Jackknife and Analytical Bias Reduction for Nonlinear Panel Models

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Abstract

Fixed effects estimator of panel models can be severely biased because of the well-known incidental parameter problems. It is shown that such bias can be reduced as T grows with n by using an analytical bias correction or by using a panel jacknife. We describe both of these approaches. We consider asymptotics where n and T grow at the same rate as an approximation that allows us to compare bias properties. Under these asymptotics the bias corrected estimators are centered at the truth, whereas the fixed effects estimator is not. This asymptotic theory shows the bias reduction given by the analytical or jacknife correction.

1 Introduction

Panel data, consisting of observations across time for different individual economic agents, allows the possibility of controlling for unobserved individual heterogeneity. Such heterogeneity can be an important phenomenon, and failure to control for it can result in misleading inferences. This problem is particularly severe when the unobserved heterogeneity is correlated with explanatory variables. Such situation arises naturally when some of the explanatory variables are decision variables.

Models and methods of controlling for unobserved heterogeneity in linear models are well established. A partial list of references is Amemiya and MaCurdy (1986), Anderson and Hsiao (1982), Bhargava and Sargan (1983), Chamberlain (1982), Hausman and Taylor (1981), and Mundlak (1978). Controlling for unobserved heterogeneity is much more difficult in nonlinear models. Conditional maximum likelihood can be used in the rare instance that there is a sufficient statistic for the individual effect. In a few cases the individual effects can be removed by transformations. However, such cases are the exception rather than the rule. Generally it is not possible to find a transformation or a sufficient statistic to remove the individual effect.

One way of attempting to control for individual effects in nonlinear models is to treat each effect as a separate parameter to be estimated. Unfortunately, such estimators are typically subject to the incidental parameters problem noted by Neyman and Scott (1948). The estimators of the parameters of interest will be inconsistent if the number of individuals n goes to infinity while the number of time periods T is held fixed. This inconsistency occurs because only a finite number of observations are available to estimate each individual effect, so that the estimate of the individual effects are random, even in the limit, and this randomness contaminates the estimates of the parameters of interest.

Even in static models the incidental parameters bias can be severe when there are few observations, e.g. see Chamberlain (1982) and Abrevaya (1997). With many time series observations the bias in static models may be small, as in the Monte Carlo studies of Heckman (1981), although the bias is more pronounced in dynamic models, e.g. Heckman (1981) and Nickell (1981). This bias is so severe that estimators are even asymptotically biased if T grows at the same rate as n. This problem is exemplified by the results of Hahn and Kuersteiner (2001), who find asymptotic bias in fixed effects estimators of linear dyanamic models under such asymptotics. We find here an analogous fixed effects asymptotic bias for nonlinear models.

The purpose of this note is to consider two approaches to reducing the bias from fixed effects estimation in nonlinear models. One approach is an analytical bias correction using the the bias formula obtained from an asymptotic expansion as T grows, similarly to Hahn and Kuersteiner (2001). This correction is obtained by generalizing a bias formula from Waterman et. al. (2000), estimating the bias term, and using this to correct the estimator. The second approach is based on a jacknife in the time series dimension of the panel. The idea here is to use the variation in the fixed effects estimators as a time period is dropped to form a bias corrected estimator. We do this by applying the Quenouilles (1956) and Tukey (1958) jacknife formula to the fixed effects estimators that drop each time period. This produces an estimator with properties similar to the analytical approach without explicit computation of the bias term.

We analyze the properties of both of these estimators under asymptotics as n and T grow at the same rate. We find that both approaches yield an estimator that is asymptotically normal and centered at the truth. This large n, large T asymptotics provides a way to formalize the idea that the bias corrected estimators should be closer to the truth. As usual, the asymptotic theory is primarily intended as an approximation to the finite sample distribution of the estimator, here especially to its bias. Under our asymptotic approximation, bias correction does not increase the asymptotic variance. This suggests that the benefit of bias reduction substantially dominates any potential increase of variance.

We conjecture that the bias corrected estimators have an even stronger property, being asymptotically normal and correctly centered as long as T grows faster than $n^{1/3}$. This property would provide further justification for the bias corrected estimators. It would mean they have little bias even when T is allowed to be much smaller than n, a situation that is often encountered in practice.

In Section 2 we use some comparatively simple calculations to describe how the bias corrections work. Section 3 gives the form of the analytical bias correction. Section 4 describes the jacknife bias correction. Asymptotic theory is given in Section 5.

2 The Effect of Bias Correction

Some expansions can be used to motivate the bias results. Let θ denote an R-dimensional parameter vector of interest with true value θ_0, and let $\theta_{[T]}$ denote the limiting value, as $n \to \infty$ with T fixed, of a fixed effects estimator $\widehat{\theta}$ of θ_0. A consequence of the incidental parameters problem is that typically $\theta_{[T]} \neq \theta_0$. Note that the bias should be small for large enough T, i.e., $\lim_{T\to\infty} \theta_{[T]} = \theta_0$. Indeed, it can be shown that in some generality,

$$\theta_{[T]} = \theta_0 + \frac{\beta}{T} + O\left(\frac{1}{T^2}\right).$$

To see the resultant bias of the fixed effects estimator, suppose that as $n/T \to \rho$ it is the case that $\sqrt{nT}\left(\widehat{\theta} - \theta_{[T]}\right) \xrightarrow{d} N\left(0, \Omega\right)$. Then

$$\sqrt{nT}\left(\widehat{\theta} - \theta_0\right) = \sqrt{nT}\left(\widehat{\theta} - \theta_{[T]}\right) + \sqrt{nT}\frac{\beta}{T} + O\left(\sqrt{\frac{n}{T^3}}\right) \xrightarrow{d} N(\beta\sqrt{\rho}, \Omega).$$

Thus we see that even when T grows as fast as n the fixed effects estimator has asymptotic bias, with its limit distribution not being centered at zero.

The first approach to bias correction consists of estimating β by some $\widehat{\beta}$ and then forming a bias corrected estimator

$$\widehat{\widehat{\theta}} \equiv \widehat{\theta} - \frac{\widehat{\beta}}{T}, \tag{1}$$

This estimator should be less biased than the fixed effects estimator $\widehat{\theta}$. Specifically, when n and T grow at the same rate as before and $\widehat{\beta} \xrightarrow{p} \beta$,

$$\sqrt{nT}\left(\widehat{\widehat{\theta}} - \theta_0\right) = \sqrt{nT}\left(\widehat{\theta} - \theta_{[T]}\right) - \sqrt{\frac{n}{T}}\left(\widehat{\beta} - \beta\right) + O\left(\sqrt{\frac{n}{T^3}}\right) \xrightarrow{d} N\left(0, \Omega\right), \tag{2}$$

so that the use of $\widehat{\widehat{\theta}}$ eliminates that bias. Thus, all we need to do is find a consistent estimator of β. We do this by plugging in consistent esitimators of components of the formula for β. We show that equation (2) holds for the $\widehat{\beta}$ we use.

The panel jacknife provides an alternative approach that avoids estimation of β. To describe it, let $\widehat{\theta}_{(t)}$ be the fixed effects estimator based on the subsample excluding the observations of the tth period. The jackknife estimator is

$$\widetilde{\theta} \equiv T\widehat{\theta} - (T-1)\sum_{t=1}^{T}\widehat{\theta}_{(t)}/T \tag{3}$$

To explain the bias correction from this estimator it is helpful to consider a further expansion

$$\theta_{[T]} = \theta + \frac{\beta}{T} + \frac{\gamma}{T^2} + O\left(\frac{1}{T^3}\right). \tag{4}$$

To see how the jacknife affects the bias we can consider the limit of $\widetilde{\theta}$ for fixed T and see how it changes with T. The estimator $\widetilde{\theta}$ will converge in probability to

$$\begin{aligned} T\theta_{[T]} - (T-1)\,\theta_{[T-1]} &= \theta + \left(\frac{1}{T} - \frac{1}{T-1}\right)\gamma + O\left(\frac{1}{T^2}\right) \\ &= \theta - \frac{1}{T\,(T-1)}\gamma + O\left(\frac{1}{T^2}\right) = \theta + O\left(\frac{1}{T^2}\right). \end{aligned}$$

Thus we see that the bias of the jacknife corrected value is of order $1/T^2$. Furthermore, as shown below, the jackknife estimator $\widetilde{\theta}$ has the same property as $\widehat{\widehat{\theta}}$ under the large n and T approximation, namely $\sqrt{nT}\left(\widetilde{\theta} - \theta_0\right) \xrightarrow{d} N\left(0, \Omega\right)$.

Our conjecture, that the analytical bias correction works if T grows faster than $n^{1/3}$, can also be explained by an analogous expansion. Suppose now that $\hat{\beta}$ is $\sqrt{nT}$ consistent, so that $\widehat{\beta} = \beta + O\left(1/\sqrt{nT}\right)$. Then expanding as in equation (2) gives

$$\sqrt{nT}\left(\widehat{\widehat{\theta}} - \theta_0\right) = \sqrt{nT}\left(\widehat{\theta} - \theta_{[T]}\right) - \sqrt{nT}\left(\widehat{\beta} - \beta\right)/T + O\left(\sqrt{\frac{n}{T^3}}\right) \xrightarrow{d} N(0, \Omega).$$

Although we do not verify this conjecture here, this expansion strongly suggests that the bias correction removes asymptotic bias under the condition that T grows at least as fast as $n^{1/3}$. This conjecture suggests that the bias corrected fixed effects estimator should be approximately unbiased when T is much smaller than n.

3 Analytical Bias Correction

We first describe the model we consider. Let the data observations be denoted by $x_{it}, (t = 1, ..., T; i = 1, ..., n)$. We assume that these observations are mutually independent and that there is a density function $f\left(x; \theta, \alpha\right)$ such that

$$x_{it} \sim f\left(\cdot; \theta_0, \alpha_{i0}\right), \qquad (t = 1, ..., T; i = 1, ..., n)$$

We assume that $\dim\left(\theta\right) = R \geq 1$ and that $\dim\left(\alpha\right) = 1$. Thus, here $X_{iT} \equiv \left(x_{i1}, ..., x_{iT}\right)$ is i.i.d. over t, while the distribution of observations across i differs only due to α_{i0}. The fixed effects estimator is given by

$$\begin{aligned}
\widehat{\theta} &= \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} q\left(X_{iT}, \theta\right), \\
q\left(X_{iT}, \theta\right) &\equiv \sum_{t=1}^{T} \log f\left(x_{it}; \theta, \widehat{\alpha}_i\left(\theta\right)\right), \\
\widehat{\alpha}_i\left(\theta\right) &\equiv \underset{\alpha_i}{\operatorname{argmax}} \sum_{t=1}^{T} \log f\left(x_{it}; \theta, \alpha\right).
\end{aligned}$$

By the usual results for extremum estimators, for fixed T the estimator $\widehat{\theta}$ will be consistent for

$$\theta_{[T]} = \underset{\theta}{\operatorname{argmax}}\, E\left[q\left(X_{iT}, \theta\right)\right] = \underset{\theta}{\operatorname{argmax}}\, E\left[\log f\left(x_{it}; \theta, \widehat{\alpha}_i\left(\theta\right)\right)\right].$$

The incidental parameters problem, with $\theta_{[T]} \neq \theta_0$, arises because of randomness of $\widehat{\alpha}_i\left(\theta\right)$. If $\widehat{\alpha}_i\left(\theta\right)$ were replaced by $\alpha_i(\theta) \equiv \operatorname{argmax}_{\alpha_i} E\left[\log f\left(x_{it}; \theta, \alpha\right)\right]$, i.e. if the sum over T were replaced by the expectation, then it is straightforward to show that $\theta_{[T]} = \theta_0$.

To describe the analytical bias correction, we need to define some additional derivatives. Let

$$\begin{aligned}
s_\theta\left(x_{it}; \theta, \alpha_i\right) &\equiv \frac{\partial}{\partial \theta} \log f\left(x_{it}; \theta, \alpha_i\right), \qquad V\left(x_{it}; \theta, \alpha_i\right) \equiv \frac{\partial}{\partial \alpha_i} \log f\left(x_{it}; \theta, \alpha_i\right), \\
V_2\left(x_{it}; \theta, \alpha_i\right) &\equiv f\left(x_{it}; \theta, \alpha_i\right)^{-1} \frac{\partial^2}{\partial \alpha_i^2} f\left(x_{it}; \theta, \alpha_i\right).
\end{aligned}$$

Also, for the fixed effect estimator $\widehat{\theta}$ and $\widehat{\alpha}_1, ..., \widehat{\alpha}_n$, let

$$\begin{aligned}
\widehat{s}_{it} &\equiv s_\theta\left(x_{it}; \widehat{\theta}, \widehat{\alpha}_i\right), \quad \widehat{V}_{it} \equiv V\left(x_{it}; \widehat{\theta}, \widehat{\alpha}_i\right), \quad \widehat{V}_{2it} \equiv V_2\left(x_{it}; \widehat{\theta}, \widehat{\alpha}_i\right), \\
\widehat{U}_{it} &\equiv \widehat{s}_{it} - \widehat{V}_{it}\left(\sum_{u=1}^{T} \widehat{V}_{it}\widehat{s}_{it}\right) \Big/ \left(\sum_{u=1}^{T} \widehat{V}_{it}^2\right).
\end{aligned}$$

Then the bias term estimator is

$$\widehat{\beta} = -\frac{1}{2}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \widehat{U}_{it}\widehat{U}_{it}'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\widehat{U}_{it}\widehat{V}_{2it} \Big/ \sum_{t=1}^{T}\widehat{V}_{it}^2\right)\right).$$

The bias corrected estimator can then be formed as in equation (1).

This object can be interpreted as local bias correction for random parameters. Note that $1/\sum_{t=1}^{T}\widehat{V}_{it}^2$ is an information approxmation to the variance of $\widehat{\alpha}_i$. From Chesher (1984), we know that the score for a random α_i with variance $1/\sum_{t=1}^{T}\widehat{V}_{it}^2$ would be

$$\widehat{D}_{it} = \frac{1}{2}\left(\frac{1}{\sum_{t=1}^{T}\widehat{V}_{it}^2}\right)\widehat{V}_{2it}.$$

Applying the Kiefer and Skoog (1984) general formula for local bias of the MLE under misspecification to the special case of random parameters then gives $\widehat{\beta}$. Thus we see that $\widehat{\beta}$ is an estimator of the local bias that results from random parameters, and so bias correcting with it makes the bias go away faster.

4 Jacknife Bias Correction

The jacknife bias correction is constructed by re-estimating θ while dropping each time series observation in turn. To describe it, for a given θ let the fixed effect estimator of α_i from all observations but the t^{th} be given by

$$\widehat{\alpha}_{i(t)}(\theta) \equiv \underset{\alpha}{\operatorname{argmax}} \sum_{s\neq t, s=1}^{T} \log f(x_{is}; \theta, \alpha).$$

The fixed effects estimator of θ excluding time period t is then given by

$$\widehat{\theta}_{(t)} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \sum_{s\neq t} \log f\left(x_{is}; \theta, \widehat{\alpha}_{i(t)}(\theta)\right)$$

The jacknife bias corrected estimator is then as given in equation (3).

The jacknife bias correction requires recomputing the estimator T times, but avoids the computation of higher-order derivatives needed for the analytic bias correction. The fixed effects estimator $\widehat{\theta}, \widehat{\alpha}_1, ..., \widehat{\alpha}_n$ provide a good starting value for these computations. Starting at the fixed effects and iterating towards $\widehat{\theta}_{(t)}$ should be straightforward computationally, requiring no additional software. Thus, the jackife provides a straightforward approach to bias correction.

A simple example may serve to illustrate this jacknife bias correction. Suppose

$$x_{it} \overset{i.i.d.}{\sim} N(\alpha_{i0}, \theta_0) \qquad t = 1, \ldots, T : i = 1, \ldots, n$$

so that

$$\log f(x_{it}; \theta, \alpha_i) = C - \frac{1}{2}\log\theta - \frac{(x_{it} - \alpha_i)^2}{2\theta}$$

The fixed effects MLE is

$$\begin{aligned}
\widehat{\alpha}_i &= \frac{1}{T}\sum_{t=1}^{T} x_{it} \equiv \overline{x}_i, \\
\widehat{\theta} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - \widehat{\alpha}_i)^2 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - \overline{x}_i)^2
\end{aligned}$$

As is well known, $E\left[\widehat{\theta}\right] = \frac{T-1}{T}\theta_0$. It is straightforward to show that

$$\widetilde{\theta} \equiv T\widehat{\theta} - \frac{T-1}{T}\sum_{t=1}^{T}\widehat{\theta}_{(t)} = \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it} - \overline{x}_i)^2 = \frac{T}{(T-1)}\widehat{\theta}.$$

In this example, the jacknife bias corrected estimator is unbiased.

5 Asymptotic Theory

The first result we consider is the asymptotic distribution of the fixed effects MLE when $n, T \to \infty$ at the same rate. We impose the following conditions:

Condition 1 *$n, T \to \infty$ such that $\frac{n}{T} \to \rho$, where $0 < \rho < \infty$.*

Condition 2 *(i) The function $\ln f(\cdot;\theta,\alpha)$ is continuous in $(\theta,\alpha) \in \Upsilon$; (ii) The parameter space Υ is compact; (iii) There exists a function function $M(x_{it})$ such that $|\log f(x_{it};\theta,\alpha_i)| \le M(x_{it})$, $\left|\frac{\partial \log f(x_{it};\theta,\alpha_i)}{\partial(\theta,\alpha_i)}\right| \le M(x_{it})$, and $\sup_i E\left[M(x_{it})^{33}\right] < 0$.*

Condition 3 *For each $\eta > 0$, $\inf_i \left[G_{(i)}(\theta_0,\alpha_{i0}) - \sup_{\{(\theta,\alpha):|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta\}} G_{(i)}(\theta,\alpha)\right] > 0$, where*

$$\widehat{G}_{(i)}(\theta,\alpha_i) \equiv T^{-1}\sum_{t=1}^{T}\log f(x_{it};\theta,\alpha_i) \equiv T^{-1}\sum_{t=1}^{T} g(x_{it};\theta,\alpha_i), \quad G_{(i)}(\theta,\alpha_i) \equiv E[\log f(x_{it};\theta,\alpha_i)]$$

Let

$$\begin{aligned} U_{it} &\equiv s_\theta(x_{it};\theta_0,\alpha_{i0}) - \rho_{i0}\cdot V_{it}, \quad V_{it} \equiv V(x_{it};\theta_0,\alpha_{i0}), \\ \rho_{i0} &\equiv E[s_\theta(x_{it};\theta_0,\alpha_{i0})V_{it}]/E[V_{it}^2], \quad V_{2it} \equiv V_2(x_{it};\theta_0,\alpha_{i0}), \\ \mathcal{I}_i &\equiv E[U_{it}U_{it}']. \end{aligned}$$

Condition 4 *(i) There exists some $M(x_{it})$ such that*

$$\left|\frac{\partial^{m_1+m_2}\log f(x_{it};\theta,\alpha_i)}{\partial\theta^{m_1}\partial\alpha_i^{m_2}}\right| \le M(x_{it}) \qquad 0 \le m_1+m_2 \le 1,\ldots,6$$

and $\sup_i E\left[M(x_{it})^Q\right] < \infty$ for some $Q > 64$; (ii) $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} E[U_{it}U_{it}'] > 0$; (iii) $\min_i E\left[V_i^2\right] > 0$.

Under these conditions we can obtain the asymptotic distribution of the fixed effects MLE as n and T grow at the same rate.

Theorem 1 *Under Conditions 1, 2, 3, and 4, we have*

$$\sqrt{nT}\left(\widehat{\theta}-\theta_0\right) \to N\left(\beta\sqrt{\rho}, \left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\right)$$

where

$$\beta \equiv -\frac{1}{2}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E[V_{2it}U_{it}]}{E[V_{it}^2]}\right)$$

Proof. See Appendix B. ■

Waterman et al (2000) establish the asymptotic distribution of $\sqrt{nT}\left(\widehat{\theta}-\theta_0\right)$ assuming that $\widehat{\theta}$ and $\widehat{\alpha}_i\left(\widehat{\theta}\right)$ are consistent for θ_0 and α_{i0}, and $\dim(\theta) = 1$. We show that $\widehat{\theta}$ and $\widehat{\alpha}_i$ are consistent for θ_0 and α_{i0}: Theorems 4 and 5 in Appendix A establish such consistency under Conditions 1, 2, and 3. We also allow for $\dim(\theta)$ to be bigger than 1.

Next, we show that the analytic bias correction eliminates the asymptotic bias term.

Theorem 2 *Under Conditions 1, 2, 3, and 4, we have $\widehat{\beta} = \beta + o_p(1)$*

Proof. See Appendix E. ■

Corollary 1 *Suppose that Conditions 1, 2, 3, and 4 hold. Let*

$$\widehat{\widehat{\theta}} \equiv \widehat{\theta} - \frac{1}{T}\widehat{\beta}$$

We then have

$$\sqrt{nT}\left(\widehat{\widehat{\theta}} - \theta_0\right) \to N\left(0, \left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\right).$$

Proof. The conclusion easily follows from

$$\sqrt{nT}\cdot\frac{1}{T}\widehat{\beta} = \sqrt{\frac{n}{T}}\widehat{\beta} = \beta\sqrt{\rho} + o_p(1)$$

■

Remark 1 *By equation (22) in Appendix E, we can see that* $\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{U}_{it}\widehat{U}_{it}'\right)^{-1}$ *is a consistent estimator of the asymptotic variance* $\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}$ *of* $\sqrt{nT}\left(\widehat{\widehat{\theta}} - \theta_0\right)$.

To compare our result with Waterman et. al. (2000), let $\widetilde{V}_{it} \equiv (V_{it}, V_{2it})'$ and

$$\begin{aligned}\widetilde{\rho}_i &\equiv \left(E\left[\widetilde{V}_{it}\widetilde{V}_{it}'\right]\right)^{-1} E\left[\widetilde{V}_{it}U_{it}\right],\\ U^*(x_{it};\theta,\alpha) &= s_\theta(x_{it};\theta,\alpha_i) - (V(x_{it};\theta,\alpha_i), V_2(x_{it};\theta,\alpha_i))\,\widetilde{\rho}_i.\end{aligned}$$

They showed the same conclusion as above with scalar θ for a nonfeasible estimator obtained by solving

$$\sum_{i=1}^{n}\sum_{t=1}^{T}U_i^*(x_{it};\theta,\alpha) = 0, \quad \sum_{t=1}^{T}V_i(x_{it};\theta,\alpha_i) = 0.$$

This estimator is not feasible because $\widetilde{\rho}_i$ is unknown. In contrast, our result shows that a feasible bias corrected estimator satisfies the conclusion.

The jacknife bias corrected estimator has the same limiting distribution as the analytic bias corrected estimator. In order to simplify the proof we only show this for scalar θ.

Theorem 3 *Suppose that* $\dim(\theta) = 1$. *Also suppose that Conditions 1, 2, 3, and 4, hold. We then have*

$$\sqrt{nT}\left(\widetilde{\theta} - \theta_0\right) \to N\left(0, \frac{1}{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^2\right]}\right).$$

Proof. See Appendix I. ■

6 Summary

We developed two methods of reducing bias of the fixed effects maximum likelihood estimator for nonlinear panel models with fixed effects, an analytic method and an automatic method based on a time series jacknife. It might be of interest consider the extra term γ in the expansion of equation (4), and seek to remove this terrm by analytical and/or jacknife methods. We expect that such could be done under yet another alternative approximation where n and T grow to infinity at a different rate. Such analysis is expected to be substantially complicated, and we leave it to future research.

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Appendix

A Consistency

Throughout this section, we assume that Conditions 1, 2, and 3 hold.

Lemma 1 *Assume that W_t are iid with $E[W_t]=0$ and $E\left[W_t^{2k}\right]<\infty$. Then,*

$$E\left[\left(\sum_{t=1}^{T} W_t\right)^{2k}\right]=C(k)T^k+o(T^k)$$

for some constant $C(k)$.

Proof. By adopting an argument in the proof of Lemma 5.1 in Lahiri (1992), we have

$$E\left[\left(\sum_{t=1}^{T} W_t\right)^{2k}\right]=\sum_{j=1}^{2k}\sum_{\alpha} C\left(\alpha_1,...,\alpha_j\right)\sum_{I} E\left[\prod_{s=1}^{j} W_{t_s}^{\alpha_s}\right], \tag{5}$$

where for each fixed $j\in\{1,...,2k\}$, $\sum_{\alpha}$ extends over all j-tuples of positive integers $(\alpha_1,...,\alpha_j)$ such that $\alpha_1+...+\alpha_j=2k$ and $\sum_I$ extends over all ordered j-tuples $(t_1,...,t_j)$ of integers such that $1\le t_j\le T$. Also, $C(\alpha_1,...,\alpha_j)$ stands for a bounded constant. Note, that if $j>k$ then at least one of the indices $\alpha_j=1$. By independence and the fact that $E[W_t]=0$ it follows that $E\left[\prod_{s=1}^{j} W_{t_s}^{\alpha_s}\right]=0$ whenever $j>k$. This shows that $\left|E\left[\left(\sum_{t=1}^{T} W_t\right)^{2k}\right]\right|\le C(k)\,T^k\cdot E\left[W_t^{2k}\right]$ for some constant $C(k)$. ■

Lemma 2 *Suppose that, for each i, $\{\xi_{it},t=1,2,\ldots\}$ is a sequence of zero mean i.i.d. random variables. We assume that $\{\xi_{it},t=1,2,3\}$ are independent across i. We also assume that $\max_i E\left[|\xi_{it}|^{16}\right]<\infty$. Finally, we assume that $n=O(T)$. We then have*

$$\max_i \Pr\left[\left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right|>\eta\right]=o\left(T^{-2}\right)$$

for every $\eta>0$.

Proof. Using Lemma 1, we obtain

$$E\left[\left|\sum_{t=1}^{T}\xi_{it}\right|^{16}\right]\le CT^8\cdot E\left[\xi_{it}^{16}\right],$$

where $C>0$ is a constant. Therefore, we have

$$T^2\Pr\left[\left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right|>\eta\right]\le T^2\frac{CT^8}{T^{16}\eta^{16}}E\left[\xi_{it}^{16}\right]$$

or

$$\max_i T^2\Pr\left[\left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right|>\eta\right]\le\frac{C}{T^6\eta^{16}}\max_i E\left[\xi_{it}^{16}\right]=o(1).$$

■

Lemma 3 *Suppose that Conditions 1 and 2 hold. We then have for all* $\eta > 0$ *that*

$$\Pr\left[\max_{1\leq i\leq n}\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|\geq\eta\right]=o\left(T^{-1}\right)$$

Proof. Let $\eta > 0$ be given. We note that

$$\Pr\left[\max_{1\leq i\leq n}\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|\geq\eta\right]\leq\sum_{i=1}^{n}\Pr\left[\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|\geq\eta\right]. \tag{6}$$

Let $\varepsilon > 0$ be chosen such that $2\varepsilon\max_i E[M(x_{it})] < \frac{\eta}{3}$. Divide Υ into subsets $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{M(\varepsilon)}$ such that $\left|(\theta,\alpha)-\left(\theta',\alpha'\right)\right| < \varepsilon$ whenever (θ,α) and $\left(\theta',\alpha'\right)$ are in the same subset. Let (θ_j,α_j) denote *some* point in Υ_j for each j. Then,

$$\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|=\max_{j}\sup_{\Upsilon_j}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|,$$

and therefore

$$\Pr\left[\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|>\eta\right]\leq\sum_{j=1}^{M(\varepsilon)}\Pr\left[\sup_{\Upsilon_j}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|>\eta\right] \tag{7}$$

For $(\theta,\alpha)\in\Upsilon_j$, we have

$$\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|\leq\left|\widehat{G}_{(i)}(\theta_j,\alpha_j)-G_{(i)}(\theta_j,\alpha_j)\right|+\frac{\varepsilon}{T}\left|\sum_{t=1}^{T}\left(M(x_{it})-E[M(x_{it})]\right)\right|+2\varepsilon E[M(x_{it})]$$

Then,

$$\begin{aligned}\Pr\left[\sup_{\Upsilon_j}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|>\eta\right] &\leq \Pr\left[\left|\widehat{G}_{(i)}(\theta_j,\alpha_j)-G_{(i)}(\theta_j,\alpha_j)\right|>\frac{\eta}{3}\right]\\ &\quad+\Pr\left[\frac{1}{T}\left|\sum_{t=1}^{T}\left(M(x_{it})-E[M(x_{it})]\right)\right|>\frac{\eta}{3\varepsilon}\right]\\ &= o\left(T^{-2}\right)\end{aligned} \tag{8}$$

by Lemma 2. Combining (6), (7), (8), and $n = O(T)$, we obtain the desired conclusion. ■

Theorem 4 *Under Conditions 1, 2, and 3,* $\Pr\left[\left|\widehat{\theta}-\theta_0\right|\geq\eta\right]=o\left(T^{-1}\right)$ *for every* $\eta > 0$.

Proof. Let η be given, and let $\varepsilon\equiv\inf_i\left[G_{(i)}(\theta_0,\alpha_{i0})-\sup_{\{(\theta,\alpha):|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta\}}G_{(i)}(\theta,\alpha)\right]>0$. With probability equal to $1-o\left(\frac{1}{T}\right)$, we have

$$\begin{aligned}\max_{|\theta-\theta_0|>\eta,\alpha_1,\ldots,\alpha_n}n^{-1}\sum_{i=1}^{n}\widehat{G}_{(i)}(\theta,\alpha_i) &\leq \max_{|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta}n^{-1}\sum_{i=1}^{n}\widehat{G}_{(i)}(\theta,\alpha_i)\\ &< \max_{|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta}n^{-1}\sum_{i=1}^{n}G_{(i)}(\theta,\alpha_i)+\frac{1}{3}\varepsilon\\ &< n^{-1}\sum_{i=1}^{n}G_{(i)}(\theta_0,\alpha_{i0})-\frac{2}{3}\varepsilon\\ &< n^{-1}\sum_{i=1}^{n}\widehat{G}_{(i)}(\theta_0,\alpha_{i0})-\frac{1}{3}\varepsilon,\end{aligned}$$

where the second and fourth inequalities are based on Lemma 3. Because

$$\max_{\theta,\alpha_1,\ldots,\alpha_n} n^{-1}\sum_{i=1}^{n}\widehat{G}_{(i)}(\theta,\alpha_i) \geq n^{-1}\sum_{i=1}^{n}\widehat{G}_{(i)}(\theta_0,\alpha_{i0})$$

by definition, we can conclude that $\Pr\left[\left|\widehat{\theta}-\theta_0\right|\geq\eta\right] = o\left(\frac{1}{T}\right)$. ■

Theorem 5 *Under Conditions 1, 2, and 3,*

$$T\Pr\left[\max_{1\leq i\leq n}\left|\widehat{\alpha}_i-\alpha_{i0}\right|\geq\eta\right] = o(1)$$

Proof. We first prove that

$$T\Pr\left[\max_{1\leq i\leq n}\sup_{\alpha}\left|\widehat{G}_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}(\theta_0,\alpha)\right|\geq\eta\right] = o(1) \tag{9}$$

for every $\eta>0$. Note that

$$\begin{aligned}
&\max_{1\leq i\leq n}\sup_{\alpha}\left|\widehat{G}_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}\left(\widehat{\theta},\alpha\right)\right| \\
\leq\;&\max_{1\leq i\leq n}\sup_{\alpha}\left|\widehat{G}_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}\left(\widehat{\theta},\alpha\right)\right| + \max_{1\leq i\leq n}\sup_{\alpha}\left|G_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}(\theta_0,\alpha)\right| \\
\leq\;&\max_{1\leq i\leq n}\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right| + \max_{1\leq i\leq n}E\left[M(x_{it})\right]\cdot\left|\widehat{\theta}-\theta_0\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T\Pr\left[\max_{1\leq i\leq n}\sup_{\alpha}\left|\widehat{G}_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}(\theta_0,\alpha)\right|\geq\eta\right] &\leq T\Pr\left[\max_{1\leq i\leq n}\sup_{(\theta,\alpha)}\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|\geq\frac{\eta}{2}\right] \\
&\quad +T\Pr\left[\left|\widehat{\theta}-\theta_0\right|\geq\frac{\eta}{2\left(1+\max_{1\leq i\leq n}E\left[M(x_{it})\right]\right)}\right] \\
&= o(1)
\end{aligned}$$

by Lemma 3 and Theorem 4.

We now get back to the proof of Theorem 5. It suffices to prove that

$$T\Pr\left[\max_{1\leq i\leq n}\left|\widehat{\alpha}_i-\alpha_{i0}\right|\geq\eta\right] = o(1)$$

for every $\eta>0$. Let η be given, and let $\varepsilon\equiv\inf_i\left[G_{(i)}(\theta_0,\alpha_{i0})-\sup_{\{\alpha_i:|\alpha_i-\alpha_{i0}|>\eta\}}G_{(i)}(\theta_0,\alpha_i)\right]>0$. Condition on the event $\left\{\max_{1\leq i\leq n}\sup_{\alpha}\left|\widehat{G}_{(i)}\left(\widehat{\theta},\alpha\right)-G_{(i)}(\theta_0,\alpha)\right|\leq\frac{1}{3}\varepsilon\right\}$, which has a probability equal to $1-o\left(\frac{1}{T}\right)$ by (9). We then have

$$\max_{|\alpha_i-\alpha_{i0}|>\eta}\widehat{G}_{(i)}\left(\widehat{\theta},\alpha_i\right) < \max_{|\alpha_i-\alpha_{i0}|>\eta}G_{(i)}(\theta_0,\alpha_i)+\frac{1}{3}\varepsilon < G_{(i)}(\theta_0,\alpha_{i0})-\frac{2}{3}\varepsilon < \widehat{G}_{(i)}\left(\widehat{\theta},\alpha_{i0}\right)-\frac{1}{3}\varepsilon$$

This is inconsistent with $\widehat{G}_{(i)}\left(\widehat{\theta},\widehat{\alpha}_i\right)\geq\widehat{G}_{(i)}\left(\widehat{\theta},\alpha_{i0}\right)$, and therefore, $\left|\widehat{\alpha}_i-\alpha_{i0}\right|\leq\eta$ for every i. ■

B Proof of Theorem 1

We can see that the MLE $\widehat{\theta}$ also solves

$$0=\sum_{i=1}^{n}\sum_{t=1}^{T}U\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\left(\widehat{\theta}\right)\right).$$

Let $F\equiv(F_1,\ldots,F_n)$ denote the collection of distribution functions. Let $\widehat{F}\equiv\left(\widehat{F}_1,\ldots,\widehat{F}_n\right)$, where $\widehat{F}_i$ denotes the empirical distribution function for the stratum i. Define $F(\epsilon)\equiv F+\epsilon\sqrt{T}\left(\widehat{F}-F\right)$ for $\epsilon\in\left[0,T^{-1/2}\right]$. For each fixed θ and ϵ, let $\alpha_i(\theta,F_i(\epsilon))$ be the solution to the estimating equation

$$0=\int V_i\left[\theta,\alpha_i(\theta,F_i(\epsilon))\right]dF_i(\epsilon),$$

and let $\theta(F(\epsilon))$ be the solution to the estimating equation

$$0=\sum_{i=1}^{n}\int U_i\left(x_{it};\theta(F(\epsilon)),\alpha_i(\theta(F_i(\epsilon)),F_i(\epsilon))\right)dF_i(\epsilon).$$

By Taylor series expansion, we have

$$\theta\left(\widehat{F}\right)-\theta(F)=\frac{1}{\sqrt{T}}\theta^{\epsilon}(0)+\frac{1}{2}\left(\frac{1}{\sqrt{T}}\right)^2\theta^{\epsilon\epsilon}(0)+\frac{1}{6}\left(\frac{1}{\sqrt{T}}\right)^3\theta^{\epsilon\epsilon\epsilon}(\widetilde{\epsilon}),$$

where $\theta^{\epsilon}(\epsilon)\equiv d\theta(F(\epsilon))/d\epsilon$, $\theta^{\epsilon\epsilon}(\epsilon)\equiv d^2\theta(F(\epsilon))/d\epsilon^2,\ldots$, and $\widetilde{\epsilon}$ is somewhere in between 0 and $T^{-1/2}$. We therefore have

$$\sqrt{nT}\left(\theta\left(\widehat{F}\right)-\theta(F)\right)=\sqrt{nT}\frac{1}{\sqrt{T}}\theta^{\epsilon}(0)+\sqrt{nT}\frac{1}{2}\left(\frac{1}{\sqrt{T}}\right)^2\theta^{\epsilon\epsilon}(0)+\frac{1}{6}\sqrt{\frac{n}{T}}\frac{1}{\sqrt{T}}\theta^{\epsilon\epsilon\epsilon}(\widetilde{\epsilon}). \tag{10}$$

Theorem 6 in Appendix D establishes that last term in (10) is $o_p(1)$ under Condition 4. It is shown later in Appendix C that

$$\begin{aligned}\sqrt{nT}\frac{1}{\sqrt{T}}\theta^{\epsilon}(0) &\rightarrow N\left(0,\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\right),\\ \sqrt{nT}\frac{1}{2}\left(\frac{1}{\sqrt{T}}\right)^2\theta^{\epsilon\epsilon}(0) &= -\frac{1}{2}\sqrt{\frac{n}{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{E\left[V_{2it}U_i\right]}{E\left[V_i^2\right]}\right)+o_p(1),\end{aligned}$$

from which Theorem 1 follows.

C Derivatives of θ

Let

$$h_i(\cdot,\epsilon)\equiv U_i\left(\cdot;\theta(F(\epsilon)),\alpha_i(\theta(F(\epsilon)),F_i(\epsilon))\right) \tag{11}$$

The first order condition may be written as

$$0=\frac{1}{n}\sum_{i=1}^{n}\int h_i(\cdot,\epsilon)\,dF_i(\epsilon) \tag{12}$$

Differentiating repeatedly with respect to ϵ, we obtain

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{dh_i(\cdot,\epsilon)}{d\epsilon}dF_i(\epsilon) + \frac{1}{n}\sum_{i=1}^{n}\int h_i(\cdot,\epsilon)\,d\Delta_{iT} \tag{13}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2h_i(\cdot,\epsilon)}{d\epsilon^2}dF_i(\epsilon) + 2\frac{1}{n}\sum_{i=1}^{n}\int \frac{dh_i(\cdot,\epsilon)}{d\epsilon}d\Delta_{iT} \tag{14}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^3h_i(\cdot,\epsilon)}{d\epsilon^3}dF_i(\epsilon) + 3\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2h_i(\cdot,\epsilon)}{d\epsilon^2}d\Delta_{iT} \tag{15}$$

where $\Delta_{iT} \equiv \sqrt{T}\left(\widehat{F}_i - F_i\right)$.

C.1 $\theta^{\epsilon}(0)$

Because

$$\frac{dh_i(\cdot,\epsilon)}{d\epsilon} = \frac{\partial h_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\epsilon}$$

we may rewrite (13) as

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int\left(\frac{\partial h_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\epsilon}\right)dF_i(\epsilon) + \frac{1}{n}\sum_{i=1}^{n}\int h_i(\cdot,\epsilon)\,d\Delta_{iT} \tag{16}$$

Evaluating at $\epsilon = 0$, and noting that $E\left[U_i^{\alpha_i}\right] = 0$, we obtain

$$\theta^{\epsilon}(0) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\int U_i d\Delta_{iT}\right) \tag{17}$$

We therefore have

$$\sqrt{nT}\frac{1}{\sqrt{T}}\theta^{\epsilon}(0) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(\frac{1}{\sqrt{n}\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}U_i\right) \to N\left(0,\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\right)$$

C.2 α_i^{θ} and α_i^{ϵ}

In the ith stratum, $\alpha_i(\theta, F_i(\epsilon))$ solves the estimating equation

$$\int V_i(\cdot;\theta,\alpha_i(\theta,F_i(\epsilon)))\,dF_i(\epsilon) = 0 \tag{18}$$

Differentiating the LHS with respect to θ and ϵ, we obtain

$$0 = \int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\theta}dF_i(\epsilon) + \left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta},$$

$$0 = \left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} + \int V_i(\cdot,\theta,\epsilon)\,d\Delta_{iT}.$$

Observe that

$$\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta} = -\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)^{-1}\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\theta}dF_i(\epsilon)\right),$$

$$\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} = -\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)^{-1}\left(\int V_i(\cdot,\theta,\epsilon)\,d\Delta_{iT}\right).$$

Equating these equations to zero and solving for derivatives of α_i evaluated at $\epsilon = 0$ gives

$$\alpha_i^\theta = -\frac{E\left[\frac{\partial V_i}{\partial \theta}\right]}{E\left[\frac{\partial V_i}{\partial \alpha_i}\right]} = \frac{E\left[\frac{\partial V_i}{\partial \theta}\right]}{E\left[V_i^2\right]} = O(1), \tag{19}$$

$$\alpha_i^\epsilon = -\frac{\sum_{t=1}^T V_{it}}{\sqrt{T} E\left[\frac{\partial V_i}{\partial \alpha_i}\right]} = \frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]} = O_p(1), \tag{20}$$

where

$$\alpha_i^\theta \equiv \frac{\partial \alpha_i(\theta, F_i(0))}{\partial \theta}, \qquad \alpha_i^\epsilon \equiv \frac{\partial \alpha_i(\theta, F_i(0))}{\partial \epsilon}.$$

C.3 $\theta^{\epsilon\epsilon}(0)$

Note that

$$\begin{aligned}
\frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2} &= \mathcal{G}_i(\cdot,\epsilon) + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\theta}{\partial\epsilon} + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\alpha_i}{\partial\epsilon}\frac{\partial\theta}{\partial\epsilon} \\
&\quad + \frac{\partial h_i(\cdot,\epsilon)}{\partial\theta'}\frac{\partial^2\theta}{\partial\epsilon^2} \\
&\quad + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\theta}{\partial\epsilon}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)^2 + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\alpha_i}{\partial\epsilon} \\
&\quad + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\left(\frac{\partial\theta'}{\partial\epsilon}\frac{\partial^2\alpha_i}{\partial\theta\partial\theta'}\right)\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\theta'\partial\epsilon}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial^2\theta}{\partial\epsilon^2} \\
&\quad + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\theta}{\partial\epsilon}\frac{\partial\alpha_i}{\partial\epsilon} + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\alpha_i}{\partial\epsilon} + \frac{\partial^2 h_i(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\epsilon}\right)^2 \\
&\quad + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\epsilon\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\epsilon^2},
\end{aligned}$$

where $\mathcal{G}_i(\cdot,\epsilon)$ is a R-dimensional column vector such that its r-th element $\mathcal{G}_i^{(r)}(\cdot,\epsilon)$ is equal to

$$\mathcal{G}_i^{(r)}(\cdot,\epsilon) = \frac{\partial\theta(\cdot,\epsilon)'}{\partial\epsilon}\frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta\partial\theta'}\frac{\partial\theta(\cdot,\epsilon)}{\partial\epsilon},$$

and $h_i^{(r)}(\cdot,\epsilon)$ denotes the r-th element of h_i.

Evaluating each term of (14) at $\epsilon = 0$, and noting that $E\left[U_i^{\alpha_i}\right] = 0$, we obtain

$$\begin{aligned}
0 &= \frac{1}{n}\sum_{i=1}^n E\left[\frac{\partial U_i}{\partial\theta'}\right]\theta^{\epsilon\epsilon}(0) \\
&\quad + \frac{2}{n}\sum_{i=1}^n \alpha_i^\epsilon \cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\theta^\epsilon(0) + \frac{2}{n}\sum_{i=1}^n \alpha_i^\epsilon\left(\theta^\epsilon(0)'\alpha_i^\theta\right)\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right] \\
&\quad + \mathcal{G} + \frac{2}{n}\sum_{i=1}^n \theta^\epsilon(0)'\alpha_i^\theta \cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\theta^\epsilon(0) + \frac{1}{n}\sum_{i=1}^n\left(\theta^\epsilon(0)'\alpha_i^\theta\right)^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right] \\
&\quad + \frac{1}{n}\sum_{i=1}^n\left(\alpha_i^\epsilon\right)^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right] \\
&\quad + \frac{2}{n}\sum_{i=1}^n\left(\int\frac{\partial U_i}{\partial\theta'}d\Delta_{iT}\right)\theta^\epsilon(0) + \frac{2}{n}\sum_{i=1}^n\left(\theta^\epsilon(0)'\alpha_i^\theta\right)\cdot\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT} + \frac{2}{n}\sum_{i=1}^n\alpha_i^\epsilon\cdot\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT}
\end{aligned}$$

where

$$\mathcal{G}=\left[\begin{array}{c}\theta^{\epsilon}(0)'\left(\frac{1}{n}\sum_{i=1}^{n}E\left[\frac{\partial^2 U_i^{(1)}}{\partial\theta\partial\theta'}\right]\right)\theta^{\epsilon}(0)\\ \vdots\\ \theta^{\epsilon}(0)'\left(\frac{1}{n}\sum_{i=1}^{n}E\left[\frac{\partial^2 U_i^{(R)}}{\partial\theta\partial\theta'}\right]\right)\theta^{\epsilon}(0)\end{array}\right]$$

from which we obtain

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)\theta^{\epsilon\epsilon}(0) &= \frac{1}{n}\sum_{i=1}^{n}(\alpha_i^{\epsilon})^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right]+\frac{2}{n}\sum_{i=1}^{n}\alpha_i^{\epsilon}\cdot\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT}\\ &+2\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\epsilon}\cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\right)\theta^{\epsilon}(0)+\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\epsilon}E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right]\left(\alpha_i^{\theta}\right)'\right)\theta^{\epsilon}(0)\\ &+2\left(\frac{1}{n}\sum_{i=1}^{n}\int\frac{\partial U_i}{\partial\theta'}d\Delta_{iT}\right)\theta^{\epsilon}(0)+\left(\frac{1}{n}\sum_{i=1}^{n}\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT}\cdot\left(\alpha_i^{\theta}\right)'\right)\theta^{\epsilon}(0)\\ &+\mathcal{G}+\frac{2}{n}\sum_{i=1}^{n}\theta^{\epsilon}(0)'\alpha_i^{\theta}\cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\theta^{\epsilon}(0)+\frac{1}{n}\sum_{i=1}^{n}\left(\theta^{\epsilon}(0)'\alpha_i^{\theta}\right)^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right]\end{aligned}\tag{21}$$

Because

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}(\alpha_i^{\epsilon})^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right] &= \frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\alpha_i\alpha_i}\right]\left(\frac{\sum_{t=1}^{T}V_{it}}{\sqrt{T}E\left[V_i^2\right]}\right)^2\\ \sum_{i=1}^{n}\alpha_i^{\epsilon}\cdot\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{t=1}^{T}V_{it}}{\sqrt{T}E\left[V_i^2\right]}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(U_i^{\alpha_i}-E\left[U_i^{\alpha_i}\right]\right)\right)\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\epsilon}\cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\right)\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\\ \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\epsilon}E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right]\left(\alpha_i^{\theta}\right)'\right)\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\\ \left(\frac{1}{n}\sum_{i=1}^{n}\int\frac{\partial U_i}{\partial\theta'}d\Delta_{iT}\right)\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\\ \left(\frac{1}{n}\sum_{i=1}^{n}\int\frac{\partial U_i}{\partial\alpha_i}d\Delta_{iT}\cdot\left(\alpha_i^{\theta}\right)'\right)\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\theta^{\epsilon}(0)'\left(\frac{1}{n}\sum_{i=1}^{n}E\left[\frac{\partial^2 U_i^{(r)}}{\partial\theta\partial\theta'}\right]\right)\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O(1)\,O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\\ \frac{1}{n}\sum_{i=1}^{n}\theta^{\epsilon}(0)'\alpha_i^{\theta}\cdot E\left[\frac{\partial^2 U_i}{\partial\theta'\partial\alpha_i}\right]\theta^{\epsilon}(0) &= O_p\left(\frac{1}{\sqrt{n}}\right)O(1)\,O_p\left(\frac{1}{\sqrt{n}}\right)=O_p\left(\frac{1}{n}\right)\\ \frac{1}{n}\sum_{i=1}^{n}\left(\theta^{\epsilon}(0)'\alpha_i^{\theta}\right)^2\cdot E\left[\frac{\partial^2 U_i}{\partial\alpha_i^2}\right] &= O_p\left(\frac{1}{\sqrt{n}}\right)^2O(1)=O_p\left(\frac{1}{n}\right)\end{aligned}$$

we may write

$$
\begin{aligned}
\theta^{\epsilon\epsilon}(0) &= \left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\alpha_i\alpha_i}\right]\left(\frac{\sum_{t=1}^{T}V_{it}}{\sqrt{T}E\left[V_i^2\right]}\right)^2 \\
&\quad +2\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{t=1}^{T}V_{it}}{\sqrt{T}E\left[V_i^2\right]}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(U_i^{\alpha_i}-E\left[U_i^{\alpha_i}\right]\right)\right) \\
&\quad +O_p\left(\frac{1}{n}\right) \\
&= 2\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sum_{t=1}^{T}V_{it}}{\sqrt{T}E\left[V_i^2\right]}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(U_i^{\alpha_i}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}V_{it}\right)\right]+o_p(1)
\end{aligned}
$$

Therefore, we have

$$
\sqrt{nT}\frac{1}{2}\left(\frac{1}{\sqrt{T}}\right)^2\theta^{\epsilon\epsilon}(0)=\sqrt{\frac{n}{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{V_{it}}{E\left[V_i^2\right]}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(U_i^{\alpha_i}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}V_{it}\right)\right]+o_p(1)
$$

The terms in square parentheses have joint limiting normal distributions by CLT. Their product is a quadratic form with mean

$$
\frac{E\left[V_{it}U_i^{\alpha_i}\right]}{E\left[V_i^2\right]}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}=\frac{E\left[V_{it}U_i^{\alpha_i}\right]}{E\left[V_i^2\right]}-\frac{E\left[V_{it}U_i^{\alpha_i}\right]}{2E\left[V_i^2\right]}=\frac{E\left[V_{it}U_i^{\alpha_i}\right]}{2E\left[V_i^2\right]}=-\frac{E\left[V_{2it}U_i\right]}{2E\left[V_i^2\right]}.
$$

Therefore, we have

$$
\sqrt{nT}\frac{1}{2}\left(\frac{1}{\sqrt{T}}\right)^2\theta^{\epsilon\epsilon}(0)=\frac{1}{2}\sqrt{\frac{n}{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right)^{-1}\left(-\frac{1}{n}\sum_{i=1}^{n}\frac{E\left[V_{2it}U_i\right]}{E\left[V_i^2\right]}\right)+o_p(1)
$$

D Remainder in (10)

D.1 $\alpha_i^{\theta\theta}$, $\alpha_i^{\theta\epsilon}$, and $\alpha_i^{\epsilon\epsilon}$

Second order differentiation $\left(\frac{\partial^2}{\partial\theta^2},\frac{\partial^2}{\partial\theta\partial\epsilon},\frac{\partial^2}{\partial\epsilon^2}\right)$ of (18) yields

$$
\begin{aligned}
0 &= \int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\theta\partial\theta'}dF_i(\epsilon)+\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i\partial\theta'}dF_i(\epsilon)\right)+\left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i\partial\theta}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta'} \\
&\quad +\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\theta'}+\left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta'},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\theta\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad +\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\epsilon}+\left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2}dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad +\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\theta}d\Delta_{iT}+\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta},
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2}+\left(\int\frac{\partial^2V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2}dF_i(\epsilon)\right)\left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 \\
&\quad +2\left(\int\frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i}d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}.
\end{aligned}
$$

These three equalities characterizes $\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\theta'}$, $\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\epsilon}$, and $\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2}$.

D.2 Some Lemmas

Lemma 4 *Suppose that, for each i, $\{\xi_{it}, t = 1, 2, \ldots\}$ is a sequence of zero mean i.i.d. random variables. We assume that $\{\xi_{it}, t = 1, 2, 3\}$ are independent across i. We also assume that $\max_i E\left[|\xi_{it}|^{16}\right] < \infty$. Finally, assume that $n = O(T)$. We then have*

$$\Pr\left[\max_i \left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right| > \eta\right] = o\left(\frac{1}{T}\right)$$

for every $\eta > 0$.

Proof. The conclusion follows from combining Lemma 2 with the observation that

$$T\Pr\left[\max_i \left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right| > \eta\right] \leq T\sum_{i=1}^{n}\Pr\left[\left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right| > \eta\right] \leq \max_i T^2\Pr\left[\left|\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right| > \eta\right].$$

■

Lemma 5 *Suppose that, for each i, $\{\xi_{it}(\phi), t = 1, 2, \ldots\}$ is a sequence of zero mean i.i.d. random variables indexed by some parameter $\phi \in \Phi$. We assume that $\{\xi_{it}(\phi), t = 1, 2, 3\}$ are independent across i. We also assume that $\sup_{\phi\in\Phi}|\xi_{it}(\phi)| \leq B_{it}$ for some sequence of random variables B_{it} that is i.i.d. across t and independent across i. Finally, we assume that $\max_i E\left[|B_{it}|^{64}\right] < \infty$, and $n = O(T)$. We then have*

$$\Pr\left[\max_i \left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{1}{10}-\upsilon}\right] = o\left(\frac{1}{T}\right)$$

for every υ such that $0 \leq \upsilon < \frac{1}{160}$. Here, $\{\phi_i\}$ is an arbitrary sequence in Φ.

Proof. This proof is a modification of Hall and Horowitz (1996, Lemma 1). Note that

$$T^2\Pr\left[\max_{1\leq t\leq T}|B_{it}| > T^{1/16}\right] \leq T^2\sum_{t=1}^{T}\Pr\left[|B_{it}| > T^{1/16}\right] = T^3\Pr\left[|B_{it}| > T^{1/16}\right]$$
$$\leq T^3\frac{E\left[|B_{it}|^{64}\right]}{\left(T^{1/16}\right)^{64}} \leq \frac{\max_i E\left[|B_{it}|^{64}\right]}{T} = o(1).$$

Condition on $\max_{1\leq t\leq T}|B_{it}| \leq T^{1/16}$. By Markov's inequality,

$$\Pr\left[\sup_{\phi\in\Phi}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{1}{10}-\upsilon}\right] = \Pr\left[\sup_{\phi\in\Phi}\left|\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{3}{5}-\upsilon}\right]$$
$$\leq \frac{E\left[\sup_{\phi\in\Phi}\left|\sum_{t=1}^{T}\xi_{it}(\phi_i)\right|^{64}\right]}{T^{\frac{3}{5}\times 64-64\upsilon}\eta^{64}} = \frac{\sup_{\phi\in\Phi}E\left[\left|\sum_{t=1}^{T}\xi_{it}(\phi_i)\right|^{64}\right]}{T^{\frac{192}{5}-64\upsilon}\eta^{64}} \leq \frac{E\left[\left|\sum_{t=1}^{T}B_{it}\right|^{64}\right]}{T^{\frac{192}{5}-64\upsilon}\eta^{64}},$$

where the last equality is based on dominated convergence. By Lahiri (1992, Lemma 5.1), we have

$$E\left[\left|\sum_{t=1}^{T}B_{it}\right|^{64}\right] \leq CT^{36},$$

where $C > 0$ is a constant. Therefore, we have

$$T^2 \Pr\left[\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{1}{10}-v}\right] \leq T^2\frac{CT^{36}}{T^{\frac{192}{5}-64v}\eta^{64}} = O\left(T^{-\frac{2}{5}+64v}\right),$$

and

$$T\Pr\left[\max_i\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{1}{10}-v}\right] \leq T\sum_{i=1}^{n}\Pr\left[\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\xi_{it}(\phi_i)\right| > T^{\frac{1}{10}-v}\right] = nT\cdot\frac{CT^{36}}{T^{\frac{192}{5}-64v}\eta^{64}} = o(1).$$

■

Lemma 6 *Under Conditions 1, 2, and 3,*

$$T\Pr\left[\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\widehat{\theta}(\epsilon)-\theta_0\right| \geq \eta\right] = o(1)$$

and

$$T\Pr\left[\max_{1\leq i\leq n}\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\widehat{\alpha}_i(\epsilon)-\alpha_{i0}\right| \geq \eta\right] = o(1)$$

for every $\eta > 0$.

Proof. Only the first assertion is proved. The second assertion can be proved similarly. Let η be given, and let $\varepsilon \equiv \inf_i\left[G_{(i)}(\theta_0,\alpha_{i0}) - \sup_{\{(\theta,\alpha):|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta\}} G_{(i)}(\theta,\alpha)\right] > 0$. Recall that

$$F(\epsilon) \equiv F + \epsilon\sqrt{T}\left(\widehat{F}-F\right), \qquad \epsilon \in \left[0,\frac{1}{\sqrt{T}}\right]$$

We have

$$\int g(\cdot;\theta,\alpha_i(\theta))\,dF_i(\epsilon) = \left(1-\epsilon\sqrt{T}\right)G_{(i)}(\theta,\alpha_i) + \epsilon\sqrt{T}\widehat{G}_{(i)}(\theta,\alpha_i)$$

and

$$\left|\int g(\cdot;\theta,\alpha_i(\theta))\,dF_i(\epsilon) - G_{(i)}(\theta,\alpha_i)\right| \leq \left(1-\epsilon\sqrt{T}\right)\left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right| \leq \left|\widehat{G}_{(i)}(\theta,\alpha)-G_{(i)}(\theta,\alpha)\right|.$$

By Lemma 3, we have

$$\Pr\left[\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\max_{1\leq i\leq n}\sup_{(\theta,\alpha)}\left|\int g(\cdot;\theta,\alpha_i(\theta))\,dF_i(\epsilon) - G_{(i)}(\theta,\alpha)\right| \geq \eta\right] = o\left(T^{-1}\right)$$

Therefore, for every $0 \leq \epsilon \leq \frac{1}{\sqrt{T}}$ with probability equal to $1 - o\left(\frac{1}{T}\right)$, we have

$$\begin{aligned}
\max_{|\theta-\theta_0|>\eta,\alpha_1,\ldots,\alpha_n} n^{-1}\sum_{i=1}^{n}\int g(\cdot;\theta,\alpha_i(\theta))\,dF_i(\epsilon) &\leq \max_{|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta} n^{-1}\sum_{i=1}^{n}\int g(\cdot;\theta,\alpha_i(\theta))\,dF_i(\epsilon) \\
&< \max_{|(\theta,\alpha)-(\theta_0,\alpha_{i0})|>\eta} n^{-1}\sum_{i=1}^{n}G_{(i)}(\theta,\alpha_i) + \frac{1}{3}\varepsilon \\
&< n^{-1}\sum_{i=1}^{n}G_{(i)}(\theta_0,\alpha_{i0}) - \frac{2}{3}\varepsilon \\
&< n^{-1}\sum_{i=1}^{n}\int g(\cdot;\theta_0,\alpha_{i0})\,dF_i(\epsilon) - \frac{1}{3}\varepsilon.
\end{aligned}$$

We also have

$$\max_{\theta,\alpha_1,\ldots,\alpha_n} n^{-1}\sum_{i=1}^{n}\int g\left(\cdot;\theta,\alpha_i\right)dF_i\left(\epsilon\right)\geq n^{-1}\sum_{i=1}^{n}\int g\left(\cdot;\theta_0,\alpha_{i0}\right)dF_i\left(\epsilon\right)$$

by definition. It follows that

$$\max_{|\theta-\theta_0|>\eta,\alpha_1,\ldots,\alpha_n} n^{-1}\sum_{i=1}^{n}\int g\left(\cdot;\theta,\alpha_i\left(\theta\right)\right)dF_i\left(\epsilon\right)<\max_{\theta,\alpha_1,\ldots,\alpha_n} n^{-1}\sum_{i=1}^{n}\int g\left(\cdot;\theta,\alpha_i\right)dF_i\left(\epsilon\right)-\frac{1}{3}\varepsilon$$

for every $0\leq\epsilon\leq\frac{1}{\sqrt{T}}$. We therefore obtain that $\Pr\left[\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\widehat{\theta}\left(\epsilon\right)-\theta_0\right|\geq\eta\right]=o\left(\frac{1}{T}\right)$. ■

Lemma 7 *Assume that Condition 4 holds. Suppose that* $K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)$ *is equal to*

$$\frac{\partial^{m_1+m_2}\log f\left(x_{it};\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)}{\partial\theta^{m_1}\partial\alpha_i^{m_2}}$$

for some $m_1+m_2\leq 1,\ldots,5$*. Then, for any* $\eta>0$*, we have*

$$\Pr\left[\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\frac{1}{n}\sum_{i=1}^{n}\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)dF_i\left(\epsilon\right)-\frac{1}{n}\sum_{i=1}^{n}E\left[K_i\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|>\eta\right]=o\left(\frac{1}{T}\right)$$

and

$$\Pr\left[\max_i\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)dF_i\left(\epsilon\right)-E\left[K_i\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|>\eta\right]=o\left(\frac{1}{T}\right).$$

Also,

$$\Pr\left[\max_i\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)d\Delta_{iT}\right|>CT^{\frac{1}{10}-\upsilon}\right]=o\left(\frac{1}{T}\right)$$

for some constant $C>0$ *and* υ *such that* $0\leq\upsilon<\frac{1}{160}$*.*

Proof. Note that we may write

$$\begin{aligned}
&\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)dF_i\left(\epsilon\right)-\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)dF_i\\
=&\int K_i\left(\cdot;\theta\left(\epsilon\right),\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)\right)dF_i\left(\epsilon\right)-\int K_i\left(\cdot;\theta\left(0\right),\alpha_i\left(\theta\left(0\right),0\right)\right)dF_i\left(\epsilon\right)\\
&+\int K_i\left(\cdot;\theta\left(0\right),\alpha_i\left(\theta\left(0\right),0\right)\right)dF_i\left(\epsilon\right)-\int K_i\left(\cdot;\theta\left(0\right),\alpha_i\left(\theta\left(0\right),0\right)\right)dF_i\\
=&\int\frac{\partial K_i\left(\cdot;\theta^*,\alpha_i^*\right)}{\partial\theta'}\cdot\left(\theta\left(\epsilon\right)-\theta\right)dF_i\left(\epsilon\right)\\
&+\int\frac{\partial K_i\left(\cdot;\theta^*,\alpha_i^*\right)}{\partial\alpha_i}\cdot\left(\alpha_i\left(\theta\left(\epsilon\right),\epsilon\right)-\alpha_i\right)dF_i\left(\epsilon\right)\\
&+\epsilon\sqrt{T}\int K_i\left(\cdot;\theta\left(0\right),\alpha_i\left(\theta\left(0\right),0\right)\right)d\left(\widehat{F}_i-F_i\right)
\end{aligned}$$

where (θ^*, α_i^*) are between (θ, α_i) and $(\theta(\epsilon), \alpha_i(\theta(\epsilon), \epsilon))$. Therefore, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}\int K_i(\cdot;\theta(\epsilon),\alpha_i(\theta(\epsilon),\epsilon))\,dF_i(\epsilon) - \frac{1}{n}\sum_{i=1}^{n}E\left[K_i(x_{it};\theta_0,\alpha_{i0})\right]\right| \\
\leq\; & |\theta(\epsilon)-\theta|\cdot\frac{1}{n}\sum_{i=1}^{n}\left(E\left[M(x_{it})\right]+\frac{1}{T}\sum_{t=1}^{T}M(x_{it})\right) \\
&+\left(\frac{1}{n}\sum_{i=1}^{n}(\alpha_i(\theta(\epsilon),\epsilon)-\alpha_i)^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(E\left[M(x_{it})\right]+\frac{1}{T}\sum_{t=1}^{T}M(x_{it})\right)^2\right)^{1/2} \\
&+\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}M(x_{it})-E\left[M(x_{it})\right]\right)\right|
\end{aligned}$$

where $M(\cdot)$ is defined in Condition 4. Using Lemmas 4 and 6, we can bound

$$\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\frac{1}{n}\sum_{i=1}^{n}\int K_i(\cdot;\theta(\epsilon),\alpha_i(\theta(\epsilon),\epsilon))\,dF_i(\epsilon)-\frac{1}{n}\sum_{i=1}^{n}E\left[K_i(x_{it};\theta_0,\alpha_{i0})\right]\right|$$

in absolute value by some $\eta > 0$ with probability $1 - o\left(\frac{1}{T}\right)$. Because

$$\begin{aligned}
&\left|\int K_i(\cdot;\theta(\epsilon),\alpha_i(\theta(\epsilon),\epsilon))\,dF_i(\epsilon)-E\left[K_i(x_{it};\theta_0,\alpha_{i0})\right]\right| \\
\leq\; & |\theta(\epsilon)-\theta|\cdot\left(E\left[M(x_{it})\right]+\frac{1}{T}\sum_{t=1}^{T}M(x_{it})\right) \\
&+|\alpha_i(\theta(\epsilon),\epsilon)-\alpha_i|\cdot\left(E\left[M(x_{it})\right]+\frac{1}{T}\sum_{t=1}^{T}M(x_{it})\right) \\
&+\left|\frac{1}{T}\sum_{t=1}^{T}M(x_{it})-E\left[M(x_{it})\right]\right|,
\end{aligned}$$

we can bound

$$\max_i\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\int K_i(\cdot;\theta(\epsilon),\alpha_i(\theta(\epsilon),\epsilon))\,dF_i(\epsilon)-E\left[K_i(x_{it};\theta_0,\alpha_{i0})\right]\right|$$

in absolute value by some $\eta > 0$ with probability $1 - o\left(\frac{1}{T}\right)$.

Using Condition 4 and Lemmas 5, we can also show that

$$\max_i\left|\int K_i(\cdot;\theta(\epsilon),\alpha_i(\theta(\epsilon),\epsilon))\,d\Delta_{iT}\right|$$

can be bounded by in absolute value by $CT^{\frac{1}{10}-\upsilon}$ for some constant $C > 0$ and υ such that $0 \leq \upsilon < \frac{1}{160}$ with probability $1 - o\left(\frac{1}{T}\right)$. ■

D.3 Bounds

Lemma 8 *Suppose that Condition 4 holds. Then, we have*

$$\begin{aligned}
\Pr\left[\max_i\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\alpha_i^{\theta}(\epsilon)\right|>C\right] &= o\left(T^{-1}\right) \\
\Pr\left[\max_i\max_{0\leq\epsilon\leq\frac{1}{\sqrt{T}}}\left|\alpha_i^{\epsilon}(\epsilon)\right|>CT^{\frac{1}{10}-\upsilon}\right] &= o\left(T^{-1}\right)
\end{aligned}$$

for some constant $C > 0$ and υ such that $0 \leq \upsilon < \frac{1}{160}$.

Proof. From Appendix C.2, we obtain

$$\begin{aligned}\frac{\partial \alpha_i\left(\theta, F_i\left(\epsilon\right)\right)}{\partial \theta} &= -\left(\int \frac{\partial V_i\left(\cdot, \theta, \epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right)^{-1}\left(\int \frac{\partial V_i\left(\cdot, \theta, \epsilon\right)}{\partial \theta} dF_i\left(\epsilon\right)\right), \\ \frac{\partial \alpha_i\left(\theta, F_i\left(\epsilon\right)\right)}{\partial \epsilon} &= -\left(\int \frac{\partial V_i\left(\cdot, \theta, \epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right)^{-1}\left(\int V_i\left(\cdot, \theta, \epsilon\right) d\Delta_{iT}\right).\end{aligned}$$

Using Lemma 7 and Condition 4, we can see that

$$\left(\int \frac{\partial V_i\left(\cdot, \theta, \epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right)^{-1}$$

is uniformly bounded away from zero. We can also see that

$$\int \frac{\partial V_i\left(\cdot, \theta, \epsilon\right)}{\partial \theta} dF_i\left(\epsilon\right)$$

is uniformly bounded by some constant C, and

$$\int V_i\left(\cdot, \theta, \epsilon\right) d\Delta_{iT}$$

is uniformly bounded by $CT^{\frac{1}{10}-\upsilon}$ for some constant C and υ such that $0 \leq \upsilon < \frac{1}{160}$. ■

Lemma 9 *Suppose that Condition 4 holds. Then, we have*

$$\Pr\left[\max_{0 \leq \epsilon \leq \frac{1}{\sqrt{T}}} \left|\theta^{\epsilon}\left(\epsilon\right)\right| > CT^{\frac{1}{10}-\upsilon}\right] = o\left(T^{-1}\right)$$

for some constant $C > 0$ and υ such that $0 \leq \upsilon < \frac{1}{160}$.

Proof. From (16), we have

$$\begin{aligned}\theta^{\epsilon}\left(\epsilon\right) = & -\left[\frac{1}{n}\sum_{i=1}^{n}\int\left(\frac{\partial h_i\left(\cdot, \epsilon\right)}{\partial \theta'} + \frac{\partial h_i\left(\cdot, \epsilon\right)}{\partial \alpha_i}\frac{\partial \alpha_i}{\partial \theta'}\right) dF_i\left(\epsilon\right)\right]^{-1} \\ & \left[\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \alpha_i}{\partial \epsilon}\left(\int \frac{\partial h_i\left(\cdot, \epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right) + \frac{1}{n}\sum_{i=1}^{n}\int h_i\left(\cdot, \epsilon\right) d\Delta_{iT}\right]\end{aligned}$$

Using Lemmas 7, 8, and Condition 4, we can bound the denominator of $\theta^{\epsilon}\left(\epsilon\right)$ by some $C > 0$, and the numerator by some $CT^{\frac{1}{10}-\upsilon}$ with probability $1 - o\left(T^{-1}\right)$. ■

Lemma 10 *Suppose that Condition 4 holds. Then, we have*

$$\begin{aligned}\Pr\left[\max_i \max_{0 \leq \epsilon \leq \frac{1}{\sqrt{T}}} \left|\alpha_i^{\theta_r \theta_{r'}}\left(\epsilon\right)\right| > C\right] &= o\left(T^{-1}\right) \\ \Pr\left[\max_i \max_{0 \leq \epsilon \leq \frac{1}{\sqrt{T}}} \left|\alpha_i^{\theta_r \epsilon}\left(\epsilon\right)\right| > CT^{\frac{1}{10}-\upsilon}\right] &= o\left(T^{-1}\right) \\ \Pr\left[\max_i \max_{0 \leq \epsilon \leq \frac{1}{\sqrt{T}}} \left|\alpha_i^{\epsilon\epsilon}\left(\epsilon\right)\right| > C\left(T^{\frac{1}{10}-\upsilon}\right)^2\right] &= o\left(T^{-1}\right)\end{aligned}$$

for some constant $C > 0$ and υ such that $0 \leq \upsilon < \frac{1}{160}$. Here, $\alpha_i^{\theta_r \theta_{r'}} \equiv \frac{\partial^2 \alpha_i}{\partial \theta_r \partial \theta_{r'}}$. We similarly define $\alpha_i^{\theta_r \epsilon}$.

Proof. From Appendix D.1, we have

$$
\begin{aligned}
0 &= \int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta\partial\theta'} dF_i(\epsilon) + \frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i\partial\theta'} dF_i(\epsilon)\right) + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i\partial\theta} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta'} \\
&\quad + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\theta'} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta'},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta\partial\epsilon} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad + \int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\theta} d\Delta_{iT} + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta},
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 \\
&\quad +2\left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}.
\end{aligned}
$$

The result then follows by applying the same argument as in the proof of Lemma 8. ■

Lemma 11 *Suppose that Condition 4 holds. Then, we have*

$$
\Pr\left[\max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\theta^{\epsilon\epsilon}(\epsilon)\right| > C\left(T^{\frac{1}{10}-\upsilon}\right)^2\right] = o\left(T^{-1}\right)
$$

for some constant $C>0$ and υ such that $0\le\upsilon<\frac{1}{160}$.

Proof. The conclusion follows by using the characterization of $\theta^{\epsilon\epsilon}(\epsilon)$ in Appendix C.3, and Lemmas 7, 8, 9, and 10. ■

Lemma 12 *Suppose that Condition 4 holds. Then, we have*

$$
\begin{aligned}
\Pr\left[\max_i \max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\alpha_i^{\theta_r\theta_{r'}\theta_{r''}}(\epsilon)\right| > C\right] &= o\left(T^{-1}\right) \\
\Pr\left[\max_i \max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\alpha_i^{\theta_r\theta_{r'}\epsilon}(\epsilon)\right| > CT^{\frac{1}{10}-\upsilon}\right] &= o\left(T^{-1}\right) \\
\Pr\left[\max_i \max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\alpha_i^{\theta_r\epsilon\epsilon}(\epsilon)\right| > C\left(T^{\frac{1}{10}-\upsilon}\right)^2\right] &= o\left(T^{-1}\right) \\
\Pr\left[\max_i \max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\alpha_i^{\epsilon\epsilon\epsilon}(\epsilon)\right| > C\left(T^{\frac{1}{10}-\upsilon}\right)^3\right] &= o\left(T^{-1}\right)
\end{aligned}
$$

for some constant $C>0$ and υ such that $0\le\upsilon<\frac{1}{160}$.

Proof. It was seen in Appendix D.1 that

$$
\begin{aligned}
0 &= \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
&\quad + \int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\theta_r} d\Delta_{iT} + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r},
\end{aligned}
$$

and

$$
\begin{aligned}
0 \quad = \quad & \left(\int \frac{\partial V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right)\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \epsilon^2} + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2} dF_i\left(\epsilon\right)\right)\left(\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \epsilon}\right)^2 \\
& +2\left(\int \frac{\partial V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i} d\Delta_{iT}\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \epsilon}.
\end{aligned}
$$

We therefore obtain

$$
\begin{aligned}
0 \quad = \quad & \int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \theta_r \partial \theta_{r'} \partial \theta_{r''}} dF_i\left(\epsilon\right) + \left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_r \partial \theta_{r'}} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r''}} \\
& + \frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r \partial \theta_{r''}}\left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_{r'}} dF_i\left(\epsilon\right)\right) + \frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r}\left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_{r'} \partial \theta_{r''}} dF_i\left(\epsilon\right)\right) \\
& + \frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r''}}\left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2 \partial \theta_{r'}} dF_i\left(\epsilon\right)\right) \\
& + \left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_r \partial \theta_{r''}} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'}} + \left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2 \partial \theta_r} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'}}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r''}} \\
& + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_r} dF_i\left(\epsilon\right)\right)\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'} \partial \theta_{r''}} \\
& + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i \partial \theta_{r''}} dF_i\left(\epsilon\right)\right)\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r \partial \theta_{r'}} + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r''}}\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r \partial \theta_{r'}} \\
& + \left(\int \frac{\partial V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i} dF_i\left(\epsilon\right)\right)\frac{\partial^3 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r \partial \theta_{r'} \partial \theta_{r''}} \\
& + \left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2 \partial \theta_{r''}} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'}} \\
& + \left(\int \frac{\partial^3 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^3} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'}}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r''}} \\
& + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2} dF_i\left(\epsilon\right)\right)\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r \partial \theta_{r''}}\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'}} \\
& + \left(\int \frac{\partial^2 V_i\left(\cdot,\theta,\epsilon\right)}{\partial \alpha_i^2} dF_i\left(\epsilon\right)\right)\frac{\partial \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_r}\frac{\partial^2 \alpha_i\left(\theta,F_i\left(\epsilon\right)\right)}{\partial \theta_{r'} \partial \theta_{r''}},
\end{aligned}
$$

which characterizes $\frac{\partial^3 \alpha_i(\theta, F_i(\epsilon))}{\partial\theta_r \partial\theta_{r'} \partial\theta_{r''}}$,

$$\begin{aligned}
0 \;=\; & \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\theta_{r'}\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
& + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}\partial\epsilon} \\
& + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_{r'}\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}}\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon} \\
& + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^3\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\theta_{r'}\partial\epsilon} \\
& + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\theta_{r'}\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
& + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^3} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
& + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\theta_{r'}}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
& + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}\partial\epsilon} \\
& + \int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\theta_{r'}} d\Delta_{iT} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}} \\
& + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_{r'}\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_{r'}} \\
& + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\theta_{r'}},
\end{aligned}$$

which characterizes $\frac{\partial^3 \alpha_i(\theta, F_i(\epsilon))}{\partial\theta_r \partial\theta_{r'} \partial\epsilon}$,

$$\begin{aligned}
0 \;=\; & \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} \\
& + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right)\frac{\partial^3\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon^2} \\
& + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i^2} dF_i(\epsilon)\right)\left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^3} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 \\
& +2\left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon} \\
& +2\left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\theta_r\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} + 2\left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} d\Delta_{iT}\right)\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r}\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \\
& +2\left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right)\frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\theta_r\partial\epsilon},
\end{aligned}$$

which characterizes $\frac{\partial^3 \alpha_i(\theta, F_i(\epsilon))}{\partial\theta_r \partial\epsilon^2}$, and

$$
\begin{aligned}
0 \;=\; & \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right) \frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right) \frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} \\
& + \left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} dF_i(\epsilon)\right) \frac{\partial^3\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^3} \\
& + \left(\int \frac{\partial^3 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^3} dF_i(\epsilon)\right) \left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^3 + \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} d\Delta_{iT}\right) \left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 \\
& +2 \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} dF_i(\epsilon)\right) \frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon} \frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2} \\
& +2 \left(\int \frac{\partial^2 V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i^2} d\Delta_{iT}\right) \left(\frac{\partial\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon}\right)^2 + 2\left(\int \frac{\partial V_i(\cdot,\theta,\epsilon)}{\partial\alpha_i} d\Delta_{iT}\right) \frac{\partial^2\alpha_i(\theta,F_i(\epsilon))}{\partial\epsilon^2},
\end{aligned}
$$

which characterizes $\frac{\partial^3 \alpha_i(\theta, F_i(\epsilon))}{\partial\epsilon^3}$. Inspecting these derivatives and applying Lemmas 7, 8, and 10, we obtain the desired result. ■

Theorem 6 *Suppose that Condition 4 holds. Then, we have*

$$
\Pr\left[\max_{0\le\epsilon\le\frac{1}{\sqrt{T}}} \left|\theta^{\epsilon\epsilon\epsilon}(\epsilon)\right| > C\left(T^{\frac{1}{10}-\upsilon}\right)^3\right] = o\left(T^{-1}\right)
$$

for some constant $C > 0$ *and* υ *such that* $0 \le \upsilon < \frac{1}{160}$.

Proof. From (15), we have

$$
0 = \frac{1}{n}\sum_{i=1}^{n} \int \frac{d^3 h_i(\cdot,\epsilon)}{d\epsilon^3} dF_i(\epsilon) + 3\frac{1}{n}\sum_{i=1}^{n} \int \frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2} d\Delta_{iT}
$$

Combining Lemmas 7, 8, 9, 10, and 11, we can bound $\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2} d\Delta_{iT}$ by $C\left(T^{\frac{1}{10}-\upsilon}\right)^3$ with probability $1 - o\left(T^{-1}\right)$. It was seen in Appendix C.3 that the r-th component $\frac{d^2 h_i^{(r)}(\cdot,\epsilon)}{d\epsilon^2}$of $\frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2}$ is equal to

$$
\begin{aligned}
\frac{d^2 h_i^{(r)}(\cdot,\epsilon)}{d\epsilon^2} \;=\; & \frac{\partial\theta(\cdot,\epsilon)'}{\partial\epsilon} \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta\partial\theta'} \frac{\partial\theta(\cdot,\epsilon)}{\partial\epsilon} + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\theta}{\partial\epsilon} + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\alpha_i}{\partial\epsilon}\frac{\partial\theta}{\partial\epsilon} \\
& + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\theta'}\frac{\partial^2\theta}{\partial\epsilon^2} \\
& + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\theta}{\partial\epsilon}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)^2 + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\alpha_i}{\partial\epsilon} \\
& + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i}\left(\frac{\partial\theta'}{\partial\epsilon}\frac{\partial^2\alpha_i}{\partial\theta\partial\theta'}\right)\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\theta'\partial\epsilon}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial^2\theta}{\partial\epsilon^2} \\
& + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\theta'\partial\alpha_i}\frac{\partial\theta}{\partial\epsilon}\frac{\partial\alpha_i}{\partial\epsilon} + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\theta'}\frac{\partial\theta}{\partial\epsilon}\right)\frac{\partial\alpha_i}{\partial\epsilon} + \frac{\partial^2 h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i^2}\left(\frac{\partial\alpha_i}{\partial\epsilon}\right)^2 \\
& + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\epsilon\partial\theta'}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial h_i^{(r)}(\cdot,\epsilon)}{\partial\alpha_i}\frac{\partial^2\alpha_i}{\partial\epsilon^2}.
\end{aligned}
$$

Using Lemmas 7, 8, 9, 10, and 11 again, we can conclude that $\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^3 h_i(\cdot,\epsilon)}{d\epsilon^3} dF_i(\epsilon)$ is equal to $\left(\frac{1}{n}\sum_{i=1}^{n}\int \frac{\partial h_i(\cdot,\epsilon)}{\partial\theta'} dF_i(\epsilon)\right)\frac{\partial^3\theta}{\partial\epsilon^3}$ plus terms that can all be bounded by $\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2} d\Delta_{iT}$ by $C\left(T^{\frac{1}{10}-\upsilon}\right)^3$

with probability $1-o\left(T^{-1}\right)$. Because $\left(\frac{1}{n}\sum_{i=1}^{n}\int\frac{\partial h_i(\cdot,\epsilon)}{\partial\theta'}dF_i\left(\epsilon\right)\right)^{-1}$ is bounded away from 0 by Lemma 7 and Condition 4, we obtain the desired conclusion. ■

E Proof of Theorem 2

Note that

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)s_\theta\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right| \\
\leq\ &\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)s_\theta\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)-\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right| \\
&+\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right| \\
\leq\ &\left(\max_i E\left[M_{it}\left(x_{it}\right)^2\right]+\left|\frac{1}{T}\sum_{t=1}^{T}\left(M_{it}\left(x_{it}\right)^2-E\left[M_{it}\left(x_{it}\right)^2\right]\right)\right|\right)\left(\left|\widehat{\theta}-\theta_0\right|+\max_i\left|\widehat{\alpha}_i-\alpha_{i0}\right|\right) \\
&+\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|
\end{aligned}$$

By Lemma 2, we obtain

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}\left(M_{it}\left(x_{it}\right)^2-E\left[M_{it}\left(x_{it}\right)^2\right]\right)\right|=o_p\left(1\right)$$

and

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|=o_p\left(1\right)$$

Combined with Lemmas 4 and 5, we obtain

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)s_\theta\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|=o_p\left(1\right)$$

We can similarly show that

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}V_{it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)^2-E\left[V_{it}\left(x_{it};\theta_0,\alpha_{i0}\right)^2\right]\right|=o_p\left(1\right)$$

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}V_{2it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)V_{it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)-E\left[V_{2it}\left(x_{it};\theta_0,\alpha_{i0}\right)V_{it}\left(x_{it};\theta_0,\alpha_{i0}\right)\right]\right|=o_p\left(1\right)$$

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}s_\theta\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)^2-E\left[s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)^2\right]\right|=o_p\left(1\right)$$

$$\max_i\left|\frac{1}{T}\sum_{t=1}^{T}s_\theta\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)V_{it}\left(x_{it};\widehat{\theta},\widehat{\alpha}_i\right)-s_\theta\left(x_{it};\theta_0,\alpha_{i0}\right)V_{it}\left(x_{it};\theta_0,\alpha_{i0}\right)\right|=o_p\left(1\right)$$

It follows that

$$\left|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{U}_{it}\widehat{U}_{it}' - \frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i\right| = o_p(1)$$

$$\left|\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{T}\sum_{t=1}^{T}\widehat{U}_{it}\widehat{V}_{2it}}{\frac{1}{T}\sum_{t=1}^{T}\widehat{V}_{it}^{2}} - \frac{1}{n}\sum_{i=1}^{n}\frac{E\left[V_{2it}U_{it}\right]}{E\left[V_{it}^{2}\right]}\right| = o_p(1)$$

from which we obtain

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{U}_{it}\widehat{U}_{it}' = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i + o_p(1) \tag{22}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{1}{T}\sum_{t=1}^{T}\widehat{U}_{it}\widehat{V}_{2it}}{\frac{1}{T}\sum_{t=1}^{T}\widehat{V}_{it}^{2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E\left[V_{2it}U_{it}\right]}{E\left[V_{it}^{2}\right]} + o_p(1) \tag{23}$$

F V-Statistics

F.1 Properties of Normalized V-Statistics

Consider a statistic of the form

$$\begin{aligned} W_{i,T} &\equiv \frac{1}{T^{m/2}}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\cdots\sum_{t_m=1}^{T}k_1\left(x_{i,t_1}\right)k_2\left(x_{i,t_2}\right)\cdots k_m\left(x_{i,t_m}\right) \\ &\equiv T^{m/2}\overline{k}_{i,1}\overline{k}_{i,2}\cdots\overline{k}_{i,m} \\ &\equiv T^{m/2}K_{i,T} \end{aligned}$$

where $E\left[k_j\left(x_{i,t}\right)\right]=0$. We will call the average

$$\frac{1}{n}\sum_{i=1}^{n}W_{i,T}$$

of such $W_{i,T}$ the normalized V-statistic of order m. We will focus on normalized V-statistic up to order 4.

Condition 5 *(i) $n=O(T)$; (ii) $E\left[k_j\left(x_{i,t}\right)\right]=0$; (iii) $\left|k_j\left(x_{i,t}\right)\right|\leq CM\left(x_{i,t}\right)$ such that $\sup_i E\left[M\left(x_{i,t}\right)^8\right]<\infty$, where C denotes a generic constant.*

Lemma 13 *Suppose that Condition 5 holds. When $m=1,2,3,4$, then $\frac{1}{n}\sum_{i=1}^{n}W_{i,T}=O_p(1)$.*

Proof. Suppose that $m=1$. By Chebyshev's inequality, we have

$$\begin{aligned} \Pr\left[\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}W_{i,T}\right|\geq\mathcal{M}\right] &= \Pr\left[\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}k_1\left(x_{i,t}\right)\right)\right|\geq\mathcal{M}\right] \\ &\leq \frac{E\left[\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}k_1\left(x_{i,t}\right)\right)\right|^2\right]}{\left(\mathcal{M}\right)^2} \\ &= \frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left[k_1\left(x_{i,t}\right)^2\right]}{\left(\mathcal{M}\right)^2} \\ &\leq \frac{C^2\sup_i E\left[M\left(x_{i,t}\right)^2\right]}{\left(\mathcal{M}\right)^2} \end{aligned}$$

It therefore follows that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{i,T} = O_p(1)$, from which we obtain the desired conclusion.

Now suppose that $m = 2$. By Markov's inequality, we have

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^2 \geq \mathcal{M}\right] \leq \frac{E\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^2\right]}{\mathcal{M}}$$
$$= \frac{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left[k_1(x_{i,t})^2\right]}{\mathcal{M}}$$
$$= \frac{C^2 \sup_i E\left[M(x_{i,t})^2\right]}{\mathcal{M}}$$

It therefore follows that $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^2 = O_p(1)$. Likewise, we have $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)^2 = O_p(1)$. Because

$$\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)\right| \leq \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)^2\right)^{1/2},$$

we obtain the desired conclusion.

Now, suppose that $m = 3$. By Markov's inequality, we have

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^4 \geq \mathcal{M}\right] \leq \frac{E\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^4\right]}{\mathcal{M}}$$

Because

$$E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^4\right] = \frac{TE\left[k_1(x_{i,t})^4\right] + 6\frac{T(T-1)}{2}E\left[k_1(x_{i,t})^2\right]^2}{T^2}$$
$$= \frac{E\left[k_1(x_{i,t})^4\right]}{T} + 3\frac{(T-1)E\left[k_1(x_{i,t})^2\right]^2}{T}$$
$$\leq \frac{E\left[M(x_{i,t})^4\right]}{T} + 3\frac{(T-1)E\left[M(x_{i,t})^4\right]}{T}$$
$$< \infty$$

we have $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)^4 = O_p(1)$. Likewise, we have $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)^4 = O_p(1)$, and $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)^4 = O_p(1)$. This implies that

$$\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right|^3 = O_p(1), \quad \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right|^3 = O_p(1), \quad \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right|^3 = O_p(1)$$

by Jensen's inequality. Because

$$\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_3(x_{i,t})\right)\right|$$
$$\leq \left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_1(x_{i,t})\right|^3\right)^{1/3}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_2(x_{i,t})\right|^3\right)^{1/3}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} k_3(x_{i,t})\right|^3\right)^{1/3}$$

by repeated application of Holder's inequality, we obtain the desired conclusion for $m = 3$. We obtain the same conclusion for $m = 4$ by the same method. ■

F.2 "Jackknifing" Normalized V-Statistics

We consider the average of delete-t estimators

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} W_{i,T,(-t)} &= \frac{1}{T}\sum_{t=1}^{T} (T-1)^{m/2} K_{i,T,(-t)} \\
&= (T-1)^{m/2}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{T-1}\sum_{t_1\neq t}^{T} k_1(x_{i,t_1})\right)\left(\frac{1}{T-1}\sum_{t_2\neq t}^{T} k_2(x_{i,t_2})\right)\cdots\left(\frac{1}{T-1}\sum_{t_m\neq t}^{T} k_m(x_{i,t_m})\right) \\
&= (T-1)^{m/2}\frac{1}{T}\sum_{t=1}^{T}\frac{T\overline{k}_{i,1}-k_1(x_{i,t})}{T-1}\frac{T\overline{k}_{i,2}-k_2(x_{i,t})}{T-1}\cdots\frac{T\overline{k}_{i,m}-k_m(x_{i,t})}{T-1}
\end{aligned}$$

Lemma 14 *Suppose that Condition 5 holds. When* $m = 1$, *then* $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n} W_{i,T,(-t)}\right) = \frac{(T-1)^{1/2}}{T^{1/2}}\frac{1}{n}\sum_{i=1}^{n} W_{i,T}$. *When* $m = 2$, *then* $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n} W_{i,T,(-t)}\right) = \frac{T-2}{T-1}\frac{1}{n}\sum_{i=1}^{n} W_{i,T} + \frac{1}{T(T-1)}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t})$. *When* $m = 3$ *or* 4, *then* $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n} W_{i,T,(-t)}\right) = \frac{1}{n}\sum_{i=1}^{n} W_{i,T} + o_p\left(\frac{1}{\sqrt{T}}\right)$.

Proof. Lemma 14 would follow if (i) when $m = 1$,

$$\frac{1}{T}\sum_{t=1}^{T} W_{i,T,(-t)} = \frac{1}{T}\sum_{t=1}^{T}(T-1)^{1/2} K_{i,T,(-t)} = (T-1)^{1/2} K_{i,T} = \frac{(T-1)^{1/2}}{T^{1/2}} W_{i,T}$$

(ii) when $m = 2$,

$$\frac{1}{T}\sum_{t=1}^{T} W_{i,T,(-t)} = \frac{T-2}{T-1} W_{i,T} + \frac{1}{T(T-1)}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t})$$

(iii) when $m = 3$, we have

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} W_{i,T,(-t)} = &\ \frac{T^3-3T^2}{T^{3/2}(T-1)^{3/2}} W_{i,T} \\
&+ \frac{T^{1/2}}{(T-1)^{3/2}}\left(T^{1/2}\overline{k}_{i,1}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_2(x_{i,t}) k_3(x_{i,t})\right) \\
&+ \frac{T^{1/2}}{(T-1)^{3/2}}\left(T^{1/2}\overline{k}_{i,2}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_3(x_{i,t}) k_1(x_{i,t})\right) \\
&+ \frac{T^{1/2}}{(T-1)^{3/2}}\left(T^{1/2}\overline{k}_{i,3}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t})\right) \\
&- \frac{1}{(T-1)^{3/2}}\left(\frac{1}{T}\sum_{t=1}^{T} k(x_{i,t}) k_2(x_{i,t}) k_3(x_{i,t})\right)
\end{aligned}$$

(iv) when $m = 4$, we have

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} W_{i,T,(-t)} &= \frac{T^2-4T}{(T-1)^2} W_{i,T} \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,1}\right)\left(T^{1/2}\overline{k}_{i,2}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_3(x_{i,t}) k_4(x_{i,t})\right) \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,1}\right)\left(T^{1/2}\overline{k}_{i,3}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_2(x_{i,t}) k_4(x_{i,t})\right) \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,1}\right)\left(T^{1/2}\overline{k}_{i,4}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_2(x_{i,t}) k_3(x_{i,t})\right) \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,2}\right)\left(T^{1/2}\overline{k}_{i,3}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_4(x_{i,t})\right) \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,2}\right)\left(T^{1/2}\overline{k}_{i,4}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_3(x_{i,t})\right) \\
&+ \frac{T}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,3}\right)\left(T^{1/2}\overline{k}_{i,4}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t})\right) \\
&- \frac{T^{1/2}}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,1}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_2(x_{i,t}) k_3(x_{i,t}) k_4(x_{i,t})\right) \\
&- \frac{T^{1/2}}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,2}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_3(x_{i,t}) k_4(x_{i,t})\right) \\
&- \frac{T^{1/2}}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,3}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t}) k_4(x_{i,t})\right) \\
&- \frac{T^{1/2}}{(T-1)^2}\left(T^{1/2}\overline{k}_{i,4}\right)\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t}) k_3(x_{i,t})\right) \\
&+ \frac{1}{(T-1)^2}\left(\frac{1}{T}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t}) k_3(x_{i,t}) k_4(x_{i,t})\right)
\end{aligned}$$

In order to prove the preceding assertions, we note the following.For $m = 1$, it suffices to note that

$$\frac{1}{T}\sum_{t=1}^{T} K_{i,T,(-t)} = \overline{k}_{i,1} = K_{i,T}$$

For $m = 2$, it suffices to note that

$$\frac{1}{T}\sum_{t=1}^{T} K_{i,T,(-t)} = \frac{T^2-2T}{(T-1)^2}\overline{k}_{i,1}\overline{k}_{i,2} + \frac{1}{T(T-1)^2}\sum_{t=1}^{T} k_1(x_{i,t}) k_2(x_{i,t})$$

For $m=3$, it suffices to note that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}K_{i,T,(-t)} &= \frac{T^3-3T^2}{(T-1)^3}\overline{k}_{i,1}\overline{k}_{i,2}\overline{k}_{i,3} \\
&+\frac{1}{(T-1)^3}\overline{k}_{i,1}\sum_{t=1}^{T}k_2(x_{i,t})k_3(x_{i,t})+\frac{1}{(T-1)^3}\overline{k}_{i,2}\sum_{t=1}^{T}k_3(x_{i,t})k_1(x_{i,t}) \\
&+\frac{1}{(T-1)^3}\overline{k}_{i,3}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t})-\frac{1}{T(T-1)^3}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t})k_3(x_{i,t})
\end{aligned}$$

For $m=4$, it suffices to note that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}K_{i,T,(-t)} &= \frac{T^4-4T^3}{(T-1)^4}\overline{k}_{i,1}\overline{k}_{i,2}\overline{k}_{i,3}\overline{k}_{i,4} \\
&+\overline{k}_{i,1}\overline{k}_{i,2}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_3(x_{i,t})k_4(x_{i,t})+\overline{k}_{i,1}\overline{k}_{i,3}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_2(x_{i,t})k_4(x_{i,t}) \\
&+\overline{k}_{i,1}\overline{k}_{i,4}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_2(x_{i,t})k_3(x_{i,t})+\overline{k}_{i,2}\overline{k}_{i,3}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_1(x_{i,t})k_4(x_{i,t}) \\
&+\overline{k}_{i,2}\overline{k}_{i,4}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_1(x_{i,t})k_3(x_{i,t})+\overline{k}_{i,3}\overline{k}_{i,4}\frac{T}{(T-1)^4}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t}) \\
&-\frac{1}{(T-1)^4}\overline{k}_{i,1}\sum_{t=1}^{T}k_2(x_{i,t})k_3(x_{i,t})k_4(x_{i,t})-\frac{1}{(T-1)^4}\overline{k}_{i,2}\sum_{t=1}^{T}k_1(x_{i,t})k_3(x_{i,t})k_4(x_{i,t}) \\
&-\frac{1}{(T-1)^4}\overline{k}_{i,3}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t})k_4(x_{i,t})-\frac{1}{(T-1)^4}\overline{k}_{i,4}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t})k_3(x_{i,t}) \\
&+\frac{1}{T(T-1)^4}\sum_{t=1}^{T}k_1(x_{i,t})k_2(x_{i,t})k_3(x_{i,t})k_4(x_{i,t})
\end{aligned}$$

■

F.3 "Jackknifing" Functions of Normalized V-Statistic

Now, consider a statistic of the form

$$W_T = W_{[1],T}W_{[2],T}\cdots W_{[L],T}$$

where

$$\begin{aligned}
W_{[\ell],T} &\equiv \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{m_\ell/2}}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\cdots\sum_{t_{m_\ell}=1}^{T}k_{[\ell,1]}(x_{i,t_1})k_{[\ell,2]}(x_{i,t_2})\cdots k_{[\ell,m_\ell]}(x_{i,t_m}) \\
&\equiv T^{m_\ell/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[\ell,1]}\overline{k}_{i,[\ell,2]}\cdots\overline{k}_{i,[\ell,m_\ell]} \\
&\equiv T^{m_\ell/2}\frac{1}{n}\sum_{i=1}^{n}K_{i,[\ell],T} \\
&\equiv T^{m_\ell/2}K_{[\ell],T}.
\end{aligned}$$

Therefore, W_T is the product of $m_1, \cdots, m_L$ normalized V-statistics. We will also call them normalized V-statistic of order $\sum_{\ell=1}^{L} m_\ell$.

We consider the average of delete-t estimators

$$\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = \frac{1}{T}\sum_{t=1}^{T} W_{[1],T,(-t)} W_{[2],T,(-t)} \cdots W_{[L],T,(-t)}$$

where

$$\begin{aligned} W_{[\ell],T,(-t)} &\equiv \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(T-1)^{m_\ell/2}} \sum_{t_1 \neq t}^{T}\sum_{t_2 \neq t}^{T} \cdots \sum_{t_{m_\ell} \neq t}^{T} k_{[\ell,1]}(x_{i,t_1}) k_{[\ell,2]}(x_{i,t_2}) \cdots k_{[\ell,m_\ell]}(x_{i,t_m}) \\ &= (T-1)^{m_\ell/2} \frac{1}{n}\sum_{i=1}^{n} \frac{T\overline{k}_{i,[\ell,1]} - k_{[\ell,1]}(x_{i,t})}{T-1} \frac{T\overline{k}_{i,[\ell,2]} - k_{[\ell,2]}(x_{i,t})}{T-1} \cdots \frac{T\overline{k}_{i,[\ell,m_\ell]} - k_{[\ell,m_\ell]}(x_{i,t})}{T-1} \\ &= \frac{1}{(T-1)^{m_\ell/2}} \frac{1}{n}\sum_{i=1}^{n} \left(T\overline{k}_{i,[\ell,1]} - k_{[\ell,1]}(x_{i,t})\right)\left(T\overline{k}_{i,[\ell,2]} - k_{[\ell,2]}(x_{i,t})\right) \cdots \left(T\overline{k}_{i,[\ell,m_\ell]} - k_{[\ell,m_\ell]}(x_{i,t})\right) \end{aligned}$$

Below, we characterize properties of "jackknifed" normalized V-statistic of order up to 4.

Lemma 15 *Suppose that Condition 5 holds. When* $\sum_\ell m_\ell = 2, \sum_\ell m_\ell = 2, L = 2, m_1 = m_2 = 1$, *we have*

$$\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = \frac{T-2}{T-1} W_T + O_p\left(\frac{1}{nT}\right).$$

Proof. We have

$$W_{[1],T} = T^{1/2}\frac{1}{n}\sum_{i=1}^{n} \overline{k}_{i,[1,1]}, \qquad W_{[2],T} = T^{1/2}\frac{1}{n}\sum_{i=1}^{n} \overline{k}_{i,[2,1]}$$

and therefore,

$$W_T = T\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}$$

and

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} &= \frac{1}{T(T-1)}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T} \left(T\overline{k}_{i_1,[1,1]} - k_{[1,1]}(x_{i_1,t})\right)\left(T\overline{k}_{i_2,[2,1]} - k_{[2,1]}(x_{i_2,t})\right) \\ &= \frac{T^2-2T}{T-1}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]} \\ &\quad + \frac{1}{T(T-1)}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T} k_{[1,1]}(x_{i_1,t}) k_{[2,1]}(x_{i_2,t}) \\ &= \frac{T-2}{T-1} W_T + \frac{1}{T(T-1)}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T} k_{[1,1]}(x_{i_1,t}) k_{[2,1]}(x_{i_2,t}) \end{aligned}$$

Note that

$$W_T = \frac{1}{n}\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{n}\left(\sqrt{T}\overline{k}_{i_1,[1,1]}\right)\right)\left(\frac{1}{\sqrt{n}}\sum_{i_2=1}^{n}\left(\sqrt{T}\overline{k}_{i_2,[2,1]}\right)\right) = O_p\left(\frac{1}{n}\right)$$

and

$$\begin{aligned}
&\frac{1}{T(T-1)}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}k_{[1,1]}(x_{i_1,t})\,k_{[2,1]}(x_{i_2,t})\\
=\;&\frac{1}{n(T-1)}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{\sqrt{n}}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)
\end{aligned}$$

Because

$$\begin{aligned}
&E\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{\sqrt{n}}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\right|^2\right]\\
\leq\;&\frac{1}{T}\sum_{t=1}^{T}E\left[\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)^2\left(\frac{1}{\sqrt{n}}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)^2\right]\\
\leq\;&E\left[\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)^4\right]^{1/2}E\left[\left(\frac{1}{\sqrt{n}}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)^4\right]^{1/2}\\
\leq\;&\frac{n\max_i E\left[k_1(x_{i,t})^4\right]+6\frac{n(n-1)}{2}\max_i E\left[k_1(x_{i,t})^2\right]^2}{n^2}\\
<\;&\infty
\end{aligned}$$

we have

$$\frac{1}{T(T-1)}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}k_{[1,1]}(x_{i_1,t})\,k_{[2,1]}(x_{i_2,t})=O_p\left(\frac{1}{nT}\right)$$

Therefore, we have

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=\frac{T-2}{T-1}W_T+O_p\left(\frac{1}{nT}\right)$$

■

Lemma 16 *Suppose that Condition 5 holds. When* $\sum_\ell m_\ell=3, L=2, m_1=1, m_2=2$, *we have*

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$W_{[1],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]},\qquad W_{[2],T}=T\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}\overline{k}_{i,[2,2]}$$

and therefore,

$$W_T=T^{3/2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}\overline{k}_{i_2,[2,2]}$$

and

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
= \quad & \frac{1}{T\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]}-k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(T\overline{k}_{i_2,[2,1]}-k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(T\overline{k}_{i_2,[2,2]}-k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
= \quad & \frac{T^3-3T^2}{T^{3/2}\left(T-1\right)^{3/2}}W_T \\
& +\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& +\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& +\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
& -\frac{1}{\left(T-1\right)^{3/2}}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \qquad (24)
\end{aligned}
$$

We now show that

$$
\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \qquad (25)
$$

For this purpose, it suffices to note that

$$
\begin{aligned}
& \left|\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right| \\
\leq \quad & \left|\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\right|\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right| \\
= \quad & O_p\left(1\right)
\end{aligned}
$$

by Lemma 13.

We now show that

$$
\begin{aligned}
\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \quad &= \quad o_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{T^{1/2}}{\left(T-1\right)^{3/2}}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) \quad &= \quad o_p\left(\frac{1}{\sqrt{T}}\right). \qquad (26)
\end{aligned}
$$

For this purpose, we observe that

$$\begin{aligned}
& \left|\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right|^2 \\
= & \left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right|^2 \\
\leq & \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)^2\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)^2 \\
\leq & \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)^2\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^2k_{[2,2]}\left(x_{i_2,t}\right)^2\right) \\
\leq & \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)^2\right)\cdot\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^2\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)^2\right) \\
\leq & \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)^2\right)\cdot\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^2\left(\frac{1}{T}\sum_{t=1}^{T}M\left(x_{i_2,t}\right)^2\right),
\end{aligned}$$

where the far RHS has an expectation equal to

$$\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}E\left[M\left(x_{i_1,t}\right)^2\right]\cdot\frac{1}{n}\sum_{i_2=1}^{n}E\left[k_{[2,1]}\left(x_{it}\right)^2\right]\left(\frac{1}{T}\sum_{t=1}^{T}E\left[M\left(x_{i_2,t}\right)^2\right]\right)=O\left(1\right)$$

Therefore, we have

$$\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

Likewise, we have

$$\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

We now show that

$$-\frac{1}{\left(T-1\right)^{3/2}}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \tag{27}$$

For this purpose, it suffices to note that

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

Using equations (24) and (25), (26), (27), we conclude that

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right)$$

■

Lemma 17 *Suppose that Condition 5 holds. When $\sum_\ell m_\ell = 3, L = 3, m_1 = 1, m_2 = 1, m_3 = 1$, we have*

$$\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = W_T + o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$W_{[1],T} = T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]}, \qquad W_{[2],T} = T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}, \qquad W_{[3],T} = T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[3,1]}$$

and therefore,

$$W_T = T^{3/2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}$$

and

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
=\ & \frac{1}{T(T-1)^{3/2}}\frac{1}{n^3}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]} - k_{[1,1]}(x_{i_1,t})\right)\left(T\overline{k}_{i_2,[2,1]} - k_{[2,1]}(x_{i_2,t})\right)\left(T\overline{k}_{i_3,[3,1]} - k_{[3,1]}(x_{i_3,t})\right) \\
=\ & \frac{T^3 - 3T^2}{T^{3/2}(T-1)^{3/2}}W_T \\
& + \frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right) \\
& + \frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right) \\
& + \frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\right) \\
& - \frac{1}{(T-1)^{3/2}}\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right)
\end{aligned} \tag{28}$$

We now show that

$$\begin{aligned}
\frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \\
\frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \\
\frac{T^{1/2}}{(T-1)^{3/2}}\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \\
\frac{1}{(T-1)^{3/2}}\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}(x_{i_1,t})\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned} \tag{29}$$

Only the first assertion is proved. Others are proved similarly. It suffices to prove that $\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]} =$

$O_p(1)$ and $\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right)=O_p(1)$. Because

$$\begin{aligned} E\left[\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)^2\right] &= \frac{1}{n^2}\sum_{i_1=1}^{n}E\left[\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)^2\right]=\frac{1}{n^2T}\sum_{i_1=1}^{n}E\left[\left(k_{[1,1]}(x_{i_1,t})\right)^2\right] \\ &\leq \frac{1}{nT}\max_i E\left[M(x_{i,t})^2\right] \end{aligned}$$

we have

$$\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}=O_p\left(\frac{1}{\sqrt{nT}}\right)$$

and the first part is proved. We also have

$$\begin{aligned} E\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}(x_{i_2,t})\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}(x_{i_3,t})\right)\right)\right|\right] &\leq \frac{1}{n^2T}\sum_{t=1}^{T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}E\left[\left|k_{[2,1]}(x_{i_2,t})\right|\left|k_{[3,1]}(x_{i_3,t})\right|\right] \\ &\leq \frac{1}{n^2T}\sum_{t=1}^{T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}E\left[M(x_{i_2,t})M(x_{i_3,t})\right] \\ &= O(1) \end{aligned}$$

from which the second part follows.

We now get back to the proof of Lemma 17. Using equations (28) and (29), we conclude that

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right)$$

■

Lemma 18 *Suppose that Condition 5 holds. When $\sum_\ell m_\ell=4, L=2, m_1=1, m_2=3$, we have*

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$W_{[1],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]},\qquad W_{[2],T}=T^{3/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}\overline{k}_{i,[2,2]}\overline{k}_{i,[2,3]}$$

and therefore,

$$W_T=T^2\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}\overline{k}_{i_2,[2,3]}$$

and

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
= \;& \frac{1}{T\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]}-k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(T\overline{k}_{i_2,[2,1]}-k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
& \quad\times\left(T\overline{k}_{i_2,[2,2]}-k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(T\overline{k}_{i_2,[2,3]}-k_{[2,3]}\left(x_{i_2,t}\right)\right) \\
= \;& \frac{T^{2}-4T}{\left(T-1\right)^{2}}W_{T} \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& +\frac{1}{\left(T-1\right)^{2}}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right) \qquad (30)
\end{aligned}
$$

We now show that

$$\begin{aligned}
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \quad (31)
\end{aligned}$$

Only the first claim is proved. Others can be proved similarly. For this purpose, it suffices to prove that

$$\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)=O_p(1)$$

Because $\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}=O_p(1)$, we need only to prove that

$$\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)=O_p(1)$$

Because

$$\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)\right|\leq\frac{1}{n}\sum_{i_2=1}^{n}\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|\left|\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right|,$$

we have

$$\begin{aligned}
&E\left[\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)\right|\right]\\
\leq\ &\frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^2\right]\right)^{1/2}\left(E\left[\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)^2\right]\right)^{1/2}\\
\leq\ &\frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[k_{[2,1]}\left(x_{i_2,t}\right)^2\right]\right)^{1/2}E\left[\left(\frac{1}{T}\sum_{t=1}^{T}M\left(x_{i_2,t}\right)^2\right)^2\right]\\
=\ &O(1).
\end{aligned}$$

The conclusion follows by Markov inequality.

We now show that

$$\begin{aligned}
\frac{T}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \quad (32)
\end{aligned}$$

Only the first claim is proved. Others can be proved similarly. For this purpose, it suffices to prove that

$$\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)=O_p(1)$$

Note that

$$\begin{aligned}
& \left|\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right| \\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|\left|T^{1/2}\overline{k}_{i_2,[2,2]}\right|\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}\left|k_{[1,1]}\left(x_{i_1,t}\right)\right|\right)\left|k_{[2,3]}\left(x_{i_2,t}\right)\right|\right) \\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|\left|T^{1/2}\overline{k}_{i_2,[2,2]}\right|\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left[\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\right] \\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)\right)^2\right]\right)^{1/2} \\
& \quad \times\left(E\left[\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^4\right]\right)^{1/4}\left(E\left[\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)^4\right]\right)^{1/4}
\end{aligned}$$

But because

$$\begin{aligned}
E\left[\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)\right)^2\right] & \leq \frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}E\left[M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^2\right] \\
& \leq \frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}\max_i E\left[M\left(x_{i,t}\right)^4\right] \\
& = \max_i E\left[M\left(x_{i,t}\right)^4\right],
\end{aligned}$$

$$\begin{aligned}
E\left[\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)^4\right] & = \frac{TE\left[k_{[2,1]}\left(x_{i_2,t}\right)^4\right]+3T\left(T-1\right)E\left[k_{[2,1]}\left(x_{i_2,t}\right)^2\right]^2}{T^2} \\
& \leq 3\max_i E\left[M\left(x_{i,t}\right)^4\right],
\end{aligned}$$

and

$$E\left[\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)^4\right]\leq 3\max_i E\left[M\left(x_{i,t}\right)^4\right],$$

we should have

$$E\left[\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\right]=O\left(1\right),$$

from which the conclusion follows.

We now show that

$$\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \quad (33)$$

For this purpose, it suffices to prove that

$$\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

This follows from

$$\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)=O_p\left(1\right)$$

and

$$\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

We now show that

$$\begin{aligned}
\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,3]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \qquad (34)
\end{aligned}$$

Only the first claim is proved. Others can be proved similarly. For this purpose, it suffices to prove that

$$\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)=O_p\left(1\right)$$

Note that

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)^2\right)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&E\left[\left|\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\right]\\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|^2\right]\right)^{1/2}\left(E\left[\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)^2\right)^2\right]\right)^{1/2}\\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[M\left(x_{i_2,t}\right)^2\right]\right)^{1/2}\left(E\left[\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^4\right]\right)^{1/2}\\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left(E\left[M\left(x_{i_2,t}\right)^2\right]\right)^{1/2}\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i_1=1}^{n}\left(E\left[M\left(x_{i_1,t}\right)^4\right]\right)^{1/2}\left(E\left[M\left(x_{i_2,t}\right)^8\right]\right)^{1/2}\right)^{1/2}\\
\leq\ & \frac{1}{n}\sum_{i_2=1}^{n}\left(\max_i E\left[M\left(x_{i,t}\right)^2\right]\right)^{1/2}\left(\max_i E\left[M\left(x_{i,t}\right)^4\right]\right)^{1/4}\left(\max_i E\left[M\left(x_{i,t}\right)^8\right]\right)^{1/4}\\
=\ & O\left(1\right),
\end{aligned}$$

from which the conclusion follows.

We now show that

$$\frac{1}{(T-1)^2}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \quad (35)$$

For this purpose, it suffices to prove that

$$\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right)=O_p\left(1\right)$$

Note that

$$\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\leq\frac{1}{n^2T}\sum_{t=1}^{T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)^3$$

Therefore, we have

$$E\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)k_{[2,3]}\left(x_{i_2,t}\right)\right)\right|\right]=O\left(1\right),$$

from which the conclusion follows.

We now get back to the proof of Lemma 18. Using equations (30) and (31), (32), (33), (34), (35), we conclude that

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right)$$

■

Lemma 19 *Suppose that Condition 5 holds. When* $\sum_\ell m_\ell=4, L=3, m_1=1, m_2=2, m_3=1,$ *we have*

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$W_{[1],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]},\qquad W_{[2],T}=T\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}\overline{k}_{i,[2,2]},\qquad W_{[3],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[3,1]}$$

and therefore,

$$W_T=T^2\frac{1}{n^3}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}\overline{k}_{i_3,[3,1]}$$

and

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
= \;& \frac{1}{T\left(T-1\right)^2}\frac{1}{n^3}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]}-k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(T\overline{k}_{i_2,[2,1]}-k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
&\quad\times\left(T\overline{k}_{i_2,[2,2]}-k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(T\overline{k}_{i_3,[3,1]}-k_{[3,1]}\left(x_{i_3,t}\right)\right) \\
= \;& \frac{T^2-4T}{\left(T-1\right)^2}W_T \\
&+\frac{T}{\left(T-1\right)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&+\frac{T}{\left(T-1\right)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&+\frac{T}{\left(T-1\right)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) \\
&+\frac{T}{\left(T-1\right)^2}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&+\frac{T}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) \\
&+\frac{T}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) \\
&-\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&-\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&-\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
&-\frac{T^{1/2}}{\left(T-1\right)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) \\
&+\frac{1}{\left(T-1\right)^2}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \qquad (36)
\end{aligned}
$$

We now show that

$$\begin{aligned}
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\frac{1}{n}\sum_{i_2=1}^{n}\left(\left(\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \qquad (37)
\end{aligned}$$

Only the first assertion will be proved. Other can be proved similarly. For this purpose, it suffices to prove that

$$\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right)=O_p\left(1\right)$$

Because

$$\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}=O_p\left(1\right),$$

we only need to prove that

$$\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right)=O_p\left(1\right)$$

Note that

$$\begin{aligned}
& E\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right|\right]\\
\leq\;& \frac{1}{n^2T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{t=1}^{T}E\left[\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|M\left(x_{i_2,t}\right)M\left(x_{i_3,t}\right)\right]\\
\leq\;& \frac{1}{n^2T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{t=1}^{T}\left(E\left[\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|^2\right]\right)^{1/2}\left(E\left[M\left(x_{i_2,t}\right)^2M\left(x_{i_3,t}\right)^2\right]\right)^{1/2}\\
=\;& O\left(1\right),
\end{aligned}$$

from which the conclusion follows.

We now show that

$$\begin{aligned}
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}\left(T^{1/2}\overline{k}_{i_3,[3,1]}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned} \tag{38}$$

Only the first assertion will be proved. Other can be proved similarly. For this purpose, it suffices to prove that

$$\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right)=O_p(1)$$

But it follows from

$$\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)=O_p(1),$$

and

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right|\\
\leq\;& \frac{1}{n^2T}\sum_{t=1}^{T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}M\left(x_{i_2,t}\right)^2M\left(x_{i_3,t}\right)\\
=\;& O_p(1)
\end{aligned}$$

We now show that

$$\frac{1}{(T-1)^2}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \tag{39}$$

For this purpose, it suffices to prove that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)=O_p(1)$$

This follows because

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right|\\
\leq\;& \frac{1}{n^2T}\sum_{t=1}^{T}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)M\left(x_{i_3,t}\right)\\
=\;& O_p(1)
\end{aligned}$$

We now get back to the proof of Lemma 19. Using equations (36) and (37), (38), (39), we conclude

$$\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = W_T + o_p\left(\frac{1}{\sqrt{T}}\right)$$

■

Lemma 20 *Suppose that Condition 5 holds. When* $\sum_\ell m_\ell = 4, L = 2, m_1 = 2, m_2 = 2$*, we have*

$$\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = W_T + o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$W_{[1],T} = T\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]}\overline{k}_{i,[1,2]}, \qquad W_{[2],T} = T\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}\overline{k}_{i,[2,2]}$$

and therefore,

$$W_T = T^2\frac{1}{n^3}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_1,[1,2]}\overline{k}_{i_2,[2,1]}\overline{k}_{i_3,[3,1]}$$

and

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
= \; & \frac{1}{T\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]}-k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(T\overline{k}_{i_1,[1,2]}-k_{[1,2]}\left(x_{i_1,t}\right)\right) \\
& \quad \times\left(T\overline{k}_{i_2,[2,1]}-k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(T\overline{k}_{i_2,[2,2]}-k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
= \; & \frac{T^2-4T}{\left(T-1\right)^{2}}W_T \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
& +\frac{1}{\left(T-1\right)^{2}}\frac{1}{n^{2}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) \qquad (40)
\end{aligned}
$$

We now show that

$$
\begin{aligned}
\frac{T}{(T-1)^2}\left(\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}\frac{1}{T}\sum_{t=1}^{T}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \quad (41)
\end{aligned}
$$

The first assertion follows from Lemma 13. For the remaining assertions, only the second will be established. Others can be proved similarly. In order to prove the first assertion, it suffices to note that

$$
\begin{aligned}
& E\left[\left|\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right)\right|\right]\\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}E\left[\left|T^{1/2}\overline{k}_{i_1,[1,1]}\right|\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)\right]\\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(E\left[\left(\left|T^{1/2}\overline{k}_{i_1,[1,1]}\right|\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|\right)^2\right]\right)^{1/2}\left(E\left[M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^2\right]\right)^{1/2}\\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(E\left|T^{1/2}\overline{k}_{i_1,[1,1]}\right|^4\right)^{1/4}\left(E\left|T^{1/2}\overline{k}_{i_2,[2,1]}\right|^4\right)^{1/4}\left(E\left[M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^2\right]\right)^{1/2}\\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(3\max_i E\left[M\left(x_{i,t}\right)^4\right]\right)^{1/4}\left(3\max_i E\left[M\left(x_{i,t}\right)^4\right]\right)^{1/4}\left(\max_i E\left[M\left(x_{i,t}\right)^4\right]\right)^{1/2}\\
=\ & O(1)
\end{aligned}
$$

We now show that

$$
\begin{aligned}
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)k_{[2,2]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right)\\
\frac{T^{1/2}}{(T-1)^2}\frac{1}{n}\sum_{i_2=1}^{n}\left(T^{1/2}\overline{k}_{i_2,[2,2]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)\right)k_{[2,1]}\left(x_{i_2,t}\right)\right) &= o_p\left(\frac{1}{\sqrt{T}}\right) \quad (42)
\end{aligned}
$$

Only the first assertion will be proved. Others can be established similarly. In order to prove the first assertion, it suffices to note that

$$\begin{aligned}
& E\left[\left|\frac{1}{n}\sum_{i_1=1}^{n}\left(T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,2]}\left(x_{i_1,t}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right)\right|\right] \\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}E\left[\left|T^{1/2}\overline{k}_{i_1,[1,1]}\right|M\left(x_{i_1,t}\right)M\left(x_{i_2,t}\right)^2\right] \\
=\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}\left(E\left[\left|T^{1/2}\overline{k}_{i_1,[1,1]}\right|^2\right]\right)^{1/2}\left(E\left[M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^4\right]\right)^{1/2} \\
=\ & O(1)
\end{aligned}$$

We now show that

$$\frac{1}{(T-1)^2}\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)=o_p\left(\frac{1}{\sqrt{T}}\right) \tag{43}$$

This follows easily from

$$\begin{aligned}
& E\left[\left|\frac{1}{n^2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}k_{[1,1]}\left(x_{i_1,t}\right)k_{[1,2]}\left(x_{i_1,t}\right)k_{[2,1]}\left(x_{i_2,t}\right)k_{[2,2]}\left(x_{i_2,t}\right)\right)\right|\right] \\
\leq\ & \frac{1}{n^2T}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{t=1}^{T}E\left[M\left(x_{i_1,t}\right)^2M\left(x_{i_2,t}\right)^2\right] \\
=\ & O(1)
\end{aligned}$$

We now get back to the proof of Lemma 20. Using equations (40) and (41), (42), (43), we conclude

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right)$$

■

Lemma 21 *Suppose that Condition 5 holds. When* $\sum_\ell m_\ell=4, L=4, m_1=1, m_2=1, m_3=1, m_4=1$, *we have*

$$\frac{1}{T}\sum_{t=1}^{T}W_{T,(-t)}=W_T+o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof. We have

$$\begin{aligned}
W_{[1],T} &= T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[1,1]}, \qquad W_{[2],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[2,1]}, \\
W_{[3],T} &= T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[3,1]}, \qquad W_{[4],T}=T^{1/2}\frac{1}{n}\sum_{i=1}^{n}\overline{k}_{i,[4,1]}
\end{aligned}$$

and therefore,

$$W_T=T^2\frac{1}{n^4}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{i_4=1}^{n}\overline{k}_{i_1,[1,1]}\overline{k}_{i_2,[2,1]}\overline{k}_{i_3,[3,1]}\overline{k}_{i_4,[4,1]}$$

and

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} \\
= \; & \frac{1}{T\left(T-1\right)^{2}}\frac{1}{n^{4}}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\sum_{i_3=1}^{n}\sum_{t=1}^{T}\left(T\overline{k}_{i_1,[1,1]}-k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(T\overline{k}_{i_2,[2,1]}-k_{[2,1]}\left(x_{i_2,t}\right)\right) \\
& \quad \times\left(T\overline{k}_{i_3,[3,1]}-k_{[3,1]}\left(x_{i_3,t}\right)\right)\left(T\overline{k}_{i_4,[4,1]}-k_{[4,1]}\left(x_{i_4,t}\right)\right) \\
= \; & \frac{T^{2}-4T}{\left(T-1\right)^{2}}W_{T} \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}T^{1/2}\overline{k}_{i_4,[4,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}T^{1/2}\overline{k}_{i_4,[4,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
& +\frac{T}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}T^{1/2}\overline{k}_{i_4,[4,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_1=1}^{n}T^{1/2}\overline{k}_{i_1,[1,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_2=1}^{n}T^{1/2}\overline{k}_{i_2,[2,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_3=1}^{n}T^{1/2}\overline{k}_{i_3,[3,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \\
& -\frac{T^{1/2}}{\left(T-1\right)^{2}}\left(\frac{1}{n}\sum_{i_4=1}^{n}T^{1/2}\overline{k}_{i_4,[4,1]}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\right) \\
& +\frac{1}{\left(T-1\right)^{2}}\left(\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i_1=1}^{n}k_{[1,1]}\left(x_{i_1,t}\right)\right)\left(\frac{1}{n}\sum_{i_2=1}^{n}k_{[2,1]}\left(x_{i_2,t}\right)\right)\left(\frac{1}{n}\sum_{i_3=1}^{n}k_{[3,1]}\left(x_{i_3,t}\right)\right)\left(\frac{1}{n}\sum_{i_4=1}^{n}k_{[4,1]}\left(x_{i_4,t}\right)\right)\right) \qquad (44)
\end{aligned}
$$

Using equation (44) and lemmas in previous subsections, we can conclude that

$$
\frac{1}{T}\sum_{t=1}^{T} W_{T,(-t)} = W_T + o_p\left(\frac{1}{\sqrt{T}}\right)
$$

■

G Derivatives of θ when $\dim(\theta) = 1$

Recall that

$$h_i(\cdot, \epsilon) \equiv U_i(\cdot; \theta(F(\epsilon)), \alpha_i(\theta(F(\epsilon)), F_i(\epsilon)))$$

The first order condition may be written as

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int h_i(\cdot, \epsilon)\, dF_i(\epsilon)$$

Differentiating repeatedly with respect to ϵ, we obtain

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{dh_i(\cdot, \epsilon)}{d\epsilon} dF_i(\epsilon) + \frac{1}{n}\sum_{i=1}^{n}\int h_i(\cdot, \epsilon)\, d\Delta_{iT} \tag{45}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2 h_i(\cdot, \epsilon)}{d\epsilon^2} dF_i(\epsilon) + 2\frac{1}{n}\sum_{i=1}^{n}\int \frac{dh_i(\cdot, \epsilon)}{d\epsilon} d\Delta_{iT} \tag{46}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^3 h_i(\cdot, \epsilon)}{d\epsilon^3} dF_i(\epsilon) + 3\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^2 h_i(\cdot, \epsilon)}{d\epsilon^2} d\Delta_{iT} \tag{47}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^4 h_i(\cdot, \epsilon)}{d\epsilon^4} dF_i(\epsilon) + 4\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^3 h_i(\cdot, \epsilon)}{d\epsilon^3} d\Delta_{iT} \tag{48}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n}\int \frac{d^5 h_i(\cdot, \epsilon)}{d\epsilon^5} dF_i(\epsilon) + 5\frac{1}{n}\sum_{i=1}^{n}\int \frac{d^4 h_i(\cdot, \epsilon)}{d\epsilon^4} d\Delta_{iT} \tag{49}$$

G.1 $\theta^{\epsilon}(0)$

Evaluating (45) at $\epsilon = 0$, and noting that $E[U_i^{\alpha_i}] = 0$, we obtain

$$\theta^{\epsilon}(0) = \frac{\frac{1}{n}\sum_{i=1}^{n}\int U_i d\Delta_{iT}}{\frac{1}{n}\sum_{i=1}^{n} E[U_i^2]} \tag{50}$$

We therefore have

$$\sqrt{nT}\frac{1}{\sqrt{T}}\theta^{\epsilon}(0) = \frac{\frac{1}{\sqrt{n}\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T} U_{it}}{\frac{1}{n}\sum_{i=1}^{n} E[U_i^2]}$$

G.2 α_i^{θ} and α_i^{ϵ}

In the ith stratum, $\alpha_i(\theta, F_i(\epsilon))$ solves the estimating equation

$$\int V_i[\theta, \alpha_i(\theta, F_i(\epsilon))]\, dF_i(\epsilon) = 0 \tag{51}$$

Differentiating the LHS with respect to θ and ϵ, we obtain

$$0 = \int V_i^{\theta} dF_i(\epsilon) + \alpha_i^{\theta}\int V_i^{\alpha_i} dF_i(\epsilon)$$

$$0 = \alpha_i^{\epsilon}\int V_i^{\alpha_i} dF_i(\epsilon) + \int V_i d\Delta_{iT}$$

where $\Delta_{iT} \equiv \frac{d}{d\epsilon} F_i(\epsilon) = \sqrt{T}\left(\widehat{F}_i - F_i\right)$. Equating these equations to zero and solving for derivatives of α_i evaluated at $\epsilon = 0$ gives

$$\alpha_i^\theta = -\frac{E\left[V_i^\theta\right]}{E\left[V_i^{\alpha_i}\right]} = \frac{E\left[V_i^\theta\right]}{E\left[V_i^2\right]} = O(1) \tag{52}$$

$$\alpha_i^\epsilon = -\frac{\sum_{t=1}^T V_{it}}{\sqrt{T} E\left[V_i^{\alpha_i}\right]} = \frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]} = O_p(1) \tag{53}$$

G.3 $\theta^{\epsilon\epsilon}(0)$

Evaluating each term of (46) at $\epsilon = 0$, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n \int \frac{d^2 h_i(\cdot,\epsilon)}{d\epsilon^2} dF_i(\epsilon)\Bigg|_{\epsilon=0} &= \theta^{\epsilon\epsilon}(0)\frac{1}{n}\sum_{i=1}^n E\left[U_i^\theta\right] \\ &+2\theta^\epsilon(0)\left(\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\alpha_i}\right]\alpha_i^\epsilon + \frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\alpha_i^\theta\alpha_i^\epsilon\right) \\ &+\left(\theta^\epsilon(0)\right)^2\left(\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\theta}\right] + 2\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\alpha_i}\right]\alpha_i^\theta + \frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\left(\alpha_i^\theta\right)^2\right) \\ &+\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\left(\alpha_i^\epsilon\right)^2\end{aligned}$$

and

$$\begin{aligned}2\frac{1}{n}\sum_{i=1}^n \int \frac{dh_i(\cdot,\epsilon)}{d\epsilon} d\Delta_{iT}\Bigg|_{\epsilon=0} &= 2\theta^\epsilon(0)\frac{1}{n}\sum_{i=1}^n \int U_i^\theta d\Delta_{iT} + 2\theta^\epsilon(0)\frac{1}{n}\sum_{i=1}^n \alpha_i^\theta \int U_i^{\alpha_i} d\Delta_{iT} \\ &+2\frac{1}{n}\sum_{i=1}^n \alpha_i^\epsilon \int U_i^{\alpha_i} d\Delta_{iT}\end{aligned}$$

from which we obtain

$$\begin{aligned}\theta^{\epsilon\epsilon}(0)\frac{1}{n}\sum_{i=1}^n E\left[U_i^2\right] &= \frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\left(\alpha_i^\epsilon\right)^2 + 2\frac{1}{n}\sum_{i=1}^n \alpha_i^\epsilon \int U_i^{\alpha_i} d\Delta_{iT} \\ &+ 2\theta^\epsilon(0)\left(\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\alpha_i}\right]\alpha_i^\epsilon + \frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\alpha_i^\theta\alpha_i^\epsilon + \frac{1}{n}\sum_{i=1}^n \int U_i^\theta d\Delta_{iT} + \frac{1}{n}\sum_{i=1}^n \alpha_i^\theta \int U_i^{\alpha_i} d\Delta_{iT}\right) \\ &+ \left(\theta^\epsilon(0)\right)^2\left(\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\theta}\right] + 2\frac{1}{n}\sum_{i=1}^n E\left[U_i^{\theta\alpha_i}\right]\alpha_i^\theta + \frac{1}{n}\sum_{i=1}^n E\left[U_i^{\alpha_i\alpha_i}\right]\left(\alpha_i^\theta\right)^2\right)\end{aligned} \tag{54}$$

G.4 $\alpha_i^{\theta\theta}$, $\alpha_i^{\theta\epsilon}$, and $\alpha_i^{\epsilon\epsilon}$

Evaluating the second order derivatives $\left(\frac{\partial^2}{\partial\theta^2}, \frac{\partial^2}{\partial\theta\partial\epsilon}, \frac{\partial^2}{\partial\epsilon^2}\right)$ of (51) at $\epsilon = 0$, we obtain

$$\alpha_i^{\theta\theta} = \frac{E\left[V_i^{\theta\theta}\right]}{E\left[V_i^2\right]} + 2\alpha_i^\theta \frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^\theta\right)^2 \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} = O(1), \tag{55}$$

$$\alpha_i^{\theta\epsilon} = \alpha_i^\epsilon \frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^\theta\alpha_i^\epsilon \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \frac{\int V_i^\theta d\Delta_{iT}}{E\left[V_i^2\right]} + \alpha_i^\theta \frac{\int V_i^{\alpha_i} d\Delta_{iT}}{E\left[V_i^2\right]} = O_p(1), \tag{56}$$

and

$$\alpha_i^{\epsilon\epsilon} = (\alpha_i^\epsilon)^2 \frac{E[V_i^{\alpha_i\alpha_i}]}{E[V_i^2]} + 2\alpha_i^\epsilon \frac{\int V_i^{\alpha_i} d\Delta_{iT}}{E[V_i^2]} = O_p(1). \tag{57}$$

G.5 $\theta^{\epsilon\epsilon\epsilon}(0)$

Evaluating each term of (47) at $\epsilon = 0$, it can be concluded that $\theta^{\epsilon\epsilon\epsilon}(0) \frac{1}{n}\sum_{i=1}^n E\left[U_i^2\right]$ can be expressed as a sum of normalized V-statistics of order 3.[1] Condition 4 along with Lemma 13 imply that $|\theta^{\epsilon\epsilon\epsilon}(0)| = O_p(1)$.

G.6 $\theta^{\epsilon\epsilon\epsilon\epsilon}(0)$

Evaluating each term of (48) at $\epsilon = 0$, it can be concluded that $\theta^{\epsilon\epsilon\epsilon\epsilon}(0) \frac{1}{n}\sum_{i=1}^n E\left[U_i^2\right]$ can be expressed as a sum of normalized V-statistics of order 4.[2] Condition 4 along with Lemma 13 imply that $|\theta^{\epsilon\epsilon\epsilon\epsilon}(0)| = O_p(1)$.

H "Jackknifing" $\widehat{\theta}$

H.1 $T\left(\theta^\epsilon(0) - \sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \theta_{(t)}^\epsilon(0)\right)$ and $\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \theta_{(t)}^\epsilon(0)$

Because

$$\theta^\epsilon(0) = \frac{\frac{1}{n}\sum_{i=1}^n \int U_i d\Delta_{iT}}{\frac{1}{n}\sum_{i=1}^n E[U_i^2]},$$

we can use Lemma 14 and obtain

$$\frac{1}{T}\sum_{t=1}^T \theta_{(t)}^\epsilon(0) = \sqrt{\frac{T-1}{T}}\theta^\epsilon(0)$$

It therefore follows that

$$T\left(\sqrt{n}\theta^\epsilon(0) - \sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \sqrt{n}\theta_{(t)}^\epsilon(0)\right) + \sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \sqrt{n}\theta_{(t)}^\epsilon(0) = \sqrt{n}\theta^\epsilon(0) \tag{58}$$

H.2 $T\left(\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}(0) - \sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}(0)\right)$ and $\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^T \sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}(0)$

Write

$$\theta^{\epsilon\epsilon}(0)\frac{1}{n}\sum_{i=1}^n E\left[U_i^2\right] = \frac{1}{n}\sum_{i=1}^n \mathcal{W}_{1,i,T} + \mathcal{W}_{2,T},$$

where

$$\begin{aligned}\mathcal{W}_{1,i,T} &\equiv E[U_i^{\alpha_i\alpha_i}](\alpha_i^\epsilon)^2 + 2\alpha_i^\epsilon \int U_i^{\alpha_i} d\Delta_{iT} \\ &= 2\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T \frac{V_{it}}{E[V_i^2]}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T \left(U_i^{\alpha_i} + \frac{E[U_i^{\alpha_i\alpha_i}]}{2E[V_i^2]}V_{it}\right)\right]\end{aligned}$$

[1] For exact expression of the terms in (47), see Supplementary Appendix, which is available upon request.

[2] For exact expression of the terms in (48), see Supplementary Appendix, which is available upon request.

and

$$\begin{aligned}
\mathcal{W}_{2,T} &\equiv 2\theta^{\epsilon}(0)\left(\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\theta\alpha_i}\right]\alpha_i^{\epsilon}+\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\alpha_i\alpha_i}\right]\alpha_i^{\theta}\alpha_i^{\epsilon}+\frac{1}{n}\sum_{i=1}^{n}\int U_i^{\theta}d\Delta_{iT}+\frac{1}{n}\sum_{i=1}^{n}\alpha_i^{\theta}\int U_i^{\alpha_i}d\Delta_{iT}\right)\\
&\quad+\left(\theta^{\epsilon}(0)\right)^2\left(\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\theta\theta}\right]+2\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\theta\alpha_i}\right]\alpha_i^{\theta}+\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^{\alpha_i\alpha_i}\right]\left(\alpha_i^{\theta}\right)^2\right)\\
&= O_p\left(\frac{1}{n}\right)
\end{aligned}$$

By Lemma 14, we obtain

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{W}_{1,i,T,(-t)}\right)=\frac{T-2}{T-1}\frac{1}{n}\sum_{i=1}^{n}\mathcal{W}_{1,i,T}+\frac{1}{T(T-1)}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}2\frac{V_{it}}{E\left[V_i^2\right]}\left(U_i^{\alpha_i}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}V_{it}\right)$$

By Lemma 15, we obtain

$$\frac{1}{T}\sum_{t=1}^{T}\mathcal{W}_{2,T,(-t)}=\frac{T-2}{T-1}\mathcal{W}_{2,T}+O_p\left(\frac{1}{nT}\right)$$

It therefore follows that

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^2\right]\right)\cdot\frac{1}{2}T\left(\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}(0)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}(0)\right)\\
&+\left(\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^2\right]\right)\cdot\frac{1}{2}\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}(0)\\
=&\left(\frac{1}{n}\sum_{i=1}^{n}E\left[U_i^2\right]\right)\cdot\frac{\sqrt{nT}}{2}\left(\theta^{\epsilon\epsilon}(0)-\frac{1}{T}\sum_{t=1}^{T}\theta_{(t)}^{\epsilon\epsilon}(0)\right)\\
=&\frac{\sqrt{nT}}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{W}_{1,i,T}\right)\left(1-\frac{T-2}{T-1}\right)\\
&-\frac{\sqrt{nT}}{2}\left(\frac{1}{T(T-1)}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}2\frac{V_{it}}{E\left[V_i^2\right]}\left(U_i^{\alpha_i}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}V_{it}\right)\right)\\
&+\frac{\sqrt{nT}}{2}\left(1-\frac{T-2}{T-1}\right)\mathcal{W}_{2,T}+O_p\left(\frac{1}{\sqrt{nT}}\right)\\
=&\frac{1}{2}\sqrt{nT}\frac{1}{T-1}\frac{1}{n}\sum_{i=1}^{n}\mathcal{W}_{1,i,T}\\
&-\frac{1}{2}\sqrt{nT}\frac{1}{T-1}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}2\frac{V_{it}}{E\left[V_i^2\right]}\left(U_i^{\alpha_i}+\frac{E\left[U_i^{\alpha_i\alpha_i}\right]}{2E\left[V_i^2\right]}V_{it}\right)\\
&+\frac{1}{2}\sqrt{nT}\frac{1}{T-1}\mathcal{W}_{2,T}+O_p\left(\frac{1}{\sqrt{nT}}\right)\\
=&o_p(1) \qquad (59)
\end{aligned}$$

H.3 $T\left(\sqrt{\frac{n}{T}}\frac{1}{\sqrt{T}}\theta^{\epsilon\epsilon\epsilon}(0)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}(0)\right)$ and $\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}(0)$

By Lemmas 14, 16, and 17, we obtain

$$\begin{aligned}\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}(0) &= \frac{\sqrt{nT}}{\sqrt{(T-1)^3}}\theta^{\epsilon\epsilon\epsilon}(0)+\frac{\sqrt{nT}}{\sqrt{(T-1)^3}}o_p\left(\frac{1}{\sqrt{T}}\right)\\ &= o_p(1) \qquad (60)\end{aligned}$$

and

$$\begin{aligned}&T\left(\sqrt{\frac{n}{T}}\frac{1}{\sqrt{T}}\theta^{\epsilon\epsilon\epsilon}(0)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}(0)\right)\\ &= T\left(\sqrt{\frac{n}{T^2}}-\frac{\sqrt{nT}}{\sqrt{(T-1)^3}}\right)\theta^{\epsilon\epsilon\epsilon}(0)+\frac{T\sqrt{nT}}{\sqrt{(T-1)^3}}o_p\left(\frac{1}{\sqrt{T}}\right)\\ &= o_p(1) \qquad (61)\end{aligned}$$

H.4 $T\left(\sqrt{\frac{n}{T}}\frac{1}{T}\theta^{\epsilon\epsilon\epsilon\epsilon}(0)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0)\right)$ and $\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0)$

By Lemmas 14, 18, 19, 20, and 21, we obtain

$$\begin{aligned}\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0) &= \frac{\sqrt{nT}}{\sqrt{(T-1)^4}}\theta^{\epsilon\epsilon\epsilon}(0)+\frac{\sqrt{nT}}{\sqrt{(T-1)^4}}o_p\left(\frac{1}{\sqrt{T}}\right)\\ &= o_p(1) \qquad (62)\end{aligned}$$

and

$$\begin{aligned}&T\left(\sqrt{\frac{n}{T}}\frac{1}{T}\theta^{\epsilon\epsilon\epsilon\epsilon}(0)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0)\right)\\ &= T\left(\sqrt{\frac{n}{T^3}}-\frac{\sqrt{nT}}{\sqrt{(T-1)^4}}\right)\theta^{\epsilon\epsilon\epsilon}(0)+\frac{T\sqrt{nT}}{\sqrt{(T-1)^4}}o_p\left(\frac{1}{\sqrt{T}}\right)\\ &= o_p(1) \qquad (63)\end{aligned}$$

I Proof of Theorem 3

Using the same argument as in the proof of Theorem 6, it can be shown that

$$\Pr\left[\left|\frac{1}{\sqrt{T}}\theta^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon}(\widetilde{\epsilon})\right|\geq\eta\right]=o\left(\frac{1}{T}\right).$$

Therefore,

$$T\cdot\frac{1}{120}\sqrt{\frac{n}{T}}\frac{1}{T\sqrt{T}}\theta^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon}(\widetilde{\epsilon})=o_p(1)$$

and

$$\left| T \cdot \frac{1}{T} \sum_{t=1}^{T} \frac{1}{120} \sqrt{\frac{n}{T-1}} \frac{1}{(T-1)\sqrt{T-1}} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon} \left(\widetilde{\epsilon}_{(t)} \right) \right| \leq \frac{1}{120} \sqrt{\frac{n}{T-1}} \frac{T}{(T-1)\sqrt{T-1}} \max_t \left| \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon} \left(\widetilde{\epsilon}_{(t)} \right) \right|$$

$$= O\left(\max_t \left| \frac{1}{\sqrt{T}} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon} \left(\widetilde{\epsilon}_{(t)} \right) \right| \right)$$

$$= o_p(1),$$

where $\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon}$ denotes the delete-t version of $\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon}$, and the second equality is based on

$$\Pr\left[\max_t \left| \frac{1}{\sqrt{T}} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon} \left(\widetilde{\epsilon}_{(t)} \right) \right| \geq \eta \right] \leq \sum_t \Pr\left[\left| \frac{1}{\sqrt{T}} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon\epsilon\epsilon} \left(\widetilde{\epsilon}_{(t)} \right) \right| \geq \eta \right] = T \cdot o\left(\frac{1}{T} \right) = o(1).$$

We may therefore write

$$\begin{aligned} \sqrt{nT}\left(\widetilde{\theta} - \theta_0 \right) &= T \left(\sqrt{nT}\left(\widehat{\theta} - \theta_0 \right) - \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{n(T-1)} \left(\widehat{\theta}_{(t)} - \theta_0 \right) \right) \\ &\quad + \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{n(T-1)} \left(\widehat{\theta}_{(t)} - \theta_0 \right) \end{aligned}$$

or

$$\begin{aligned} \sqrt{nT}\left(\widetilde{\theta} - \theta_0 \right) &= T \left(\sqrt{n}\theta^{\epsilon}(0) - \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{n}\theta_{(t)}^{\epsilon}(0) \right) \\ &\quad + \frac{1}{2} T \left(\sqrt{\frac{n}{T}} \theta^{\epsilon\epsilon}(0) - \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \theta_{(t)}^{\epsilon\epsilon}(0) \right) \\ &\quad + \frac{1}{6} T \left(\sqrt{\frac{n}{T}} \frac{1}{\sqrt{T}} \theta^{\epsilon\epsilon\epsilon}(0) - \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \frac{1}{\sqrt{T-1}} \theta_{(t)}^{\epsilon\epsilon\epsilon}(0) \right) \\ &\quad + \frac{1}{24} T \left(\sqrt{\frac{n}{T}} \frac{1}{T} \theta^{\epsilon\epsilon\epsilon\epsilon}(0) - \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \frac{1}{T-1} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0) \right) \\ &\quad + \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{n}\theta_{(t)}^{\epsilon}(0) \\ &\quad + \frac{1}{2} \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \theta_{(t)}^{\epsilon\epsilon}(0) \\ &\quad + \frac{1}{6} \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \frac{1}{\sqrt{T-1}} \theta_{(t)}^{\epsilon\epsilon\epsilon}(0) \\ &\quad + \frac{1}{24} \sqrt{\frac{T}{T-1}} \frac{1}{T} \sum_{t=1}^{T} \sqrt{\frac{n}{T-1}} \frac{1}{T-1} \theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}(0) \\ &\quad + o_p(1) \end{aligned}$$

Using equations (58), (59), (60), (61), (62), and (63) in Section H, we have

$$T\left(\sqrt{n}\theta^{\epsilon}\left(0\right)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{n}\theta_{(t)}^{\epsilon}\left(0\right)\right)+\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{n}\theta_{(t)}^{\epsilon}\left(0\right)=\sqrt{n}\theta^{\epsilon}\left(0\right)$$

$$\frac{1}{2}T\left(\sqrt{\frac{n}{T}}\theta^{\epsilon\epsilon}\left(0\right)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}\left(0\right)\right)+\frac{1}{2}\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\theta_{(t)}^{\epsilon\epsilon}\left(0\right)=o_{p}\left(1\right)$$

$$T\left(\sqrt{\frac{n}{T}}\frac{1}{\sqrt{T}}\theta^{\epsilon\epsilon\epsilon}\left(0\right)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}\left(0\right)\right)+\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{\sqrt{T-1}}\theta_{(t)}^{\epsilon\epsilon\epsilon}\left(0\right)=o_{p}\left(1\right)$$

$$T\left(\sqrt{\frac{n}{T}}\frac{1}{T}\theta^{\epsilon\epsilon\epsilon\epsilon}\left(0\right)-\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}\left(0\right)\right)+\sqrt{\frac{T}{T-1}}\frac{1}{T}\sum_{t=1}^{T}\sqrt{\frac{n}{T-1}}\frac{1}{T-1}\theta_{(t)}^{\epsilon\epsilon\epsilon\epsilon}\left(0\right)=o_{p}\left(1\right)$$

Supplementary Appendix for “Jackknife and Analytical Bias Reduction for Nonlinear Panel Models” available upon request

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A Derivatives of h

A.1 First order derivative of h

$$\frac{dh_i(\cdot,\epsilon)}{d\epsilon} = \theta^{\epsilon}\left(U_i^{\theta} + U_i^{\alpha_i}\alpha_i^{\theta}\right) + U_i^{\alpha_i}\alpha_i^{\epsilon}$$

A.2 Second order derivative of h

$$\begin{aligned}\frac{d^2h_i(\cdot,\epsilon)}{d\epsilon^2} \quad = \quad & \theta^{\epsilon\epsilon}\left(U_i^{\theta} + U_i^{\alpha_i}\alpha_i^{\theta}\right) \\ & +2\theta^{\epsilon}\left(U_i^{\theta\alpha_i}\alpha_i^{\epsilon} + U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon} + U_i^{\alpha_i}\alpha_i^{\theta\epsilon}\right) \\ & +\left(\theta^{\epsilon}\right)^2\left(U_i^{\theta\theta} + 2U_i^{\theta\alpha_i}\alpha_i^{\theta} + U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2 + U_i^{\alpha_i}\alpha_i^{\theta\theta}\right) \\ & +U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2 + U_i^{\alpha_i}\alpha_i^{\epsilon\epsilon}\end{aligned}$$

A.3 Third order derivative of h

A.3.1 Coefficient of $\theta^{\epsilon\epsilon\epsilon}$

$$U_i^{\alpha_i}\alpha_i^{\theta} + U_i^{\theta}$$

A.3.2 Coefficient of $\theta^{\epsilon\epsilon}$

$$3U_i^{\theta\alpha_i}\alpha_i^{\epsilon} + 3U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon} + 3U_i^{\alpha_i}\alpha_i^{\theta\epsilon}$$

A.3.3 Coefficient of $\theta^{\epsilon\epsilon}\theta^{\epsilon}$

$$3U_i^{\theta\theta} + 3U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2 + 3U_i^{\alpha_i}\alpha_i^{\theta\theta} + 6U_i^{\theta\alpha_i}\alpha_i^{\theta}$$

A.3.4 Coefficient of θ^{ϵ}

$$3U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon} + 3U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2 + 3U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon} + 6U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon} + 3U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta} + 3U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2$$

A.3.5 Coefficient of $\left(\theta^{\epsilon}\right)^2$

$$3U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon} + 3U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon} + 6U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon} + 6U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon} + 3U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon} + 3U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon} + 6U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\theta\epsilon}$$

A.3.6 Coefficient of $\left(\theta^{\epsilon}\right)^3$

$$U_i^{\alpha_i}\alpha_i^{\theta\theta\theta} + U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3 + 3U_i^{\theta\theta\alpha_i}\alpha_i^{\theta} + 3U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2 + 3U_i^{\theta\alpha_i}\alpha_i^{\theta\theta} + 3U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta} + U_i^{\theta\theta\theta}$$

A.3.7 "Constant" term

$$U_i^{\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon} + 3U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon} + U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^3$$

A.4 Fourth order derivative of h

A.4.1 Coefficient of $\theta^{\epsilon\epsilon\epsilon\epsilon}$

$$U_i^{\alpha_i}\alpha_i^{\theta}+U_i^{\theta}$$

A.4.2 Coefficient of $\theta^{\epsilon\epsilon\epsilon}$

$$4U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+4U_i^{\theta\alpha_i}\alpha_i^{\epsilon}+4U_i^{\alpha_i}\alpha_i^{\theta\epsilon}$$

A.4.3 Coefficient of $\theta^{\epsilon\epsilon\epsilon}\theta^{\epsilon}$

$$4U_i^{\theta\theta}+4U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+8U_i^{\theta\alpha_i}\alpha_i^{\theta}+4U_i^{\alpha_i}\alpha_i^{\theta\theta}$$

A.4.4 Coefficient of $\theta^{\epsilon\epsilon}$

$$6U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2+6U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon}+6U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}+6U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2+6U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}$$

A.4.5 Coefficient of $\theta^{\epsilon\epsilon}\theta^{\epsilon}$

$$12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}+24U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\theta\epsilon}+24U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+12U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}+24U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon}+12U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon}$$
$$+12U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon}$$

A.4.6 Coefficient of $\theta^{\epsilon\epsilon}\left(\theta^{\epsilon}\right)^2$

$$6U_i^{\theta\theta\theta}+18U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}+6U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3+18U_i^{\theta\alpha_i}\alpha_i^{\theta\theta}+18U_i^{\theta\theta\alpha_i}\alpha_i^{\theta}+18U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2$$
$$+6U_i^{\alpha_i}\alpha_i^{\theta\theta\theta}$$

A.4.7 Coefficient of $\left(\theta^{\epsilon\epsilon}\right)^2$

$$3U_i^{\alpha_i}\alpha_i^{\theta\theta}+6U_i^{\theta\alpha_i}\alpha_i^{\theta}+3U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+3U_i^{\theta\theta}$$

A.4.8 Coefficient of θ^{ϵ}

$$4U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}+4U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^3+4U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon}+12U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon}+4U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^3$$
$$+4U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon\epsilon}+12U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\epsilon}\right)^2+12U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}$$

A.4.9 Coefficient of $\left(\theta^{\epsilon}\right)^2$

$$6U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon}+6U_i^{\theta\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2+6U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon\epsilon}+12U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon\epsilon}+12U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta\epsilon}\right)^2+24U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}$$
$$+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta}+6U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon\epsilon}+6U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\left(\alpha_i^{\epsilon}\right)^2$$
$$+6U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\epsilon}\right)^2+12U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2+12U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}+24U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}+6U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}$$

A.4.10 Coefficient of $(\theta^{\epsilon})^3$

$$12U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\epsilon}+4U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\epsilon}+12U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\epsilon}+4U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\epsilon}+12U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}\alpha_i^{\epsilon}+4U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3\alpha_i^{\epsilon}$$
$$+4U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta\epsilon}+12U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\alpha_i}\alpha_i^{\theta}+12U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}$$
$$+12U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+24U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}+12U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}+12U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}$$

A.4.11 Coefficient of $(\theta^{\epsilon})^4$

$$U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^4+6U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\theta}+6U_i^{\theta\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+4U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\theta}+4U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\theta}+4U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3$$
$$+U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\theta}+3U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta\theta}\right)^2+4U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\theta}+6U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\theta}\right)^2+12U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}+U_i^{\theta\theta\theta\theta}$$

A.4.12 "Constant" term

$$6U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^2+4U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\epsilon}+3U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\epsilon\epsilon}\right)^2+U_i^{\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon\epsilon}+U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^4$$

A.5 Fifth Order Derivative of h

A.5.1 Coefficient of $\theta^{\epsilon\epsilon\epsilon\epsilon\epsilon}$

$$U_i^{\alpha_i}\alpha_i^{\theta}+U_i^{\theta}$$

A.5.2 Coefficient of $\theta^{\epsilon\epsilon\epsilon\epsilon}$

$$5U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+5U_i^{\alpha_i}\alpha_i^{\theta\epsilon}+5U_i^{\theta\alpha_i}\alpha_i^{\epsilon}$$

A.5.3 Coefficient of $\theta^{\epsilon\epsilon\epsilon\epsilon}\theta^{\epsilon}$

$$5U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+5U_i^{\alpha_i}\alpha_i^{\theta\theta}+10U_i^{\theta\alpha_i}\alpha_i^{\theta}+5U_i^{\theta\theta}$$

A.5.4 Coefficient of $\theta^{\epsilon\epsilon\epsilon}$

$$10U_i^{\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}+20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}+10U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2+10U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon}+10U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2+10U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon}$$

A.5.5 Coefficient of $\theta^{\epsilon\epsilon\epsilon}\theta^{\epsilon}$

$$20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}+20U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}+40U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}+40U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+20U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon}+40U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon}+20U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon}$$

A.5.6 Coefficient of $\theta^{\epsilon\epsilon\epsilon}\left(\theta^{\epsilon}\right)^2$

$$10U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3+30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}+10U_i^{\alpha_i}\alpha_i^{\theta\theta\theta}+30U_i^{\theta\theta\alpha_i}\alpha_i^{\theta}+30U_i^{\theta\alpha_i}\alpha_i^{\theta\theta}+30U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+10U_i^{\theta\theta\theta}$$

A.5.7 Coefficient of $\theta^{\epsilon\epsilon\epsilon}\theta^{\epsilon\epsilon}$

$$10U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2+10U_i^{\alpha_i}\alpha_i^{\theta\theta}+20U_i^{\theta\alpha_i}\alpha_i^{\theta}+10U_i^{\theta\theta}$$

A.5.8 Coefficient of $\theta^{\epsilon\epsilon}$

$$30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}+30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon}+10U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}+10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^3$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\epsilon}\right)^2+10U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon}+30U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}+10U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}+10U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^3$$

A.5.9 Coefficient of $\theta^{\epsilon\epsilon}\theta^{\epsilon}$

$$30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}+60U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}+60U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta}+30U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon\epsilon}+120U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\epsilon}\right)^2+60U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta\epsilon}\right)^2+120U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}+60U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2+60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\left(\alpha_i^{\epsilon}\right)^2+30U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon}+60U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon\epsilon}+30U_i^{\theta\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^2+30U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon\epsilon}$$

A.5.10 Coefficient of $\theta^{\epsilon\epsilon}\left(\theta^{\epsilon}\right)^2$

$$30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\epsilon}+90U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\theta}\right)^2+90U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\theta}+90U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}\alpha_i^{\epsilon}$$
$$+90U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}+90U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}+180U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}+90U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3\alpha_i^{\epsilon}+90U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}+30U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\epsilon}+30U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\epsilon}+90U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\epsilon}$$
$$+90U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\epsilon}$$

A.5.11 Coefficient of $\theta^{\epsilon\epsilon}\left(\theta^{\epsilon}\right)^3$

$$40U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\theta}+30U_i^{\alpha_i\alpha_i}\left(\alpha_i^{\theta\theta}\right)^2+120U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}+60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\theta}\right)^2$$
$$+10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^4+10U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\theta}+60U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\theta}+60U_i^{\theta\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2$$
$$+40U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\theta}+40U_i^{\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3+10U_i^{\theta\theta\theta\theta}+40U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\theta}$$

A.5.12 Coefficient of $\left(\theta^{\epsilon\epsilon}\right)^2$

$$15U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon}+15U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}+15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}+30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}+30U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}$$
$$+15U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon}+30U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon}$$

A.5.13 Coefficient of $\left(\theta^{\epsilon\epsilon}\right)^2\theta^{\epsilon}$

$$45U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}+15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3+15U_i^{\alpha_i}\alpha_i^{\theta\theta\theta}+45U_i^{\theta\alpha_i}\alpha_i^{\theta\theta}+45U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2$$
$$+45U_i^{\theta\theta\alpha_i}\alpha_i^{\theta}+15U_i^{\theta\theta\theta}$$

A.5.14 Coefficient of θ^{ϵ}

$$30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon\epsilon}+60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}+20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon}\alpha_i^{\epsilon}+20U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}+20U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^2+15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon\epsilon}\right)^2\alpha_i^{\theta}+20U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\epsilon}\right)^3$$
$$+5U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\alpha_i^{\theta}+5U_i^{\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon\epsilon}+30U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^2+20U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\epsilon}+5U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^4$$
$$+5U_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^4+5U_i^{\theta\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon\epsilon}+15U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\epsilon\epsilon}\right)^2$$

A.5.15 Coefficient of $(\theta^{\epsilon})^2$

$$10U_i^{\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon\epsilon} + 60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon} + 30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon}\alpha_i^{\epsilon} + 20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon}\alpha_i^{\theta} + 60U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta\epsilon}\right)^2\alpha_i^{\epsilon}$$
$$+60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta} + 30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon} + 10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2(\alpha_i^{\epsilon})^3 + 60U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon\epsilon} + 10U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\theta} + 60U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta} + 60U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}(\alpha_i^{\epsilon})^2$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2 + 10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}(\alpha_i^{\epsilon})^3 + 60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon} + 60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon}$$
$$+60U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}(\alpha_i^{\epsilon})^2 + 20U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta} + 20U_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}(\alpha_i^{\epsilon})^3\alpha_i^{\theta} + 10U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}(\alpha_i^{\epsilon})^2 + 20U_i^{\theta\alpha_i}\alpha_i^{\theta\epsilon\epsilon\epsilon} + 10U_i^{\theta\theta\alpha_i\alpha_i\alpha_i}(\alpha_i^{\epsilon})^3 + 10U_i^{\theta\theta\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon} + 30U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon}$$

A.5.16 Coefficient of $(\theta^{\epsilon})^3$

$$60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon} + 20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta\epsilon}\alpha_i^{\epsilon} + 10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3(\alpha_i^{\epsilon})^2 + 30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon}\alpha_i^{\theta}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta}(\alpha_i^{\epsilon})^2 + 60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta} + 30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta} + 60U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\theta\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\theta} + 10U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\epsilon\epsilon} + 120U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta} + 60U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}$$
$$+10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^3 + 60U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta\epsilon}\right)^2\alpha_i^{\theta} + 10U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\epsilon\epsilon} + 30U_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta}(\alpha_i^{\epsilon})^2$$
$$+60U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon} + 30U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta} + 30U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}(\alpha_i^{\epsilon})^2 + 30U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}$$
$$+60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta} + 30U_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2(\alpha_i^{\epsilon})^2 + 30U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2 + 60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2 + 10U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}(\alpha_i^{\epsilon})^2 + 30U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\epsilon\epsilon} + 60U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta\epsilon}\right)^2$$
$$+10U_i^{\theta\theta\theta\alpha_i\alpha_i}(\alpha_i^{\epsilon})^2 + 10U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\epsilon\epsilon} + 30U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\epsilon\epsilon}$$

A.5.17 Coefficient of $(\theta^{\epsilon})^4$

$$5U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta\theta}\alpha_i^{\epsilon} + 60U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta\epsilon}\alpha_i^{\theta} + 5U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^4\alpha_i^{\epsilon} + 15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta\theta}\right)^2\alpha_i^{\epsilon}$$
$$+30U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon} + 20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\theta\epsilon} + 30U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\theta\theta} + 60U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon}\alpha_i^{\theta}$$
$$+20U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\epsilon}\alpha_i^{\theta} + 20U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\theta}\right)^3 + 30U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\left(\alpha_i^{\theta}\right)^2 + 20U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta\epsilon}\alpha_i^{\theta}$$
$$+5U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\theta\epsilon} + 30U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\epsilon} + 30U_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon} + 60U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\alpha_i^{\theta}$$
$$+60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta\epsilon} + 60U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta\epsilon}\alpha_i^{\theta} + 60U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\epsilon}\left(\alpha_i^{\theta}\right)^2 + 20U_i^{\theta\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta}\alpha_i^{\epsilon}$$
$$+20U_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3\alpha_i^{\epsilon} + 20U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\epsilon} + 30U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\theta\epsilon} + 20U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\theta\epsilon} + 5U_i^{\theta\theta\theta\theta\alpha_i}\alpha_i^{\epsilon}$$
$$+20U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\theta\epsilon}$$

A.5.18 Coefficient of $(\theta^{\epsilon})^5$

$$U_i^{\theta\theta\theta\theta\theta} + 5U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta\theta}\alpha_i^{\theta} + U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^5 + 10U_i^{\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\theta\theta} + 10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\theta}\right)^3$$
$$+15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta\theta}\right)^2\alpha_i^{\theta} + 30U_i^{\theta\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta}\alpha_i^{\theta} + 20U_i^{\theta\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\alpha_i^{\theta} + 30U_i^{\theta\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta}\left(\alpha_i^{\theta}\right)^2$$
$$+10U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\theta\theta\theta}\left(\alpha_i^{\theta}\right)^2 + U_i^{\alpha_i}\alpha_i^{\theta\theta\theta\theta\theta} + 5U_i^{\theta\theta\theta\theta\alpha_i}\alpha_i^{\theta} + 5U_i^{\theta\alpha_i}\alpha_i^{\theta\theta\theta\theta} + 10U_i^{\theta\theta\alpha_i}\alpha_i^{\theta\theta\theta}$$
$$+5U_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^4 + 15U_i^{\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta\theta}\right)^2 + 10U_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^3 + 10U_i^{\theta\theta\theta\alpha_i\alpha_i}\left(\alpha_i^{\theta}\right)^2 + 10U_i^{\theta\theta\theta\alpha_i}\alpha_i^{\theta\theta}$$

A.5.19 "Constant" term

$$U_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon}\right)^5+10U_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^3+15U_i^{\alpha_i\alpha_i\alpha_i}\left(\alpha_i^{\epsilon\epsilon}\right)^2\alpha_i^{\epsilon}+5U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\alpha_i^{\epsilon}$$
$$+10U_i^{\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\epsilon\epsilon}+10U_i^{\alpha_i\alpha_i\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon}\left(\alpha_i^{\epsilon}\right)^2+U_i^{\alpha_i}\alpha_i^{\epsilon\epsilon\epsilon\epsilon\epsilon}$$

B "Derivatives" of α_i

Recall that, in the ith stratum, $\alpha_i\left(\theta,F_i\left(\epsilon\right)\right)$ solves the estimating equation

$$\int V_i\left[\theta,\alpha_i\left(\theta,F_i\left(\epsilon\right)\right)\right]dF_i\left(\epsilon\right)=0 \tag{B1}$$

B.1 $\alpha_i^{\theta\theta}$, $\alpha_i^{\theta\epsilon}$, and $\alpha_i^{\epsilon\epsilon}$

Second order differentiation $\left(\frac{\partial^2}{\partial\theta^2},\frac{\partial^2}{\partial\theta\partial\epsilon},\frac{\partial^2}{\partial\epsilon^2}\right)$ of (B1) yields

$$\begin{aligned}
0 &= \int V_i^{\theta\theta}dF_i\left(\epsilon\right)+2\alpha_i^{\theta}\int V_i^{\theta\alpha_i}dF_i\left(\epsilon\right)+\alpha_i^{\theta\theta}\int V_i^{\alpha_i}dF_i\left(\epsilon\right)+\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i}dF_i\left(\epsilon\right)\\
0 &= \alpha_i^{\epsilon}\int V_i^{\theta\alpha_i}dF_i\left(\epsilon\right)+\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i}dF_i\left(\epsilon\right)+\alpha_i^{\theta}\alpha_i^{\epsilon}\int V_i^{\alpha_i\alpha_i}dF_i\left(\epsilon\right)+\int V_i^{\theta}d\Delta_{iT}+\alpha_i^{\theta}\int V_i^{\alpha_i}d\Delta_{iT}\\
0 &= \alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i}dF_i\left(\epsilon\right)+\left(\alpha_i^{\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i}dF_i\left(\epsilon\right)+2\alpha_i^{\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}
\end{aligned}$$

Evaluating at $\epsilon=0$, we obtain

$$\begin{aligned}
0 &= E\left[V_i^{\theta\theta}\right]+2\frac{E\left[V_i^{\theta}\right]}{E\left[V_i^2\right]}E\left[V_i^{\theta\alpha_i}\right]-\alpha_i^{\theta\theta}E\left[V_i^2\right]+\left(\frac{E\left[V_i^{\theta}\right]}{E\left[V_i^2\right]}\right)^2E\left[V_i^{\alpha_i\alpha_i}\right]\\
0 &= \frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]}E\left[V_i^{\theta\alpha_i}\right]-\alpha_i^{\theta\epsilon}E\left[V_i^2\right]+\frac{E\left[V_i^{\theta}\right]}{E\left[V_i^2\right]}\frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]}E\left[V_i^{\alpha_i\alpha_i}\right]\\
&\quad +T^{-1/2}\sum_{t=1}^T\left(V_i^{\theta}-E\left[V_i^{\theta}\right]\right)+\frac{E\left[V_i^{\theta}\right]}{E\left[V_i^2\right]}T^{-1/2}\sum_{t=1}^T\left(V_i^{\alpha_i}-E\left[V_i^{\alpha_i}\right]\right)\\
0 &= -\alpha_i^{\epsilon\epsilon}E\left[V_i^2\right]+\left(\frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]}\right)^2E\left[V_i^{\alpha_i\alpha_i}\right]+2\frac{T^{-1/2}\sum_{t=1}^T V_{it}}{E\left[V_i^2\right]}T^{-1/2}\sum_{t=1}^T\left(V_i^{\alpha_i}-E\left[V_i^{\alpha_i}\right]\right)
\end{aligned}$$

from which we obtain

$$\alpha_i^{\theta\theta}=\frac{E\left[V_i^{\theta\theta}\right]}{E\left[V_i^2\right]}+2\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}=O\left(1\right),$$

$$\alpha_i^{\theta\epsilon}=\alpha_i^{\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}+\alpha_i^{\theta}\alpha_i^{\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\frac{\int V_i^{\theta}d\Delta_{iT}}{E\left[V_i^2\right]}+\alpha_i^{\theta}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}=O_p\left(1\right),$$

and

$$\alpha_i^{\epsilon\epsilon}=\left(\alpha_i^{\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+2\alpha_i^{\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}=O_p\left(1\right).$$

B.2 $\alpha_i^{\theta\theta\theta}$, $\alpha_i^{\theta\theta\epsilon}$, $\alpha_i^{\theta\epsilon\epsilon}$, and $\alpha_i^{\epsilon\epsilon\epsilon}$

Third order differentiation $\left(\frac{\partial^3}{\partial\theta^3}, \frac{\partial^3}{\partial\theta^2\partial\epsilon}, \frac{\partial^3}{\partial\theta\partial\epsilon^2}, \frac{\partial^3}{\partial\epsilon^3}\right)$ of (B1) yields

$$\begin{aligned} 0 &= \int V_i^{\theta\theta\theta} dF(\epsilon) + 3\alpha_i^\theta \int V_i^{\theta\theta\alpha_i} dF(\epsilon) + 3\left(\alpha_i^\theta\right)^2 \int V_i^{\theta\alpha_i\alpha_i} dF(\epsilon) + 3\alpha_i^{\theta\theta} \int V_i^{\theta\alpha_i} dF(\epsilon) \\ &+ \left(\alpha_i^\theta\right)^3 \int V_i^{\alpha_i\alpha_i\alpha_i} dF(\epsilon) + 3\alpha_i^\theta\alpha_i^{\theta\theta} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\theta\theta\theta} \int V_i^{\alpha_i} dF(\epsilon) \end{aligned}$$

$$\begin{aligned} 0 &= \alpha_i^\epsilon \int V_i^{\theta\theta\alpha_i} dF(\epsilon) + 2\alpha_i^\epsilon\alpha_i^\theta \int V_i^{\theta\alpha_i\alpha_i} dF(\epsilon) + 2\alpha_i^{\theta\epsilon} \int V_i^{\theta\alpha_i} dF(\epsilon) \\ &+ \alpha_i^\epsilon \left(\alpha_i^\theta\right)^2 \int V_i^{\alpha_i\alpha_i\alpha_i} dF(\epsilon) + 2\alpha_i^\theta\alpha_i^{\theta\epsilon} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) \\ &+ \alpha_i^\epsilon\alpha_i^{\theta\theta} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\theta\theta\epsilon} \int V_i^{\alpha_i} dF(\epsilon) + \int V_i^{\theta\theta} d\Delta_{iT} \\ &+ 2\alpha_i^\theta \int V_i^{\theta\alpha_i} d\Delta_{iT} + \left(\alpha_i^\theta\right)^2 \int V_i^{\alpha_i\alpha_i} d\Delta_{iT} + \alpha_i^{\theta\theta} \int V_i^{\alpha_i} d\Delta_{iT} \end{aligned}$$

$$\begin{aligned} 0 &= \left(\alpha_i^\epsilon\right)^2 \int V_i^{\theta\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\epsilon\epsilon} \int V_i^{\theta\alpha_i} dF(\epsilon) \\ &+ \left(\alpha_i^\epsilon\right)^2 \alpha_i^\theta \int V_i^{\alpha_i\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\epsilon\epsilon}\alpha_i^\theta \int V_i^{\alpha_i\alpha_i} dF(\epsilon) \\ &+ 2\alpha_i^\epsilon\alpha_i^{\theta\epsilon} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\theta\epsilon\epsilon} \int V_i^{\alpha_i} dF(\epsilon) + 2\alpha_i^\epsilon \int V_i^{\theta\alpha_i} d\Delta_{iT} \\ &+ 2\alpha_i^\epsilon\alpha_i^\theta \int V_i^{\alpha_i\alpha_i} d\Delta_{iT} + 2\alpha_i^{\theta\epsilon} \int V_i^{\alpha_i} d\Delta_{iT} \end{aligned}$$

$$\begin{aligned} 0 &= \left(\alpha_i^\epsilon\right)^3 \int V_i^{\alpha_i\alpha_i\alpha_i} dF(\epsilon) + 3\alpha_i^\epsilon\alpha_i^{\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) + \alpha_i^{\epsilon\epsilon\epsilon} \int V_i^{\alpha_i} dF(\epsilon) \\ &+ 3\left(\alpha_i^\epsilon\right)^2 \int V_i^{\alpha_i\alpha_i} d\Delta_{iT} + 3\alpha_i^{\epsilon\epsilon} \int V_i^{\alpha_i} d\Delta_{iT} \end{aligned}$$

Evaluating at $\epsilon = 0$, we obtain

$$\begin{aligned} \alpha_i^{\theta\theta\theta} &= \frac{E\left[V_i^{\theta\theta\theta}\right]}{E\left[V_i^2\right]} + 3\alpha_i^\theta \frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 3\left(\alpha_i^\theta\right)^2 \frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\theta} \frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} \\ &+ \left(\alpha_i^\theta\right)^3 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^\theta\alpha_i^{\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \end{aligned}$$

$$\begin{aligned} \alpha_i^{\theta\theta\epsilon} &= \alpha_i^\epsilon \frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^\epsilon\alpha_i^\theta \frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\theta\epsilon} \frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} \\ &+ \alpha_i^\epsilon \left(\alpha_i^\theta\right)^2 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^\theta\alpha_i^{\theta\epsilon} \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^\epsilon\alpha_i^{\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\ &+ \frac{\int V_i^{\theta\theta} d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^\theta \frac{\int V_i^{\theta\alpha_i} d\Delta_{iT}}{E\left[V_i^2\right]} + \left(\alpha_i^\theta\right)^2 \frac{\int V_i^{\alpha_i\alpha_i} d\Delta_{iT}}{E\left[V_i^2\right]} + \alpha_i^{\theta\theta} \frac{\int V_i^{\alpha_i} d\Delta_{iT}}{E\left[V_i^2\right]} \end{aligned}$$

$$\begin{aligned}
\alpha_i^{\theta\epsilon\epsilon} &= \left(\alpha_i^{\epsilon}\right)^2 \frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^2 \alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad +\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad +2\alpha_i^{\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}$$

$$\begin{aligned}
\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}dF\left(\epsilon\right) &= \left(\alpha_i^{\epsilon}\right)^3 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad +3\left(\alpha_i^{\epsilon}\right)^2 \frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}$$

B.3 $\alpha_i^{\theta\theta\theta\theta}$, $\alpha_i^{\theta\theta\theta\epsilon}$, $\alpha_i^{\theta\theta\epsilon\epsilon}$, $\alpha_i^{\theta\epsilon\epsilon\epsilon}$, and $\alpha_i^{\epsilon\epsilon\epsilon\epsilon}$

Fourth order differentiation $\left(\frac{\partial^4}{\partial\theta^4}, \frac{\partial^4}{\partial\theta^3\partial\epsilon}, \frac{\partial^4}{\partial\theta^2\partial\epsilon^2}, \frac{\partial^4}{\partial\theta\partial\epsilon^3}, \frac{\partial^4}{\partial\epsilon^4}\right)$ of (B1) yields

$$\begin{aligned}
0 &= 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + 12\alpha_i^{\theta}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF\left(\epsilon\right) + 4\alpha_i^{\theta}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF\left(\epsilon\right) \\
&\quad +3\left(\alpha_i^{\theta\theta}\right)^2\int V_i^{\alpha_i\alpha_i}dF\left(\epsilon\right) + \left(\alpha_i^{\theta}\right)^4\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + 4\left(\alpha_i^{\theta}\right)^3\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) \\
&\quad +\alpha_i^{\theta\theta\theta\theta}\int V_i^{\alpha_i}dF\left(\epsilon\right) + 6\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\theta\alpha_i\alpha_i}dF\left(\epsilon\right) + 6\alpha_i^{\theta\theta}\int V_i^{\theta\theta\alpha_i}dF\left(\epsilon\right) \\
&\quad +4\alpha_i^{\theta}\int V_i^{\theta\theta\theta\alpha_i}dF\left(\epsilon\right) + \int V_i^{\theta\theta\theta\theta}dF\left(\epsilon\right) + 4\alpha_i^{\theta\theta\theta}\int V_i^{\theta\alpha_i}dF\left(\epsilon\right)
\end{aligned}$$

$$\begin{aligned}
0 &= 3\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + 3\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF\left(\epsilon\right) + 3\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\theta\theta\alpha_i\alpha_i}dF\left(\epsilon\right) \\
&\quad +3\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + \alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + 6\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF\left(\epsilon\right) \\
&\quad +3\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF\left(\epsilon\right) + \alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF\left(\epsilon\right) + 3\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}dF\left(\epsilon\right) \\
&\quad +3\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF\left(\epsilon\right) + \alpha_i^{\epsilon}\int V_i^{\theta\theta\theta\alpha_i}dF\left(\epsilon\right) + 3\alpha_i^{\theta\epsilon}\int V_i^{\theta\theta\alpha_i}dF\left(\epsilon\right) \\
&\quad +3\alpha_i^{\theta\theta\epsilon}\int V_i^{\theta\alpha_i}dF\left(\epsilon\right) + \alpha_i^{\theta\theta\theta\epsilon}\int V_i^{\alpha_i}dF\left(\epsilon\right) \\
&\quad +\int V_i^{\theta\theta\theta}d\Delta_{iT} + 3\alpha_i^{\theta}\int V_i^{\theta\theta\alpha_i}d\Delta_{iT} + 3\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT} + 3\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i}d\Delta_{iT} \\
&\quad +\left(\alpha_i^{\theta}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT} + 3\alpha_i^{\theta}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT} + \alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i}d\Delta_{iT}
\end{aligned}$$

$$
\begin{aligned}
0 &= 2\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+2\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+4\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)\\
&+\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+2\alpha_i^{\theta}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\theta\theta\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)+\left(\alpha_i^{\epsilon}\right)^2\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\theta\alpha_i}dF(\epsilon)\\
&+2\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon)+2\left(\alpha_i^{\theta\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+4\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+2\alpha_i^{\epsilon}\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}+4\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i}d\Delta_{iT}+2\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}+2\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
&+4\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+4\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\alpha_i^{\epsilon}\right)^3\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon)\\
&+\left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\theta\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)\\
&+3\left(\alpha_i^{\epsilon}\right)^2\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+3\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\alpha_i}d\Delta_{iT}+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
&+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+3\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\alpha_i^{\epsilon}\right)^4\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\left(\alpha_i^{\epsilon\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
&+4\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)\\
&+4\left(\alpha_i^{\epsilon}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+12\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+4\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}
\end{aligned}
$$

Evaluating at $\epsilon = 0$, we obtain

$$
\begin{aligned}
\alpha_i^{\theta\theta\theta\theta} &= 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+12\alpha_i^{\theta}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+4\alpha_i^{\theta}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\left(\alpha_i^{\theta\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+\left(\alpha_i^{\theta}\right)^4\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+4\left(\alpha_i^{\theta}\right)^3\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+4\alpha_i^{\theta}\frac{E\left[V_i^{\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]}+\frac{E\left[V_i^{\theta\theta\theta\theta}\right]}{E\left[V_i^2\right]}+4\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}
\end{aligned}
$$

$$\alpha_i^{\theta\theta\theta\epsilon} = 3\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+3\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon}\frac{E\left[V_i^{\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+\frac{\int V_i^{\theta\theta\theta}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\theta}\frac{\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\theta}\right)^2\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\theta}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$
$$+\left(\alpha_i^{\theta}\right)^3\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\theta}\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + \alpha_i^{\theta\theta\theta}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$

$$\alpha_i^{\theta\theta\epsilon\epsilon} = 2\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+2\alpha_i^{\theta}\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^2\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+2\left(\alpha_i^{\theta\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+2\alpha_i^{\epsilon}\frac{\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 4\alpha_i^{\theta\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^{\theta\theta\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$
$$+4\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$

$$\alpha_i^{\theta\epsilon\epsilon\epsilon} = \left(\alpha_i^{\epsilon}\right)^3\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+3\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+3\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+3\left(\alpha_i^{\epsilon}\right)^2\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$
$$+6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$

$$\alpha_i^{\epsilon\epsilon\epsilon\epsilon} = \left(\alpha_i^{\epsilon}\right)^4\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}$$
$$+4\left(\alpha_i^{\epsilon}\right)^3\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 12\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}$$

B.4 $\alpha_i^{\theta\theta\theta\theta\theta}$, $\alpha_i^{\theta\theta\theta\theta\epsilon}$, $\alpha_i^{\theta\theta\theta\epsilon\epsilon}$, $\alpha_i^{\theta\theta\epsilon\epsilon\epsilon}$, $\alpha_i^{\theta\epsilon\epsilon\epsilon\epsilon}$, and $\alpha_i^{\epsilon\epsilon\epsilon\epsilon\epsilon}$

Fifth order differentiation $\left(\frac{\partial^5}{\partial\theta^5}, \frac{\partial^5}{\partial\theta^4\partial\epsilon}, \frac{\partial^5}{\partial\theta^3\partial\epsilon^2}, \frac{\partial^5}{\partial\theta^2\partial\epsilon^3}, \frac{\partial^5}{\partial\theta\partial\epsilon^4}, \frac{\partial^5}{\partial\epsilon^5}\right)$ of (B1) yields

$$\begin{aligned}
0 \;=\; & 10\alpha_i^{\theta\theta}\int V_i^{\theta\theta\theta\alpha_i}dF(\epsilon) + 5\alpha_i^{\theta}\int V_i^{\theta\theta\theta\theta\alpha_i}dF(\epsilon) + 30\alpha_i^{\theta}\alpha_i^{\theta\theta}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon) \\
& +20\alpha_i^{\theta}\alpha_i^{\theta\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon) + 30\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 10\alpha_i^{\theta\theta}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) \\
& +5\alpha_i^{\theta}\alpha_i^{\theta\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) + 15\alpha_i^{\theta}\left(\alpha_i^{\theta\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 10\left(\alpha_i^{\theta}\right)^3\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +10\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 10\alpha_i^{\theta\theta\theta}\int V_i^{\theta\theta\alpha_i}dF(\epsilon) + 5\alpha_i^{\theta\theta\theta\theta}\int V_i^{\theta\alpha_i}dF(\epsilon) \\
& +\left(\alpha_i^{\theta}\right)^5\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 5\left(\alpha_i^{\theta}\right)^4\int V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 15\left(\alpha_i^{\theta\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon) \\
& +10\left(\alpha_i^{\theta}\right)^3\int V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon) + \alpha_i^{\theta\theta\theta\theta\theta}\int V_i^{\alpha_i}dF(\epsilon) + 10\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\theta\theta\alpha_i\alpha_i}dF(\epsilon) + \int V_i^{\theta\theta\theta\theta\theta}dF(\epsilon)
\end{aligned}$$

$$\begin{aligned}
0 \;=\; & 6\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon) + 4\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3\int V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +12\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 12\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon) + 12\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon) \\
& +4\alpha_i^{\theta}\alpha_i^{\theta\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) + \alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^4\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 4\left(\alpha_i^{\theta}\right)^3\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 4\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) + 3\alpha_i^{\epsilon}\left(\alpha_i^{\theta\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +6\alpha_i^{\theta\theta}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) + 6\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon) + 4\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\theta\theta\theta\alpha_i\alpha_i}dF(\epsilon) \\
& +4\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon) + \alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon) + 12\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +4\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon) + 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon) \\
& +\alpha_i^{\theta\theta\theta\theta\epsilon}\int V_i^{\alpha_i}dF(\epsilon) + 4\alpha_i^{\theta\theta\theta\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon) + \alpha_i^{\epsilon}\int V_i^{\theta\theta\theta\theta\alpha_i}dF(\epsilon) \\
& +4\alpha_i^{\theta\epsilon}\int V_i^{\theta\theta\theta\alpha_i}dF(\epsilon) + 6\alpha_i^{\theta\theta\epsilon}\int V_i^{\theta\theta\alpha_i}dF(\epsilon) \\
& +4\alpha_i^{\theta}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT} + 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT} + 12\alpha_i^{\theta}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT} \\
& +3\left(\alpha_i^{\theta\theta}\right)^2\int V_i^{\alpha_i\alpha_i}d\Delta_{iT} + 6\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT} + 4\left(\alpha_i^{\theta}\right)^3\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT} \\
& +4\alpha_i^{\theta\theta\theta}\int V_i^{\theta\alpha_i}d\Delta_{iT} + \alpha_i^{\theta\theta\theta\theta}\int V_i^{\alpha_i}d\Delta_{iT} + 4\alpha_i^{\theta}\int V_i^{\theta\theta\theta\alpha_i}d\Delta_{iT} + 6\alpha_i^{\theta\theta}\int V_i^{\theta\theta\alpha_i}d\Delta_{iT} \\
& +\left(\alpha_i^{\theta}\right)^4\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT} + \int V_i^{\theta\theta\theta\theta}d\Delta_{iT}
\end{aligned}$$

$$
\begin{aligned}
0 \;=\;& 6\alpha_i^{\theta}\left(\alpha_i^{\theta\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\int V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon)\\
&+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\theta\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
&+3\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+6\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta\epsilon}\int \alpha_i^{\theta\theta\epsilon}V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
&+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+3\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\theta}\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+6\alpha_i^{\theta\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+12\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
&+6\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\left(\alpha_i^{\theta\epsilon}\right)^2\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+\left(\alpha_i^{\epsilon}\right)^2\int V_i^{\theta\theta\theta\alpha_i\alpha_i}dF(\epsilon)\\
&+\alpha_i^{\theta\theta\theta\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\theta\theta\alpha_i}dF(\epsilon)+3\alpha_i^{\theta\theta\epsilon\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\theta\theta\alpha_i}dF(\epsilon)\\
&+12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+2\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
&+6\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT}\\
&+6\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
&+6\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\theta\epsilon}\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}+6\alpha_i^{\theta\theta\epsilon}\int V_i^{\theta\alpha_i}d\Delta_{iT}\\
&+2\alpha_i^{\theta\theta\theta\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}+2\alpha_i^{\epsilon}\int V_i^{\theta\theta\theta\alpha_i}d\Delta_{iT}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & 3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+2(\alpha_i^{\epsilon})^3\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +6(\alpha_i^{\epsilon})^2\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+(\alpha_i^{\epsilon})^3\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +2\alpha_i^{\theta\epsilon\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+(\alpha_i^{\epsilon})^3\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +2\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+3(\alpha_i^{\epsilon})^2\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +6\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta}(\alpha_i^{\epsilon})^2\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +(\alpha_i^{\epsilon})^3\int V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\theta\theta\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)+2\alpha_i^{\theta\epsilon\epsilon\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\theta\theta\alpha_i}dF(\epsilon)\\
& +6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\epsilon}\left(\alpha_i^{\theta\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
& +6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+3\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\theta\alpha_i\alpha_i}dF(\epsilon)\\
& +6(\alpha_i^{\epsilon})^2\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\theta}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}\\
& +3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+3(\alpha_i^{\epsilon})^2\alpha_i^{\theta\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+3\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
& +3(\alpha_i^{\epsilon})^2\left(\alpha_i^{\theta}\right)^2\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+12\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
& +3\alpha_i^{\theta\theta\epsilon\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}+3\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}+6\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\theta\alpha_i}d\Delta_{iT}+6\left(\alpha_i^{\theta\epsilon}\right)^2\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}\\
& +3(\alpha_i^{\epsilon})^2\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & 6(\alpha_i^{\epsilon})^2\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+4(\alpha_i^{\epsilon})^3\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)+6(\alpha_i^{\epsilon})^2\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +6(\alpha_i^{\epsilon})^2\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i\alpha_i}dF(\epsilon)+4\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)\\
& +\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+3(\alpha_i^{\epsilon\epsilon})^2\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+(\alpha_i^{\epsilon})^4\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +4\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}dF(\epsilon)+12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)+\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\int V_i^{\theta\alpha_i}dF(\epsilon)\\
& +\alpha_i^{\theta\epsilon\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}dF(\epsilon)+4\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+4\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +3(\alpha_i^{\epsilon\epsilon})^2\int V_i^{\theta\alpha_i\alpha_i}dF(\epsilon)+(\alpha_i^{\epsilon})^4\int V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}dF(\epsilon)\\
& +12\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}+12(\alpha_i^{\epsilon})^2\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+4(\alpha_i^{\epsilon})^3\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
& +4\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+12\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}+4(\alpha_i^{\epsilon})^3\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}\\
& +4\alpha_i^{\epsilon\epsilon\epsilon}\int V_i^{\theta\alpha_i}d\Delta_{iT}+4\alpha_i^{\theta\epsilon\epsilon\epsilon}\int V_i^{\alpha_i}d\Delta_{iT}+12\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}+12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \alpha_i^{\epsilon\epsilon\epsilon\epsilon\epsilon} \int V_i^{\alpha_i} dF(\epsilon) + 10 \left(\alpha_i^{\epsilon}\right)^2 \alpha_i^{\epsilon\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i\alpha_i} dF(\epsilon) + \left(\alpha_i^{\epsilon}\right)^5 \int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i} dF(\epsilon) \\
&\quad + 10 \left(\alpha_i^{\epsilon}\right)^3 \alpha_i^{\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i} dF(\epsilon) + 10 \alpha_i^{\epsilon\epsilon} \alpha_i^{\epsilon\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) + 5 \alpha_i^{\epsilon} \alpha_i^{\epsilon\epsilon\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i} dF(\epsilon) \\
&\quad + 15 \alpha_i^{\epsilon} \left(\alpha_i^{\epsilon\epsilon}\right)^2 \int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i} dF(\epsilon) \\
&\quad + 5 \left(\alpha_i^{\epsilon}\right)^4 \int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i} d\Delta_{iT} + 15 \left(\alpha_i^{\epsilon\epsilon}\right)^2 \int V_i^{\alpha_i\alpha_i} d\Delta_{iT} + 5 \alpha_i^{\epsilon\epsilon\epsilon\epsilon} \int V_i^{\alpha_i} d\Delta_{iT} \\
&\quad + 30 \left(\alpha_i^{\epsilon}\right)^2 \alpha_i^{\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i\alpha_i} d\Delta_{iT} + 20 \alpha_i^{\epsilon} \alpha_i^{\epsilon\epsilon\epsilon} \int V_i^{\alpha_i\alpha_i} d\Delta_{iT}
\end{aligned}
$$

Evaluating at $\epsilon = 0$, we obtain

$$
\begin{aligned}
\alpha_i^{\theta\theta\theta\theta\theta} &= 10 \alpha_i^{\theta\theta} \frac{E\left[V_i^{\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 5 \alpha_i^{\theta} \frac{E\left[V_i^{\theta\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 30 \alpha_i^{\theta} \alpha_i^{\theta\theta} \frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 20 \alpha_i^{\theta} \alpha_i^{\theta\theta\theta} \frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad + 30 \left(\alpha_i^{\theta}\right)^2 \alpha_i^{\theta\theta} \frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 10 \alpha_i^{\theta\theta} \alpha_i^{\theta\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 5 \alpha_i^{\theta} \alpha_i^{\theta\theta\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad + 15 \alpha_i^{\theta} \left(\alpha_i^{\theta\theta}\right)^2 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 10 \left(\alpha_i^{\theta}\right)^3 \alpha_i^{\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 10 \left(\alpha_i^{\theta}\right)^2 \alpha_i^{\theta\theta\theta} \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad + 10 \alpha_i^{\theta\theta\theta} \frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 5 \alpha_i^{\theta\theta\theta\theta} \frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\theta}\right)^5 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 5 \left(\alpha_i^{\theta}\right)^4 \frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad + 15 \left(\alpha_i^{\theta\theta}\right)^2 \frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 10 \left(\alpha_i^{\theta}\right)^3 \frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 10 \left(\alpha_i^{\theta}\right)^2 \frac{E\left[V_i^{\theta\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
&\quad + \frac{E\left[V_i^{\theta\theta\theta\theta\theta}\right]}{E\left[V_i^2\right]}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i^{\theta\theta\theta\epsilon} \quad = \quad & 6\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2 \frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3 \frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +12\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 12\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 12\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^4 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\left(\alpha_i^{\theta}\right)^3\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\left(\alpha_i^{\theta\theta}\right)^2 \frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +6\alpha_i^{\theta\theta}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 12\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon}\frac{E\left[V_i^{\theta\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 4\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\theta}\alpha_i^{\theta\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 12\alpha_i^{\theta}\alpha_i^{\theta\theta}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +3\left(\alpha_i^{\theta\theta}\right)^2 \frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\left(\alpha_i^{\theta}\right)^2 \frac{\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 4\left(\alpha_i^{\theta}\right)^3 \frac{\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +4\alpha_i^{\theta\theta\theta}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + \alpha_i^{\theta\theta\theta\theta}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 4\alpha_i^{\theta}\frac{\int V_i^{\theta\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\alpha_i^{\theta\theta}\frac{\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +\left(\alpha_i^{\theta}\right)^4 \frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + \frac{\int V_i^{\theta\theta\theta\theta}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i^{\theta\theta\theta\epsilon\epsilon} \;=\;& 6\alpha_i^{\theta}\left(\alpha_i^{\theta\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\theta\theta\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+3\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^3\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^3\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\theta\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+3\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\theta\theta}\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\theta\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+12\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\theta\theta\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\left(\alpha_i^{\theta\epsilon}\right)^2\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^2\frac{E\left[V_i^{\theta\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+3\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]}+3\alpha_i^{\theta\theta\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}+\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+12\alpha_i^{\theta}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+2\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^3\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\left(\alpha_i^{\theta}\right)^2\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\theta}\alpha_i^{\theta\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\alpha_i^{\theta\epsilon}\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+2\alpha_i^{\epsilon}\alpha_i^{\theta\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+6\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\alpha_i^{\theta\epsilon}\frac{\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+6\alpha_i^{\theta\theta\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+2\alpha_i^{\theta\theta\theta\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+2\alpha_i^{\epsilon}\frac{\int V_i^{\theta\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i^{\theta\theta\epsilon\epsilon\epsilon} = \;& 3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +\alpha_i^{\epsilon\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^3\left(\alpha_i^{\theta}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\theta\epsilon\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 2\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +3\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +3\alpha_i^{\epsilon}\alpha_i^{\theta\theta}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +6\alpha_i^{\theta}\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\left(\alpha_i^{\theta}\right)^2\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + \left(\alpha_i^{\epsilon}\right)^3\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +2\alpha_i^{\theta\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]} + \alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon}\left(\alpha_i^{\theta\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +6\alpha_i^{\theta\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]} \\
& +6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\frac{\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\alpha_i^{\theta}\alpha_i^{\theta\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +6\alpha_i^{\epsilon}\alpha_i^{\theta\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\theta}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +3\alpha_i^{\epsilon\epsilon}\left(\alpha_i^{\theta}\right)^2\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\left(\alpha_i^{\theta}\right)^2\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +12\alpha_i^{\theta}\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\theta\theta\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\theta\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 6\alpha_i^{\theta\epsilon\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} \\
& +6\left(\alpha_i^{\theta\epsilon}\right)^2\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]} + 3\left(\alpha_i^{\epsilon}\right)^2\frac{\int V_i^{\theta\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}
$$

$$\begin{aligned}
\alpha_i^{\theta\epsilon\epsilon\epsilon\epsilon} &= 6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+4\left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+6\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+4\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+6\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\left(\alpha_i^{\epsilon\epsilon}\right)^2\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^4\alpha_i^{\theta}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+4\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon}\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+4\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+4\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+3\left(\alpha_i^{\epsilon\epsilon}\right)^2\frac{E\left[V_i^{\theta\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^4\frac{E\left[V_i^{\theta\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+12\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\theta\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+12\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+4\left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\theta}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+4\alpha_i^{\epsilon\epsilon\epsilon}\alpha_i^{\theta}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+12\alpha_i^{\epsilon}\alpha_i^{\theta}\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+4\left(\alpha_i^{\epsilon}\right)^3\frac{\int V_i^{\theta\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+4\alpha_i^{\epsilon\epsilon\epsilon}\frac{\int V_i^{\theta\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+4\alpha_i^{\theta\epsilon\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+12\alpha_i^{\epsilon\epsilon}\alpha_i^{\theta\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+12\alpha_i^{\epsilon}\alpha_i^{\theta\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}$$

$$\begin{aligned}
\alpha_i^{\epsilon\epsilon\epsilon\epsilon\epsilon} &= 10\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+\left(\alpha_i^{\epsilon}\right)^5\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+10\left(\alpha_i^{\epsilon}\right)^3\alpha_i^{\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+10\alpha_i^{\epsilon\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+5\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\frac{E\left[V_i^{\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}+15\alpha_i^{\epsilon}\left(\alpha_i^{\epsilon\epsilon}\right)^2\frac{E\left[V_i^{\alpha_i\alpha_i\alpha_i}\right]}{E\left[V_i^2\right]}\\
&+5\left(\alpha_i^{\epsilon}\right)^4\frac{\int V_i^{\alpha_i\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+15\left(\alpha_i^{\epsilon\epsilon}\right)^2\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+5\alpha_i^{\epsilon\epsilon\epsilon\epsilon}\frac{\int V_i^{\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}\\
&+30\left(\alpha_i^{\epsilon}\right)^2\alpha_i^{\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}+20\alpha_i^{\epsilon}\alpha_i^{\epsilon\epsilon\epsilon}\frac{\int V_i^{\alpha_i\alpha_i}d\Delta_{iT}}{E\left[V_i^2\right]}
\end{aligned}$$