Econ. C103, 2003 Daniel McFadden

MECHANISM DESIGN, DIRECT SELLING MECHANISMS, EFFICIENT AUCTIONS

The theory of mechanism design provides some general insights into the construction of resource allocation mechanisms that achieve specified objectives. One of its tools is to establish a correspondence between possibly complex auction designs and relatively straightforward but somewhat abstract constructs called direct selling mechanisms. Then, theoretical properties of direct selling mechanisms can be translated into propositions about auction designs. Further, the properties of direct selling mechanisms may be directly useful in suggesting the form of implementations that accomplish specific purposes.

A primitive of mechanism design is the *environment* in which the allocation mechanism is considered. For current discussion, consider an environment in which one indivisible item is available for trade. This item is owned by one seller, indexed $j = 0$, and there are J potential buyers, indexed $j = 1,...,J$. Assume that all the players have independent private values for the item, and denote these $v_0, v_1, ..., v_J$. Each value v_j is known to its holder with certainty, but is unknown to all other players. However, it is common knowledge to all players that the value v_j of player j is drawn from a cumulative probability distribution function G_j with $G_j(0) = 0$. Another possible environment, which we will not consider here, is that players are uncertain about their value of the item, and the values are jointly distributed across the different players.

A *direct selling mechanism* is defined by (1) *assignment probabilities* $p_j(v_0,v_1,...,v_J)$ that are non-negative and sum to one, and give the probabilities that the item will be assigned to each player $j = 0,1,...,J$; (2) *cost functions* $c_j(v_0,v_1,...,v_J)$ that give the amount paid by each player $j = 1,...,J$ to the seller $j = 0$; (3) a *revelation* step in which each player j, knowing his value v_j, and knowing the functions $p_k(\cdot)$ and $c_k(\cdot)$ for $k = 0,...,J$, sends a message to an auctioneer with a reported value r_j; and (4) an *execution* step in which an assignment of the item is made using the probabilities $p_j(r_0,...,r_J)$, and payments $c_j(r_0,...,r_J)$ are made to the seller. Note that in a direct selling mechanism, payments may be required whether or not a player wins the item.

Example 1: A second-price, sealed-bid auction in which the seller also acts as a bidder has the assignment probabilities $p_j(v_0,v_1,...,v_J) = 1(v_j > \max_{i\neq j}v_i)$ for $j = 0,...,J$ and the cost functions $c_j(v_0,v_1,...,v_J) = 1(v_j > \max_{i\neq j}v_i)\cdot\max_{i\neq j}v_i$ for $j = 1,...,J$.

Example 2: A symmetric first-price, sealed bid auction in which the seller has value zero, and all buyers have the same value distribution G, and each bids the conditional second value $CSV(v_j) = E\{\max_{i\neq j}v_i|v_j > \max_{i\neq j}v_i\}$, has for $j = 1,...,J$ the assignment probabilities $p_j(v_0,v_1,...,v_J) = 1(v_j > \max_{i\neq j}v_i)$ and the cost functions $c_j(v_0,v_1,...,v_J) = 1(v_j > \max_{i\neq j}v_i)\cdot CSV(v_j)$.

These examples show that second-price and symmetric first-price sealed bid auctions map into direct sales mechanisms that have the same assignment probabilities and payoffs as the original auctions, and hence have equivalent expected payoffs. This mapping is not special to these auctions. In fact, any auction mechanism, no matter how complex, with players who select Nash equilibrium strategies, will produce assignment probabilities and cost functions, and these in turn define a direct selling mechanism with the same expected payoffs. This correspondence allows properties of direct selling mechanisms to be translated into corresponding properties of families of auction mechanisms.

Definition: A direct selling mechanism is *incentive compatible* or *truth-revealing* if each buyer finds it optimal to report his own value truthfully when all other players are doing so. A direct selling mechanism is individually rational if each player has a non-negative expected payoff from participating.

Definition: For player 1, the expected probability of winning is

(1) $$P_1(v_1) = \int_{v_2=0}^{\infty}\cdots\int_{v_J=0}^{\infty} p_1(v_0,v_1,v_2,\ldots,v_J)G_0(dv_0)G_2(dv_2)\cdots G_J(dv_J),$$

and the expected cost is

(2) $$C_1(v_1) = \int_{v_2=0}^{\infty}\cdots\int_{v_J=0}^{\infty} c_1(v_0,v_1,v_2,\ldots,v_J)G_0(dv_0)G_2(dv_2)\cdots G_J(dv_J).$$

The expected payoff from reporting value r_1 when the truth is v_1 is then

(3) $$U_1(r_1,v_1) = P_1(r_1)\cdot v_1 - C_1(r_1).$$

Incentive-compatibility requires that $U_1(r_1,v_1) \le U_1(v_1,v_1)$ for all v_1, and individual rationality requires that $\max_r U_1(r,v_1) \ge 0$. There are analogous expressions for each of the other players.

Example 1 (continued): The second-price sealed bid auction has

$$P_1(v) = \int_{v_2=0}^{v}\cdots\int_{v_J=0}^{v} G_0(dv_0)G_2(dv_2)\cdots G_J(dv_J) = \prod_{i\neq 1} G_i(v),$$

and

$$C_1(v_1) = \int_{v_2=0}^{v}\cdots\int_{v_J=0}^{v} \max_{i\neq j} v_i \, G_0(dv_0)G_2(dv_2)\cdots G_J(dv_J) = CSV_1(v)P_1(v).$$

Theorem. A direct selling mechanism is incentive-compatible and individually rational if and only if (a) $P_j(v)$ is non-decreasing for j = 0,...,J, (b) $C_j(0) \leq 0$ for j = 1,...,J, and (c) the expected cost satisfies

(4) $$C_j(v) = C_j(0) + P_j(v)\cdot v - \int_{s=0}^{v} P_j(s)ds.$$

Remark: This theorem shows that the requirements of incentive compatibility and individual rationality completely determine the cost function, once the assignment probabilities are established. Then, two mechanisms with these properties, the same assignment probabilities, and the same costs at value zero, will necessarily yield the same revenue to the seller.

Proof of the theorem: First assume that conditions (a)-(c) hold. Then

(4) $$U_j(r,v) = -C_j(0) + P_j(r)(v-r) + \int_{s=0}^{r} P_j(s)ds = -C_j(0) + \int_{s=0}^{v} P_j(s)ds + \int_{s=v}^{r} [P_j(s)-P_j(r)]ds.$$

The first two terms on the right-hand-side of (4) are non-negative. The last term is zero at r = v, and non-positive otherwise. (Check the cases v < r and v > r separately, and use the fact that P(s) is non-decreasing in s.) Hence, $U_j(r,v) \leq U_j(v,v)$, implying incentive compatibility, and $U_j(v,v) \geq 0$, implying individual rationality.

To prove the "only if" part of the theorem, assume that the mechanism is incentive-compatible and individually rational. Then,

(5) $$P_j(r)\cdot v - C_j(r) \leq P_j(v)\cdot v - C_j(v) \qquad \text{when v is true,}$$

(6) $$P_j(v)\cdot r - C_j(v) \leq P_j(r)\cdot r - C_j(r) \qquad \text{when r is true.}$$

Adding these inequalities, $(P_j(v) - P_j(r))\cdot(v - r) \geq 0$. Then, v > r implies $P_j(v) \geq P_j(r)$, and (a) holds. Next, break the interval [0,v] into subintervals [v(k-1)/K,vk/K] for k = 1,...,K. The inequality (5) applied to the end points of these subintervals gives

(7) $$\begin{aligned} C_j(vk/K) - C_j(v(k-1)/K) &\leq [P_j(vk/K) - P_j(v(k-1)/K)]vk/K \\ &= P_j(vk/K)vk/K - P_j(v(k-1)/K)v(k-1)/K - P_j(v(k-1)/K)v/K. \end{aligned}$$

Similarly, the inequality (6) applied to the end points of these subintervals gives

(8) $$\begin{aligned} C_j(vk/K) - C_j(v(k-1)/K) &\geq [P_j(vk/K) - P_j(v(k-1)/K)]v(k-1)/K \\ &= P_j(vk/K)vk/K - P_j(v(k-1)/K)v(k-1)/K - P_j(vk/K)v/K. \end{aligned}$$

Adding the inequalities (7) over k = 1,...,K, and similarly adding the inequalities (8), gives

(9) $\quad P_j(v)v - \sum_{k=1}^{K} P_j(vk/K)v/K \le C_j(v) - C_j(0) \le P_j(v)v - \sum_{k=1}^{K} P_j(v(k-1)/K)v/K.$

Letting $K \to \infty$, both ends of (9) converge to $P_j(v)v - \int_{s=0}^{v} P_j(s)ds$. Hence, this establishes that $C_j(v) = C_j(0) + P_j(v)v - \int_{s=0}^{v} P_j(s)ds = C_j(0) + \int_{s=0}^{v} [P_j(v)-P_j(s)]ds \ge C_j(0)$, implying (c). The individual rationality condition $0 \le \max_r U_j(r,0) = -C_j(0)$ implies (b). □

Theorem [*Revenue Equivalence*] If two individually rational, incentive-compatible direct selling mechanisms have the same assignment probabilities, then they yield the same revenue to the seller.

Corollary: All auctions that are *efficient* (i.e., assign the item to the highest-value bidder with probability one) are revenue equivalent. Then, in particular, the four standard auctions with symmetric bidders, which have this assignment probability, are revenue equivalent.

Suppose that the value distribution $G_j(v)$ for buyer j has a density $g_j(v)$, so that $G_j(dv) = g_j(v)dv$, and consider the case where the seller has with certainty value zero for the item. The seller's expected revenue from buyer j in an individually rational, incentive compatible direct selling mechanism is

$$
\begin{aligned}
(10) \quad R_j &= \int_{v=0}^{\infty} C_j(v)G_j(dv) = C_j(0) + \int_{v=0}^{\infty} P_j(v)vG_j(dv) - \int_{v=0}^{\infty}\int_{s=0}^{v} P_j(s)dsG_j(dv) \\
&= C_j(0) + \int_{v=0}^{\infty} P_j(v)vG_j(dv) - \int_{s=0}^{\infty}\int_{v=s}^{\infty} P_j(s)dsG_j(dv) \\
&= C_j(0) + \int_{v=0}^{\infty} P_j(v)vG_j(dv) - \int_{v=0}^{\infty} P_j(v)[1-G_j(v)]dv \\
&= C_j(0) + \int_{v=0}^{\infty} P_j(v)[v - (1-G_j(v))/g_j(v)]G_j(dv) \\
&= C_j(0) + \int_{v_1=0}^{\infty}\cdots\int_{v_J=0}^{\infty} p_j(v_1,\ldots,v_J)[v_j - (1-G_j(v_j))/g_j(v_j)]G_1(dv_1)\cdots G_J(dv_J).
\end{aligned}
$$

The seller's total expected revenue R is the sum of (10) over j = 1,...,J, or

$$(11) \quad R_j = \sum_{j=1}^{J} C_j(0) + \int_{v_1=0}^{\infty}\cdots\int_{v_J=0}^{\infty} \sum_{j=1}^{J} p_j(v_1,\ldots,v_J)[v_j - (1-G_j(v_j))/g_j(v_j)]G_1(dv_1)\cdots G_J(dv_J)$$

This is maximized by setting $p_j(v_1,\ldots,v_J) = 1$ for the j that maximizes $v_j - (1-G_j(v_j))/g_j(v_j)$, with the seller retaining the item when all the terms $v_j - (1-G_j(v_j))/g_j(v_j)$ are negative. Since $C_j(0) \le 0$, it is revenue-maximizing to set $C_j(0) = 0$. The assignment probability just defined will, by construction, be consistent with incentive compatibility and individual rationality, provided that the expected assignment probabilities $P_j(v)$ are all non-decreasing in v. A sufficient condition for this is that $v_j - (1-G_j(v_j))/g_j(v_j)$ be non-decreasing in v_j. This is a condition on the value distribution that is satisfied by many, but not all, common probability distributions on the non-negative real line. The table below gives a few examples.

Distribution	**G(v)**	**v - (1-G(v))/g(v)**	**Condition satisfied?**
Exponential	$1 - e^{-av}$, $v > 0$, $a > 0$	$v - 1/a$	Yes
Power	v^a, $0 < v < 1$, $a > 0$	$(1+1/a)v - v^{1-a}/a$	Yes if $a \geq 1$, No if $a<1$
Ratio	$v/(a+v)$, $v>0$, $a>0$	$-1 + (1-1/a)v$	Yes if $a \geq 1$, No if $a<1$

When the consistency condition above is satisfied, it is possible to implement the revenue-optimal direct selling mechanism above as a form of sealed bid auction, with rules for the winner and the payment matching those in the direct selling mechanism. This is somewhat cumbersome, as it involves the value distribution of each player. We assumed these are known to everyone, so it is valid to have the seller use them in setting the auction rules. However, in reality, a seller may be less than completely confident that these are known exactly. There is considerable simplification, however, when the buyers' value distributions are symmetric. In this case, one can have the inequality $v_j - (1-G_1(v_j))/g_1(v_j) > v_i - (1-G_1(v_i))/g_1(v_i)$ only if $v_j > v_i$, so that if the item is sold, it goes to the buyer with the highest value. Suppose we attempt to implement the optimal revenue mechanism in this symmetric case via a second-price sealed bid auction with a reservation price ρ^* that satisfies $0 = \rho^* - (1-G_1(\rho^*))/g_1(\rho^*)$. As in our previous analysis of a second-price, sealed bid auction, each bidder has a dominant strategy of stating his true value, so the highest value bidder will win provided this value exceeds ρ^*, and the winner pays the larger of ρ^* and the second highest value. One can show that the expected revenue from this auction design is precisely the expected revenue from the revenue-optimal individually rational, incentive compatible direct selling mechanism. Then, the second-price sealed bid auction with this reserve price is revenue-optimal among all possible mechanism designs meeting these conditions.