THE WINNER'S CURSE

Consider an auction for a single item whose value to a buyer is not known with certainty, but must be estimated. Each player's bid will be based on his estimate, which in turn will be based on his own information, and any information that he can obtain from others that he considers reliable. This in turn introduces new strategic elements in the game. If a player knows that in an extensive game his actions may influence the beliefs of others, then he may modify his actions to convey information that gives himself the most benefit.

A very simple circumstance in which the implications of an estimated value can be seen is a second-price sealed bid auction for a single item. This item in truth has a common value v, the same for any buyer, which is unknown to the buyers. Suppose each bidder j =1,...,J has an estimate t_j of v. Suppose that these estimates are independent and identically distributed with a cumulative distribution function $F(t_j|v)$. We will assume that higher v will shift the distribution of t_1 upward, so that when v is large, t_1 will tend to be larger. Technically, we will require that F(t|v) be decreasing in v; the condition that $F(t|v_1) \geq F(t|v_2)$ whenever $v_1 < v_2$, with strict inequality at some arguments, is called *stochastic dominance*. An implication of stochastic dominance is that the mean of $F(t|v_2)$ is larger than the mean of $F(t|v_1)$, and all the quantiles of $F(t|v_2)$, such as the median, are at least as large as the corresponding quantiles of $F(t|v_1)$. For example, if the estimates are uniformly distributed on the interval [0,2v], then $F(t|v) = ½t/v$ for $0 < t < 2v$, which satisfies the stochastic dominance property and has **E**t = v. Note that if the family F(t|v) displays stochastic dominance as v increases, so does the family $F(t|v)^J$ for any positive integer J.

Suppose for the moment that the players bid their estimates of the value of v; i.e., they behave as if the t_j were independent private values, and follow the bidding strategies that would be optimal under these circumstances in a second price auction. If we let $t_{(j)}$ denote the j^{th} highest estimate, then the auction will award the item to the buyer with the estimate $t_{(1)}$ at the price $t_{(2)}$. Now, the cumulative distribution function of $t_{(1)}$ is $F(t_{(1)}|v)^J$, the probability that all J estimates are no higher than this value. The cumulative distribution function of $t_{(2)}$ is the probability that all J estimates are no higher than this value plus the probability that one of them is, or

$$H(t_{(2)}|v) = F(t_{(2)}|v)^J + J\, F(t_{(2)}|v)^{J-1}(1-F(t_{(2)}|v)).$$

These formulas can be found in any probability theory text under "order statistics". The expected revenue to the seller is then

$$\mathbf{E}\, t_{(2)} = \int_0^\infty tH'(t|v)dt = \int_0^\infty [1-H(t|v)]dt,$$

where the last formula is obtained by integration by parts. When this expectation is no greater than one, the winner on average pays less than the value of the item, and is a net gainer from the auction. However, when the expectation exceeds v, the winner on average pays more than the value of the item. This is called the *winner's curse*. Due to the properties of order statistics, the winner's curse will tend to occur when J is moderately large, even if the individual estimates of value are on average correct. Consider the uniform distribution example $F(t|v) = \frac{1}{2}t/v$, which yields

$$H(t|v) = (t/2v)^J + J(t/2v)^{J-1}(1-t/2v).$$

Then,

$$\mathbf{E}\, t_{(2)} = \int_0^{2v} [(t/2v)^J + J(t/2v)^{J-1}(1-t/2v)]dt = 2v(J-1)/(J+1).$$

For J = 2, $E\, t_{(2)} = 2v/3$, and for J = 3, $E\, t_{(2)} = v$, so there is no winner's curse. However, for J > 3, $E\, t_{(2)} > 1$, and the winner's curse is operating.

Rational players facing uncertainty about value will be aware of the winner's curse, and will respond to it by shaving their bids in situations where it would otherwise operate. Then, the bids in a second-price sealed bid auction where players are uncertain about the value of the item will tend to be lower than their estimates of the value. To solve for the Nash equilibrium bid function, it is necessary to consider the bidders' beliefs about v. Before obtaining their estimates t_j, the bidders have a prior belief that v is distributed with a density g(v). Assume this is the same for all bidders, and all bidders know this. Suppose b = B(t) is the bid that a player will make, given his estimate t of the value, and assume that B is increasing and differentiable. Let t = V(b) denote its inverse. Consider bidder 1. The maximum T of the remaining bids, given v, has the cumulative distribution function $F(T|v)^{J-1}$. The payoff to bidder 1 from bid b_1 given v and T is

$$(v-B(T))\mathbf{1}(b_1>B(T)) = (v-B(T))\mathbf{1}(V(b)>T).$$

The joint density of v and T is $g(v)(J-1)F(T|v)^{J-2}f(T|v)$. Hence, the expected payoff to bidder 1 is

$$\int_{v=0}^{\infty}\int_{T=0}^{V(b_1)} (v-B(T))g(v)(J-1)F(T|v)^{J-2}f(T|v)dTdv$$

This is maximized when b_1 satisfies the first-order condition

$$0 = V'(b_1)\int_{v=0}^{\infty} (v-B(V(b_1)))g(v)(J-1)F(V(b_1)|v)^{J-2}f(V(b_1)|v)dv.$$

In a symmetric Nash equilibrium, the first-order condition is met at $b_1 = B(t_1)$, or $V(b_1) = t_1$, implying

$$B(t) = \int_{v=0}^{\infty} vg(v)F(t|v)^{J-2}f(t|v)dv/\int_{v=0}^{\infty} g(v)F(t|v)^{J-2}f(t|v)dv.$$

But $g(v)F(t|v)^{J-2}f(t|v)dv/\int_{v=0}^{\infty} g(v)F(t|v)^{J-2}f(t|v)dv$ is just the posterior conditional distribution of v, given an estimate t that wins the auction, and B(t) is therefore the mean of this posterior conditional distribution, given an estimate t that wins the auction. The first-order stochastic dominance of F(t|v) with increasing v ensures that the mean of the posterior conditional distribution is increasing in t, satisfying the assumption made earlier that B(t) was increasing in t.

As an example, assume again that F(t|v) = t/2v for 0 < t/v < 2, and assume that the prior beliefs have an exponential density, $g(v) = e^{-v/\mu}/\mu$, which has mean μ. We need to evaluate integrals of the form

$$\int_{v=0}^{\infty} v^i g(v)(J-1)F(t_1|v)^{J-2}f(t_1|v)dv = \int_{t_1/2}^{\infty} v^i\, e^{-v/\mu}(J-1)(t_1/2v)^{J-2}dv/2v\mu$$
$$= [(J-1)t_1^{J-2}/(2\mu)^{J-1}]\mu^i\Gamma(i+2-J,t_1/2\mu),$$

where $\Gamma(-k,c) = \int_c^{\infty} x^{-k-1}e^{-x}dx$ is an incomplete gamma function.[1] Then, the bid function for J > 3 satisfies

$$B(t) = \mu\Gamma(3-J,t/2\mu)/\Gamma(2-J,t/2\mu)$$
$$= \mu + \mu/[(J-3)\Gamma(0,t/2\mu)(t/2\mu)^{J-3}e^{t/2\mu} + (J-3)\sum_{n=0}^{J-5} (-1)^{J-4-n}n!(t/2\mu)^{J-n-4} + ((J-3)!-1)].$$

When t/2μ = α is fixed, so that the estimate t is at the α^{th} percentile of the distribution of estimates, then the bid function is homogeneous of degree one in the mean value μ. For fixed μ, when α is near zero, B(t) is approximately t/2, and when α is near one, B(t) is approximately μ. Thus, for J > 3 there is substantial bid shaving compared to each bidder's unbiased estimate of value.

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[1]Using integration by parts, Λ(-k,c) satisfies the recursion $\Lambda(-k,c) = c^{-k}e^{-c}/k + \Lambda(-k-1,c)/k$, which can be iterated to give $\Gamma(-k,c) = \Gamma(0,c) + e^{-c}\sum_{n=0}^{k-1} (-1)^{k-1+n}n!/c^{n+1}$. Finally, Γ(0,c) has a continued fraction expansion $\Gamma(0,c) = e^{-c}[1+1/c+2/c+3/c+...]$.