Existence of Walrasian Equilibrium

Theorem (Grandmont-McFadden, 1972)

Define the closed unit simplex $\mathbf{U}^* = \{\mathbf{p}\in\mathbb{R}^m \mid \mathbf{p} \ge \mathbf{0} \text{ and } \mathbf{1}\cdot\mathbf{p} = 1\}$ and the open unit simplex $\mathbf{U}^0 = \{\mathbf{p}\in\mathbf{U}^* \mid \mathbf{p}>>\mathbf{0}\}$. Suppose there exists a set $\mathbf{U}$ with $\mathbf{U}^0 \subseteq \mathbf{U} \subseteq \mathbf{U}^*$ and an excess demand correspondence ζ that maps $\mathbf{U}$ into non-emply subsets of $\mathbb{R}^m$ and satisfies

(a) ζ is bounded below; i.e., there exists $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{b} \le \mathbf{x}$ for all $\mathbf{x} \in \zeta(\mathbf{p})$, $p \in \mathbf{U}$;

(b) For each $\mathbf{p} \in \mathbf{U}$, $\zeta(\mathbf{p})$ is a convex set, and $\mathbf{p}\cdot\mathbf{x} \le 0$ for all $\mathbf{x} \in \zeta(\mathbf{p})$;

(c) ζ is upper hemicontinuous on $\mathbf{U}$; i.e., the graph $\{(\mathbf{p},\mathbf{x})\in\mathbf{U}\times\mathbb{R}^m \mid \mathbf{x}\in\zeta(\mathbf{p})\}$ is a closed subset of $\mathbb{R}^m\times\mathbb{R}^m$.

Then there exists a $\mathbf{p}^* \in \mathbf{U}$ and a $\mathbf{x}^* \in \zeta(\mathbf{p}^*)$ such that $\mathbf{x}^* \le \mathbf{0}$.

Proof: Let $\mathbf{U}^k = \{\mathbf{p}\in\mathbf{U}^0 \mid p \ge (1/mk,...,1/mk)\}$; then the $\mathbf{U}^k$ are convex and compact and their union is $\mathbf{U}^0$. Let $\mathbf{X}^k$ denote the closed convex hull of $\{\zeta(\mathbf{p}) \mid \mathbf{p}\in\mathbf{U}^k\}$. Property (a), property (b) that $\mathbf{p}\cdot\mathbf{x} \le 0$ for all $\mathbf{x} \in \zeta(\mathbf{p})$, and the definition of $\mathbf{U}^k$ imply that $\mathbf{X}^k$ is bounded, and hence compact. For $(\mathbf{x},\mathbf{p}) \in \mathbf{X}^k\times\mathbf{U}^k$, define a mapping η into non-empty subsets of $\mathbf{X}^k\times\mathbf{U}^k$ by

$$\eta(\mathbf{x},\mathbf{p}) = \{(\mathbf{x}',\mathbf{p}')\in\mathbf{X}^k\times\mathbf{U}^k \mid \mathbf{x}'\in\zeta(\mathbf{p}) \text{ and } \mathbf{p}'\cdot\mathbf{x} \ge \mathbf{p}''\cdot\mathbf{x} \text{ for all } \mathbf{p}''\in\mathbf{U}^k\}.$$

The maximands of a linear function $\mathbf{p}''\cdot\mathbf{x}$ on the compact convex set $\mathbf{U}^k$ form an upper hemicontinuous, convex valued correspondence. Together with properties (b) and (c) of ζ, this implies that η is an upper hemicontinuous, convex-valued correspondence on $\mathbf{X}^k\times\mathbf{U}^k$. A fixed point theorem of Kakutani (1941) then guarantees that there exists $(\mathbf{x}^k,\mathbf{p}^k)$ such that $(\mathbf{x}^k,\mathbf{p}^k) \in \eta(\mathbf{x}^k,\mathbf{p}^k)$. Then $\mathbf{x}^k \in \zeta(\mathbf{p}^k)$ and $0 \ge \mathbf{p}^k\cdot\mathbf{x}^k \ge \mathbf{p}\cdot\mathbf{x}^k$ for all $\mathbf{p} \in \mathbf{U}^k$. Consider the sequence $(\mathbf{x}^k,\mathbf{p}^k)$, k = 1,2,... . Property (a) and $\mathbf{p}^1\cdot\mathbf{x}^k \le 0$ imply that this sequence is bounded. Hence, it has a subsequence converging to a limit point $(\mathbf{x}^0,\mathbf{p}^0)$. Property (c) implies $\mathbf{x}^0 \in \zeta(\mathbf{p}^0)$, while the property $0 \ge \mathbf{p}\cdot\mathbf{x}^k$ for all $\mathbf{p} \in \mathbf{U}^k$ implies $0 \ge \mathbf{p}\cdot\mathbf{x}^0$ for $\mathbf{p} \in \mathbf{U}^0$, since each $\mathbf{p} \in \mathbf{U}^0$ is contained in $\mathbf{U}^k$ for k sufficiently large. This in turn implies $\mathbf{0} \ge \mathbf{x}^0$. □