PROBLEM SET II

1. *This question refers to the notation and equations of Appendix A.1 of D. McFadden "Definite Quadratic Forms Subject to Constraints" in M. Fuss and D. McFadden* <u>Production Economics</u>, *Vol. 1.*
   Prove that if $G_x(x)$ in (SND) does not have maximal rank, then $L_{x,p}(\bar{x},\bar{p})$ has zero determinant - evaluated at the critical point $(\bar{x},\bar{p})$.

2. Define $f : R^2 \to R$ by $f(x,y) = ax^2 + by^2 + 2cxy + d$. For what values of *a*,*b*,*c* and *d* is *f* concave?

3. Let $\{f_i : i \in I\}$ be a set (finite or infinite) of functions from a convex set $D \subseteq R^n$ to *R* which are convex and bounded on *D*. Show that the function *f* defined by: $f(x) = \sup_{i \in I} f_i(x)$ is also convex. What about the function *g* defined by: $g(x) = \inf_{i \in I} f_i(x)$? Is this one convex? Why or why not?

4. Let $T \geq 1$ be a finite integer. Consider the following problem:
   $$\max \sum_{t=1}^{T} u(c_t)$$
   subject to
   $$c_1 + x_1 = x$$
   $$c_t + x_t = f(x_{t-1}) \qquad t = 2,..,T$$
   $$c_t, x_t \geq 0 \qquad t = 1,..,T$$
   where $x \in R_+$ and *u*, *f* are non-decreasing continuous functions of the form $R_+ \to R_+$.
   a) Write the problem in matrix notation.
   b) Derive the Lagrangean first-order conditions for this problem.
   c) Explain under what circumstances these conditions would be sufficient.

5. A consumer with a fixed income of $I > 0$ consumes two commodities. If he purchases q_i units of commodity *i* ($i = 1,\ 2$), the price he pays would be $p_i(q_i)$, where $p_i(\cdot)$ is a strictly increasing C^1 function. The consumer's utility function is given:
   $$u(q_1, q_2) = \ln q_1 + \ln q_2$$
   a) Write down the consumer's utility maximization problem.
   b) Does the Weierstrass theorem apply to yield existence of a maximum? Write down the Lagrangean first-order conditions for a maximum.
   c) Under what conditions on $p_1(\cdot)$ and $p_2(\cdot)$ are these conditions also sufficient? (Specify the most general conditions that you can think of).
   d) Suppose $p_1(q_1) = \sqrt{q_1}$ and $p_2(q_2) = \sqrt{q_2}$. Are the sufficient conditions you have given in part (b) met by this specification? Calculate the optimal consumption bundle in this case.

6. An agent who consumes three commodities has a utility function given by:
   $$u(x_1, x_2, x_3) = x_1^3 + \min\{x_2, x_3\}$$
   Given an income of *I* and prices p_1, p_2 and p_3, write down the consumer's utility maximization problem. Can the Weierstrass and/or the Lagrangean theorems be used to obtain and characterize a solution? (Explain why or why not).