos(\pi k) = (-1)^k$, $\sin(\omega+2\pi) = \sin(\omega)$, $\sin(\pi k) = 0$, and $\cos(\omega) = \sin(\omega+\pi/2)$ for $k = \pm1,\pm2,\ldots.$ Define the complex valued function $\exp(\iota\omega s) = \cos(\omega s) + \iota\cdot\sin(\omega s)$, where $\iota = (-1)^{1/2}$. Then $\exp(\iota(\omega+2\pi)) = \exp(\iota\omega)$ and $\exp(\iota\pi k) = (-1)^k$. Here are some other useful relationships —

(1) $$\cos(\omega) = \frac{e^{\iota\omega} + e^{-\iota\omega}}{2} \quad \text{and} \quad \sin(\omega) = \frac{e^{\iota\omega} - e^{-\iota\omega}}{2\iota}$$

(2) $$\int_{-\pi}^{\pi} \cos(\omega k)d\omega = \int_{-\pi}^{\pi} \sin(\omega k)d\omega = \int_{-\pi}^{\pi} \exp(\iota\omega k)d\omega = 0 \quad \text{for } k = \pm1,\pm2,\ldots$$

(3) $$\int_{-\pi}^{\pi} \cos(0)d\omega = \int_{-\pi}^{\pi} \exp(0)d\omega = 2\pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(0)d\omega = 0$$

(4) $$\int_{-\pi}^{\pi} \cos(\omega k)^2 d\omega = \int_{-\pi}^{\pi} \sin(\omega k)^2 d\omega = \pi \quad \text{for } k = \pm1,\pm2,\ldots$$

(5) $$\int_{-\pi}^{\pi} \exp(\iota\omega k)\exp(-\iota\omega k)d\omega = 2\pi \quad \text{for } k = \pm1,\pm2,\ldots$$

(6) $$\int_{-\pi}^{\pi} \exp(\iota\omega k)\exp(-\iota\omega m)d\omega = 0 \quad \text{for } k,m = 0,\pm1,\pm2,\ldots \text{ and } k \neq m$$

(7) $$\int_{-\pi}^{\pi} \cos(\omega k)\cos(\omega m)d\omega = \int_{-\pi}^{\pi} \sin(\omega k)\sin(\omega m)d\omega = 0 \quad \text{for } k,m = 0,\pm1,\pm2,\ldots \text{and } k \neq m$$

(8) $$\int_{-\pi}^{\pi} \cos(\omega k)\sin(\omega m)d\omega = 0 \quad \text{for } k,m = 0,\pm1,\pm2,\ldots$$

These formulas are found in handbooks of mathematical functions, and are demonstrated in textbooks on orthogonal polynomials or on Fourier analysis.

Suppose T > 1is an integer, and define n = [T/2], the largest integer satisfying $n \le T/2$. Define the system of functions $\psi_k(t) = (T)^{-1/2}\exp(\iota 2\pi tk/T)$ for t = 1,...,T and k = -n,-n+1,...,0,...,n-1 for T even or k = -n+1,...,0,...,n-1 for T odd.

Every complex-valued function h(t) can be written as $h(t) = h_1(t) + \iota h_2(t)$ with h_1 and h_2 real-valued. The *complex conjugate* of h is $h^*(t) = h_1(t) - \iota h_2(t)$, and the product $h(t)h^*(t) = h_1(t)^2 + h_2(t)^2$. Apply the formula for geometric sums to show that

$$(9) \qquad \sum_{t=1}^{T} \psi_k(t)\psi_m^*(t) = \mathbf{1}(k=m).$$

Then the system of circular functions $\psi_k(t)$ form an *orthonormal basis* for $\mathbb{R}^T$. Suppose $y_1,..,y_T$ is a sequence of numbers, which may be deterministic or may be a realization from some stochastic process. This sequence can be represented in terms of the system of circular functions. Hereafter, assume T even and n = T/2. (Analogous formulas hold when T is odd, n = (T+1)/2, and the k = -n term in the sums below are dropped.) The relationship is

$$(10) \quad y_t = \sum_{k=-n}^{n-1} \psi_k(t)x_k$$

with

$$(11) \quad x_k = \sum_{t=1}^{T} \psi_k^*(t)y_t.$$

Verify that these formulas follow from the projection of $(y_1,...,y_T)$ on the space spanned by the vectors $(\psi_k(1),...,\psi_k(T))$ for k = -n,...,n-1; i.e., the regression of $(y_1,...,y_T)$ on these vectors. The vector $(x_1,...,x_T)$ is termed the *Fourier representation* of $(y_1,...,y_T)$. Write out the real and imaginary parts of (10) and (11) to get the equivalent formulas

$$(12) \quad y_t = \sum_{k=-n}^{n-1} \cos(2\pi kt/T)\cdot a_k + \sum_{k=-n}^{n-1} \sin(2\pi kt/T)\cdot b_k$$

with

$$(13) \quad a_k = T^{-1} \sum_{t=1}^{T} \cos(2\pi kt/T)y_t \text{ and } b_k = T^{-1} \sum_{t=1}^{T} \sin(2\pi kt/T)y_t.$$

Show that $\sum_{t=1}^{T} y_t^2 = \sum_{k=-n}^{n-1} x_k x_k^*$.

2. Suppose h is a real-valued function on an interval $[-\pi,\pi]$. For T a large even integer and $n = T/2$, define $y_t = h(-\pi+2\pi t/T)\cdot T^{-1/2}$. Let x_k be the Fourier coefficient given by (11), and define $z_k = 2\pi e^{\iota\pi k}x_k$. The Fourier representation of the sequence y_t, from (11), is

$$(14) \quad x_k = \sum_{t=1}^{T} \psi_k^*(t)y_t = T^{-1} \sum_{t=1}^{T} e^{-\iota 2\pi kt/T}h(-\pi+2\pi t/T),$$

implying

$$(15) \quad z_k = \frac{2\pi}{T} \sum_{t=1}^{T} e^{-\iota 2\pi kt/T+\iota\pi k}h(-\pi+2\pi t/T)$$

and, from (10),

$$(16) \quad h(-\pi+2\pi t/T) = \sum_{k=-n}^{n-1} e^{\iota 2\pi kt/T-\iota\pi k}\cdot z_k/2\pi.$$

Now let $T \to \infty$. Suppose h is of bounded variation (i.e., can be written as the difference of two increasing bounded functions). Then it is continuous except at most at a countable number of points, and is square integrable. Then (15) converges to

$$(17) \quad z_k = 2\pi \int_0^1 e^{-\iota 2\pi ks+\iota\pi k}\cdot h(-\pi+2\pi s)ds.$$

A further change of variable to $r = -\pi+2\pi s$, implying $-\iota 2\pi ks+\iota\pi k = -\iota kr$, yields

$$(18) \quad z_k = \int_{-\pi}^{\pi} e^{-\iota kr}\cdot h(r)dr.$$

Show that the z_k satisfy $\sum_{k=-n}^{n-1} z_k z_k^* = (4\pi^2/T) \sum_{t=1}^{T} h(-\pi+2\pi t/T)^2 \to 2\pi \int_{-\pi}^{\pi} h(r)^2dr$. Then, the limit of (16), evaluated at $t = [T(r+\pi)/2\pi]$, as $n \to \infty$ exists for $r > -\pi$ and equals

$$(19) \quad h(r) = \sum_{k=-\infty}^{+\infty} e^{\iota kr}\cdot z_k/2\pi$$

at all continuity points of h. The pair (18) and (19) give a Fourier representation of a function on a bounded interval. If the function is periodic with $h(r\pm 2\pi) = h(r)$ for all r, then the Fourier representation holds for all r. Using orthogonality properties of $e^{\iota kr}$, show directly that if z_k is a square summable sequence, then applying (19) then (18) reproduces the sequence. Note that if h(z) is a sum of sines and cosines with frequencies that are multiples of $1/2\pi$, then the Fourier representation will have non-zero z_k's only for the k's corresponding to these frequencies. Then, the z_k series may be thought of as extracting the frequencies appearing in h(r).

3. Suppose h(r) is a square integrable real-valued function on the real line. For a large constant M, apply the Fourier representation in the previous question to the function $M\cdot h(Mr)$ for $-\pi \le r \le \pi$ to obtain (18) and (19). Define a variable $\omega = k/M$, or $k = \omega M$, and a function $H_M(\omega)$ on the real line by

(20) $$H_M(\omega) = \int_{-\pi M}^{+\pi M} e^{-\iota\omega s}\cdot h(s)ds \rightarrow H(\omega) \equiv \int_{-\infty}^{+\infty} e^{-\iota\omega s}\cdot h(s)ds.$$

Note that $z_k = H_T(k/M) = \int_{-\pi M}^{+\pi M} e^{-\iota s(k/M)}\cdot h(s)ds$, so that (19) can be written

(21) $$h(Mr) = \frac{1}{2\pi M} \cdot \sum_{k=-\infty}^{+\infty} e^{\iota kr}\cdot H_M(k/M).$$

Letting $s = Mr$ and $\omega = k/M$, the limit of (21) as $M \rightarrow \infty$, if it exists, becomes

(22) $$h(s) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{\iota\omega s}\cdot H(\omega)d\omega .$$

The pair consisting of (22) and

(23) $$H(\omega) \equiv \int_{-\infty}^{+\infty} e^{-\iota\omega s}\cdot h(s)ds$$

are *Fourier transforms*. This construction shows that Fourier transforms are obtained as limits of Fourier representations, and also shows that when the limits exist, the Fourier representations from Question 1 can be used to approximate the Fourier transforms. Show that if (22) and (23) are satisfied, then

(24) $\int_{-\infty}^{+\infty} h(s)^2 ds = \int_{-\infty}^{+\infty} H(\omega)H^*(\omega)d\omega.$

4. For the Fourier transforms (22) and (23), verify the following conditions:

(1) h even implies H real and even

(2) h odd implies H imaginary and odd

(3) [time scaling] for $c > 0$, $h(cs)$ transforms to $c^{-1}H(\omega/c)$

(4) [frequency scaling] for $c > 0$, $H(c\omega)$ transforms to $c^{-1}h(s/c)$

(5) [time shifting] $h(s-\tau)$ transforms to $H(\omega)\cdot e^{\iota\omega\tau}$

(6) [convolution] if g and h are real functions and G and H are their transforms, and if $(g*h)(s) \equiv \int_{-\infty}^{+\infty} g(t)h(s-t)dt$, then the transform of $g*h$ is $G(\omega)\cdot H(\omega)$.

(7) [covariation] if g and h are real functions and $\text{cov}(g,h) = \int_{-\infty}^{+\infty} g(s)h(s)ds$, then $\text{cov}(g,h) = \int_{-\infty}^{+\infty} G(\omega)H^*(\omega)d\omega$.

(8) [Parseval's theorem] $\int_{-\infty}^{+\infty} h(s)^2 ds = \int_{-\infty}^{+\infty} H(\omega)H^*(\omega)d\omega.$