Analysis and Linear Algebra

Lectures 1-3 on the mathematical tools that will be used in C103

Set Notation

A , B	sets
$A \cup B$	union
$A \cap B$	intersection
$A \setminus B$	the set of objects in A that are not in B
ϕ.	Empty set
$A \subseteq B$	inclusion (A is contained in B)
. A^c	the complement of a set A (which may be relative to a set B that contains it)
$a \in A$	a is a member of A
$a \notin A$	a is not a member of A.

A family of sets is <u>disjoint</u> if the intersection of each pair is empty.

Sequences

If a_i is a sequence of real numbers indexed by $i = 1,2,...,$ then the sequence is said to have a limit (equal to a_o) if for each $\varepsilon > 0$, there exists n such that $|a_i - a_o| < \varepsilon$ for all $i \geq n$; the notation for a limit is $\lim_{i\to\infty} a_i = a_o$ or $a_i \to a_o$. The Cauchy criterion says that a sequence a_i has a limit if and only if, for each $\varepsilon > 0$, there exists n such that $|a_i - a_j| < \varepsilon$ for $i,j \geq n$.

Functions

A function $f{:}A \rightarrow B$ is a mapping from each object a in the domain A into an object $b = f(a)$ in the range B. The terms function, mapping, and transformation are interchangeable. The symbol f(C), termed the image of C, is the set of all objects f(a) for $a \in C$. For $D \subseteq B$, the symbol $f^{-1}(D)$ denotes the inverse image of D: the set of all $a \in A$ such that $f(a) \in D$.

Surjective, Bijective

The function f is onto [surjective] if B = f(A); it is one-to-one [bijective] if it is onto and if a,c $\in$ A and a $\neq$ c implies f(a) $\neq$ f(c). When f is one-to-one, the mapping f^{-1} is a function from B onto A. If C $\subseteq$ A, define the indicator function for C, denoted $1_{\mathbb{C}}$:A $\rightarrow$ $\mathbb{R}$, by $1_{\mathbb{C}}(a) = 1$ for a $\in$ C, and $1_{\mathbb{C}}(a) = 0$ otherwise. The notation $1(a\in C)$ is also used for the indicator function $1_{\mathbb{C}}$. A function is termed real-valued if its range is $\mathbb{R}$.

Sup, Max, Argmax

sup A — The supremum or least upper bound of a set of real numbers A,

$\sup_{c \in C} f(c)$ — A typical application, $f: C \rightarrow \mathbb{R}$ and $A = f(C)$

$f(d) = \max_{c \in C} f(c)$ — supremum is achieved by an object $d \in C$, so $f(d) = \sup_{c \in C} f(c)$

$\mathrm{argmax}_{c \in C} f(c)$ — set of maximizing elements, or a selection from this set

Analogous definitions hold for the infimum and minimum, denoted inf, min, and for argmin.

Limsup

The notation $\limsup_{i\to\infty} a_i$ means the limit of the suprema of the sets $\{a_i, a_{i+1}, \ldots\}$; because it is nonincreasing, it always exists (but may equal $+\infty$ or $-\infty$). An analogous definition holds for liminf.

Metrics and Metric Spaces

A real-valued function $\rho(a,b)$ defined for pairs of objects in a set A is a distance function if it is non-negative, gives a positive distance between all distinct points of A, has $\rho(a,b) = \rho(b,a)$, and satisfies the triangle inequality $\rho(a,b) \leq \rho(a,c) + \rho(c,b)$. A set A with a distance function ρ is termed a metric space. ¶

Topology of Metric Space

A (ε-)*neighborhood* of a point a in a metric space A is a set of the form $\{b \in A \mid \rho(a,b) < \varepsilon\}$. A set $C \subseteq A$ is *open* if for each point in C, some neighborhood of this point is also contained in C. A set $C \subseteq A$ is *closed* if its complement is open. The family of all the neighborhoods of the form above defines the *metric* or *strong* topology of A, the neighborhoods that define the standard for judging if points in A are close.

Compactness

The closure of a set C is the intersection of all closed sets that contain C. The interior of C is the union of all open sets contained in C; it can be empty. A covering of a set C is a family of open sets whose union contains C. The set C is said to be compact if every covering contains a finite sub-family which is also a covering.

Finite Intersection Property

A family of sets is said to have the finite intersection property if every finite sub-family has a non-empty intersection. A characterization of a compact set is that every family of closed subsets with the finite intersection property has a non-empty intersection.

Sequential Compactness

A metric space A is separable if there exists a countable subset B such that every neighborhood contains a member of B. All of the metric spaces encountered in economic analysis will be separable. A sequence a_i in a separable metric space A is convergent (to a point a_o) if the sequence is eventually contained in each neighborhood of a_0; we write $a_i \to a_o$ or $\lim_{i\to\infty} a_i = a_o$ to denote a convergent sequence. A set $C \subseteq A$ is compact if and only if every sequence in C has a convergent subsequence (which converges to a cluster point of the original sequence).

Continuity

Consider separable metric spaces A and B, and a function $f: A \to B$. The function f is <u>continuous</u> on A if the inverse image of every open set is open. Another characterization of continuity is that for any sequence satisfying $a_i \to a_o$, one has $f(a_i) \to f(a_o)$; the function is said to be continuous on $C \subseteq A$ if this property holds for each $a_o \in C$. Stated another way, f is continuous on C if for each $\varepsilon > 0$ and $a \in C$, there exists $\delta > 0$ such that for each b in a δ-neighborhood of a, f(b) is in a ε-neighborhood of f(a).

Sequential Continuity

For real valued functions on separable metric spaces, the concepts of supremum and limsup defined earlier for sequences have a natural extension: $\sup_{a\in A} f(a)$ denotes the least upper bound on the set $\{f(a) \mid a\in A\}$, and $\limsup_{a\to b} f(a)$ denotes the limit as $\varepsilon \to 0$ of the suprema of f(a) on ε-neighborhoods of b. Analogous definitions hold for inf and liminf. A real-valued function f is continuous at b if

$$\limsup_{a\to b} f(a) = \liminf_{a\to b} f(a).$$

Continuity-Preserving Operations

Continuity of real-valued functions f and g is preserved by the operations of absolute value $|f(a)|$, multiplication $f(a) \cdot g(a)$, addition $f(a)+g(a)$, and maximization $\max\{f(a),g(a)\}$ and minimization $\min\{f(a),g(a)\}$.

Uniform Continuity, Lipschitz

The function f is uniformly continuous on C if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $a \in C$ and $b \in A$ with b in a δ-neighborhood of a, one has $f(b)$ in a ε-neighborhood of $f(a)$. The distinction between continuity and uniform continuity is that for the latter a single $\delta > 0$ works for all $a \in C$. A function that is continuous on a compact set is uniformly continuous. The function f is Lipschitz on C if there exist $L > 0$ and $\delta > 0$ such that $|f(b) - f(a)| \leq L \cdot \rho(a,b)$ for all $a \in C$ and $b \in A$ with b in a δ-neighborhood of a.

Differentiability

Consider a real-valued function f on $\mathbb{R}$. The derivative of f at a_o, denoted $f'(a_o)$, $\nabla f(a_o)$, or $df(a_o)/da$, has the property if it exists that

$$|f(b) - f(a_o) - f'(a_o)(b-a_o)| \leq R(b-a_o),$$

where $\lim_{c\to 0} R(c)/c = 0$. The function is continuously differentiable at a_o if f' is a continuous function at a_o.

Taylor's Expansion

If a function is k-times continuously differentiable in a neighborhood of a point a_o, then for b in this neighborhood it has a Taylor's expansion

$$f(b) = \sum_{i=0}^{k} f^{(i)}(a_o) \cdot \frac{(b-a_o)^i}{i!} +$$

$$\left\{f^{(k)}(\lambda b + (1-\lambda)a_o) - f^{(k)}(a_o)\right\} \cdot \frac{(b-a_o)^k}{k!},$$

where $f^{(i)}$ denotes the i-th derivative, and λ is a scalar between zero and one.

Linear Spaces, Norms

A set C is termed a linear space if for every pair of elements $a, b \in C$ and real numbers α, β, the linear combination $\alpha a + \beta b \in C$. For example, the real line $\mathbb{R}$ is a linear space. For linear spaces A and B, a function $f: A \rightarrow B$ is linear if $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$ for all $a, b \in A$ and real numbers α, β. A real-valued function $\|a\|$ defined for objects in a linear space A is a norm if $\|a - b\|$ has the properties of a distance function, and $\|\lambda a\| = \lambda \|a\|$ for every $a \in A$ and scalar $\lambda \geq 0$.

Finite-Dimensional Linear Space

A finite-dimensional linear space is a set such that (a) linear combinations of points in the set are defined and are again in the set, and (b) there is a finite number of points in the set (a basis) such that every point in the set is a linear combination of this finite number of points. The dimension of the space is the minimum number of points needed to form a basis. A point x in a linear space of dimension n has a ordinate representation $x = (x_1, x_2, \ldots, x_n)$, given a basis for the space $\{b_1, \ldots, b_n\}$, where $x_1, \ldots, x_n$ are real numbers such that $x = x_1 b_1 + \ldots + x_n b_n$. The point x is called a vector, and $x_1, \ldots, x_n$ are called its components. The notation $(x)_i$ will sometimes also be used for component i of a vector x.

Real Finite-Dimensional Space

In economics, we work mostly with real finite-dimensional space. When this space is of dimension n, it is denoted $\mathbb{R}^n$. Points in this space are vectors of real numbers $(x_1,...,x_n)$; this corresponds to the previous terminology with the basis for $\mathbb{R}^n$ being the unit vectors (1,0,..,0), (0,1,0,..,0),..., (0,..,0,1).

The coordinate representation of a vector depends on the particular basis chosen for a space. Sometimes this fact can be used to choose bases in which vectors and transformations have particularly simple coordinate representations.

Euclidean Space

The Euclidean norm of a vector x is

$$\|x\|_2 = (x_1^2+...+x_n^2)^{1/2}.$$

This norm can be used to define the distance between vectors, or neighborhoods of a vector. Other possible norms include $\|x\|_1 = |x_1|+...+|x_n|$, $\|x\|_\infty = \max\{|x_1|,...,|x_n|\}$, or for $1 \le p < +\infty$, $\|x\|_p = \left[|x_1|^p+...+|x_n|^p\right]^{1/p}$.

Each norm defines a topology on the linear space, based on neighborhoods of a vector that are less than each positive distance away. The space $\mathbb{R}^n$ with the norm $\|x\|_2$ and associated topology is called Euclidean n-space.

Vector Products

The vector product of x and y in $\mathbb{R}^n$ is defined as $x \cdot y = x_1y_1+...+x_ny_n$. Other notations for vector products are <x,y> or (when x and y are interpreted as row vectors) xy' or (when x and y are interpreted as column vectors) $x'y$.

Linear Subspaces

A linear subspace of a linear space such as $\mathbb{R}^n$ is a subset that has the property that all linear combinations of its members remain in the subset. Examples of linear subspaces in $\mathbb{R}^3$ are the plane $\{(a,b,c) \mid b = 0\}$ and the line $\{(a,b,c) \mid a = b = 2\cdot c\}$. The linear subspace spanned by a set of vectors $\{x_1,...,x_J\}$ is the set of all linear combinations of these vectors, $L = \{x_1\alpha_1+...+x_J\alpha_J \mid (\alpha_1,...,\alpha_J) \in \mathbb{R}^J\}$.

Linear Independence

The vectors $\{x_1,...,x_J\}$ are linearly independent if and only if none can be written as a linear combination of the remainder. The linear subspace that is spanned by a set of J linearly independent vectors is said to be of dimension J. Conversely, each linear space of dimension J can be represented as the set of linear combinations of J linearly independent vectors, which are in fact a basis for the subspace.

Orthogonality

A linear subspace of dimension one is a line (through the origin), and a linear subspace of dimension (n-1) is a hyperplane (through the origin). If L is a subspace, then $L^{\perp} = \{x \in \mathbb{R}^n | \ x \cdot y = 0$ for all $y \in L\}$ is termed the complementary subspace. Subspaces L and M with the property that $x \cdot y = 0$ for all $y \in L$ and $x \in M$ are termed orthogonal, and denoted $L \perp M$. The angle θ between subspaces L and M is defined by

$$\cos \theta = \text{Min } \{x \cdot y | \ y \in L, \|y\|_2 = 1, x \in M, \|x\|_2 = 1\}.$$

Then, the angle between orthogonal subspaces is π/2, and the angle between subspaces that have a nonzero point in common is zero.

Affine Subspaces

A subspace that is translated by adding a nonzero vector c to all points in the subspace is termed an affine subspace. A hyperplane is a set $H = \{x \in \mathbb{R}^n | p \cdot x = \alpha\}$, where p is a vector that is not identically zero and α is a scalar. It is an affine subspace, and is a level set of the linear function $f(x) = p \cdot x$. The vector p is called a direction or normal vector for H. The sets $H_- = \{x \in \mathbb{R}^n | p \cdot x < \alpha\}$ and $H_+ = \{x \in \mathbb{R}^n | p \cdot x > \alpha\}$ are called, respectively, the open lower and upper half-spaces bounded by H; the closures of these sets are called the closed lower and upper half-spaces.

Separating Hyperplanes

Two sets A and B are said to be separated by a hyperplane H if one is entirely contained in the lower closed half-space bounded by H and the other is entirely contained in the upper closed half-space. They are said to be strictly separated by H if one is contained in an open half-space bounded by H. The affine subspace spanned by a set A is the set of all finite linear combinations of points in A. A point x is in the relative interior of a set A if there is a neighborhood of x whose intersection with the affine subspace spanned by A is contained in A. When A spans the entire space, then its interior and relative interior coincide.

Convex Sets

If $\{x_1,...,x_m\}$ are points in a linear vector space C, then $\alpha_1 x_1+...+\alpha_m x_m$, where $\alpha_j \geq 0$ for $j = 1,...,m$ and $\alpha_1+,,,+\alpha_m = 1$, is termed a convex combination of these points. A set A in a normed linear vector space C is convex if every convex combination of points from A is contained in A. If A and B are convex, then $A \cap B$, αA for a scalar α, and $A+B = \{z \in C | z=x+y , x \in A, y \in B\}$ are also convex. The cartesian product of two vector spaces C and D is defined as $C \times D = \{(x,y) | x \in C, y \in D\}$. If $A \subseteq C$ and $B \subseteq D$ are convex, then $A \times B$ is convex.

Separating Hyperplane Theorem

The relative interior of a convex set is always non-empty. The convex hull of any set $A \in C$ is the intersection of all convex sets that contain it, or equivalently the set formed from all convex combinations of points in A. If A is compact, its convex hull is compact. The closure of a convex set is convex, and its relative interior is convex. Two convex sets A and B can be separated by a linear function (i.e., there exists $p \neq 0$ such that $p \cdot x \geq p \cdot z$ for all $x \in A$, $z \in B$) if and only if their relative interiors are disjoint; this is called the separating hyperplane theorem. If A and B are disjoint and convex, and either (1) A is open or (2) A is compact and B is closed, then they can be strictly separated (i.e., there exists p such that $p \cdot x > p \cdot z$ for all $x \in A$, $z \in B$)

Linear Transformations

A mapping A from one linear space (its domain) into another (its range) is a linear transformation if it satisfies A(x+z) = A(x) + A(z) for any x and z in the domain. When the domain and range are finite-dimensional linear spaces, a linear transformation can be represented as a matrix. Specifically, a linear transformation A from $\mathbb{R}^n$ into $\mathbb{R}^m$ can be represented by a m×n array **A** with elements a_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$, with y = A(x) having components $y_i = \sum_{j=1}^{n} a_{ij} x_j$ for $1 \leq i \leq m$. In matrix notation, this is written **y** = **Ax**.

The set $\mathbb{N} = \{x \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$ is termed the <u>null space</u> of the transformation A. The subspace $\mathbb{N}^\perp$ containing all linear combinations of the column vectors of **A** is termed the <u>column space</u> of **A**; it is the complementary subspace to $\mathbb{N}$.

If A denotes a m×n matrix, then A′ denotes its n×m <u>transpose</u> (rows become columns and vice versa). The <u>identity matrix</u> of dimension n is n×n with one's down the diagonal, zero's elsewhere, and is denoted $\mathbf{I}_n$, or **I** if the dimension is clear from the context. A matrix of zeros is denoted **0**, and a n×1 vector of ones is denoted $\mathbf{1}_n$. A permutation matrix is obtained by permuting the columns of an identity matrix.

If **A** is a m×n matrix and **B** is a n×p matrix, then the matrix product **C** = **AB** is of dimension m×p with elements $c_{ik} \equiv \sum_{j=1}^{n} a_{ij}b_{jk}$ for $1 \le i \le m$ and $1 \le k \le p$.

For the matrix product to be defined, the number of columns in **A** must equal the number of rows in **B** (i.e., the matrices must be <u>commensurate</u>).

A matrix **A** is square if it has the same number of rows and columns. A square matrix **A** is symmetric if $\mathbf{A} = \mathbf{A}'$, diagonal if all off-diagonal elements are zero, upper (lower) triangular if all its elements below (above) the diagonal are zero, and idempotent if it is symmetric and $\mathbf{A}^2 = \mathbf{A}$. A matrix **A** is column orthonormal if $A'A = I$; simply orthonormal if it is both square and column orthonormal.

Each column of a n×m matrix **A** is a vector in $\mathbb{R}^n$. The rank of **A**, denoted r = ρ(**A**), is the largest number of columns that are linearly independent. Then **A** is of rank m if and only if **x** = **0** is the only solution to **Ax** = **0**. If **A** is of rank r, then orthonormalization applied to the linearly independent columns of **A** can be interpreted as defining a r×m lower triangular matrix **U** such that **AU**′ is column orthonormal. A n×m matrix **A** is of full rank if ρ(**A**) = min(n,m). A n×n matrix **A** of full rank is termed nonsingular. A nonsingular n×n matrix **A** has an inverse matrix $\mathbf{A}^{-1}$ such that both $\mathbf{A}\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\mathbf{A}$ equal the identity matrix I_n. An orthonormal matrix **A** satisfies $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$, implying that $\mathbf{A}' = \mathbf{A}^{-1}$, and hence $\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}' = \mathbf{I}_n$. The trace tr(**A**) of a square matrix **A** is the sum of its diagonal elements.

TABLE 2.1. BASIC OPERATIONS

Name	Notation	Definition
1. Matrix Product	**C** = **AB**	For m×n **A** and n×p **B**: $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$
2. Scalar Multiplication	**C** = b**A**	$c_{ij} = ba_{ij}$ for scalar b
3. Matrix Sum	**C** = **A**+**B**	$c_{ij} = a_{ij} + b_{ij}$ for **A** and **B** m×n
4. Transpose	C = A′	$c_{ij} = a_{ji}$ for m×n A:
5. Matrix Inverse	$C = A^{-1}$	$AA^{-1} = I_m$ for A n×n nonsingular
6. Trace	c = tr(A)	$c = \sum_{i=1}^{n} a_{ii}$ for n×n A

The <u>determinant</u> of a n×n matrix A is denoted $|A|$ or det(A), and has a geometric interpretation as the volume of the parallelepiped formed by the column vectors of A. The matrix A is <u>nonsingular</u> if and only if det(A) $\neq$ 0. A <u>minor</u> of a matrix A (of order r) is the determinant of a submatrix formed by striking out n-r rows and columns. A <u>principal minor</u> is formed by striking out symmetric rows and columns of A. A <u>leading principal minor</u> (of order r) is formed by striking out the last n-r rows and columns.

The minor of an element a_{ij} of A is the determinant of the submatrix A^{ij} formed by striking out row i and column j of A. Determinants satisfy the recursion relation

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A^{ij}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A^{ij}),$$

with the first equality holding for any j and the second holding for any i. This formula can be used as a recursive definition of determinants, starting from the result that the determinant of a scalar is the scalar. A useful related formula is

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ik} \det(A^{ij})/\det(A) = \delta_{kj},$$

where δ_{kj} is one if k = j and zero otherwise.

Elementary Properties of Matrices

(1) $(A')' = A$.
(2) If A^{-1} exists, then $(A^{-1})^{-1} = A$.
(3) If A^{-1} exists, then $(A')^{-1} = (A^{-1})'$.
(4) $(AB)' = B'A'$.
(5) If A,B are square, nonsingular, and commensurate, then $(AB)^{-1} = B^{-1}A^{-1}$.
(6) If A is m×n, then $\text{Min}\,\{m,n\} \geq \rho(A) = \rho(A') = \rho(A'A) = \rho(AA')$.
(7) If A is m×n and B is m×r, then $\rho(AB) \leq \min(\rho(A),\rho(B))$.
(8) If A is m×n with $\rho(A) = m$, and B is m×r, then $\rho(AB) = \rho(B)$.

(9) $\rho(A+B) \leq \rho(A) + \rho(B)$.
(10) If A is n×n, then $\det(A) \neq 0$ if and only if $\rho(A) = n$.
(11) If B and C are nonsingular and commensurate with A, then $\rho(BAC) = \rho(A)$.
(12) If A, B are n×n, then $\rho(AB) \geq \rho(A) + \rho(B) - n$.
(13) $\det(AB) = \det(A) \cdot \det(B)$.
(14) If c is a scalar and A is n×n, then $\det(cA) = c^n \det(A)$
(15) The determinant of a matrix is unchanged if a scalar times one column (row) is added to another column (row).
(16) If A is n×n and diagonal or triangular, then det(A) is the product of the diagonal elements.
(17) $\det(A^{-1}) = 1/\det(A)$.
(18) If A is n×n and $B = A^{-1}$, then
$b_{ij} = (-1)^{i+j}\det(A^{ij})/\det(A)$.

(19) The determinant of an orthonormal matrix is +1 or -1.
(20) If A is m×n and B is n×m, then tr(AB) = tr(BA).
(21) $tr(I_n) = n$.
(22) tr(A+B) = tr(A) + tr(B).
(23) A permutation matrix P is orthonormal; hence, $P' = P^{-1}$.
(24) The inverse of a (upper) triangular matrix is (upper) triangular, and the inverse of a diagonal matrix D is diagonal, with $(D^{-1})_{ii} = 1/D_{ii}$.
(25) The product of orthonormal matrices is orthonormal, and the product of permutation matrices is a permutation matrix.

Concave Functions

A real-valued function $f:A \to \mathbb{R}$ on a convex set A in a vector space C is concave if for every $x,y \in A$ and scalar α with $0 < \alpha < 1$, $f(\alpha x+(1-\alpha)y) \geq \alpha f(x)+(1-\alpha)f(y)$. Equivalently, a function f is concave if the set $\{(\lambda,x)\in\mathbb{R}\times A \mid \lambda\leq f(x)\}$, called the epigraph of f, is convex. Concave functions are shaped like overturned bowls. A function f is strictly concave if for every $x,y \in A$ with $x \neq y$ and scalar α with $0 < \alpha < 1$, $f(\alpha x+(1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)$.

Convex Functions

The function f on the convex set A is convex if for every $x,y \in A$ and scalar α with $0 < \alpha < 1$, $f(\alpha x+(1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$, or the epigraph $\{(\lambda,x)\in\mathbb{R}\times A| \lambda\geq f(x)\}$ is convex. Convex functions are shaped like upright bowls.

A function on a convex set is concave if and only if its negative is convex, and strictly concave if and only if its negative is strictly convex. Sums and maxima of collections of convex functions are again convex, while sums and minima of collections of concave functions are again concave.

Correspondences

The concept of a real-valued function on a set A in a finite-dimensional Euclidean vector space C can be generalized to mappings from A into a family of subsets of A; e.g., the family of non-empty closed subsets of A. This is called a set-valued function or correspondence, and can be written $f:A \to 2^A$. A correspondence f is upper hemicontinuous at $x \in A$ if for every open neighborhood N of f(x), there exists an open neighborhood M of x such that $f(y) \subseteq N$ for all $y \in M$.

Maximand Correspondence

Consider a real-valued function $g:A\times D \to \mathbb{R}$, where A and D are subsets of finite-dimensional Euclidean spaces with A compact, and g is continuous on $A\times D$. Let $f:D \to 2^A$ denote the set of maximands over $x \in B$ of $g(x,y)$, where B is a non-empty closed subset of A. This is also written $f(y) = \text{argmax}_{x\in B} g(x,y)$. Then f is a upper hemicontinuous correspondence on **D** whose images are non-empty compact sets. If in addition, A is convex and g is a concave function in x for each $y \in D$, then f is a convex-valued correspondence.