Part IV

Empirical Applications of Production Theory:

Microeconomic Data

Chapter IV.1

ESTIMATION TECHNIQUES FOR THE ELASTICITY OF SUBSTITUTION AND OTHER PRODUCTION PARAMETERS*

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1. Introduction

This paper develops a general analytic framework in which present estimators of the elasticity of capital–labor substitution and other production parameters can be examined for specification bias. Econometric methods are devised for testing the production model against specific alternatives of interest. In particular, tests are developed for the following sources of possible bias in the usual cross-section estimates of industry production functions:

(1) non-constant returns to scale and non-homotheticity in production;
(2) inclusion of productive factors other than capital and labor, particularly raw materials;
(3) variations in output prices faced by firms; and
(4) non-constancy of the elasticity of substitution.

An application of these techniques to the steam-electric generating industry suggests (1) fuel inputs have a very low degree of substitutability with capital and labor, making "third-input" effects negligible; (2)

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the elasticity of substitution between capital and labor is *constant*, with an approximate value of 0.75; and (3) production is *not* homothetic, exhibiting bias toward high capital–labor ratios at high scale levels, and also generally increasing returns to scale.

Empirical studies of the Constant Elasticity of Substitution (C.E.S.) production function and related formulae have emphasized its crucial role in determining the behavior of a variety of economic models. In particular, this elasticity is central to the determination of factor demand elasticities and the trend of relative factor shares over time.[1]

Although a variety of techniques have been developed for estimating the elasticity of substitution, most empirical studies up to 1971 used a log-linear formula introduced by ACMS,

$$\log(V/L) = A + \sigma \log w + u, \tag{1.1}$$

where value-added (V), labor quantity (L), and the real wage rate (w) are observed variables for a firm or industry, u is an unobserved shock, and the parameter σ is (under certain assumptions) the elasticity of substitution between capital and labor. Adaptation of this formula for econometric study avoids the use of often unavailable data on capital services (K) or service price (r). Alternately, when these last variables can be observed, a second useful formula holds,

$$\log(K/L) = B + \sigma \log(w/r) + v, \tag{1.2}$$

where v is an unobserved shock.

This paper examines the assumptions underlying the derivation of these formulae, and the possible sources of specification error. Tests of specification error against specific alternatives of interest are devised, and are applied to the steam-electric generating industry.

Section 2 of this paper gives a useful reformulation of production possibilities in terms of cost functions. The elasticity of substitution is defined in Section 3. The assumptions underlying the usual estimation procedures are reviewed in detail in Section 4, and a brief review is given of evidence in the literature concerning their validity.

[1]Arrow–Chenery–Minhas–Solow [hereafter ACMS (1961)] in their seminal article on C.E.S. functions list several economic issues whose resolution depends on the elasticity of substitution. A review of the empirical literature on C.E.S. functions by Nerlove (1967) expands this list.

In Section 5, we develop a number of exact relations between variables which can in principle be observed. These relations are valid under quite general conditions, and can be used as a basis for formulating powerful econometric tests of the assumptions presented in the previous section.

An application of the theoretical results to the steam-electric generating industry is made in Section 6. Data for this industry are given in an appendix.

2. Cost Functions

A firm faced with a given technology will generally be able to specify the minimum cost, at quoted competitive input prices, of producing a given output bundle. This procedure determines a *cost function* of the given output bundle and input prices. The cost function identifies all the economically relevant characteristics of the firm's technology, and is particularly useful in the formulation of econometric models: it is a concave function in prices, and its price derivatives, when they exist, equal the derived damands for the respective inputs (see Chapter I.1).

Consider a set of production possibilities which yield an output (Y) from inputs of capital services (K), labor (L), and raw materials (M).[2] We term these production possibilities *classical* if they can be represented by a production function $Y = F(K,L,M)$ with the following properties:

(a) increasing in inputs.
(b) strictly quasi-concave.[3]
(c) continuous for all non-negative input bundles, zero when all inputs are zero, and unbounded when all inputs are unbounded.

Given positive prices r,w,m for capital services, labor, and raw materials, respectively, one can define the *cost function*,

$$C(Y,r,w,m) = \underset{K,L,M}{\text{Min}}\{rK + wL + mM | F(K,L,M) \geqq Y\}, \tag{2.1}$$

specifying the least cost of producing the output quantity Y. With each set of classical production possibilities $Y = F(K,L,M)$ is associated a

[2]The conclusions of this section are unchanged if M represents a vector of inputs rather than a single raw materials input.

[3]Strict quasi-concavity requires that the upper contour sets of the production function be strictly convex. (Heuristically, marginal rates of substitution are strictly decreasing.)

cost function with the following properties:[4]

(I) C is concave, non-decreasing, positive linear homogeneous, and continuously differentiable in positive input prices for each level of output.

(II) C is continuous and increasing in output, is zero at zero output, and is unbounded for unbounded output.

(III) The derivatives of C with respect to input prices equal the unique cost minimizing demands for the respective inputs, hereafter denoted by

$$\partial C/\partial r \equiv C_r(Y,r,w,m) = \hat{K}(Y,r,w,m),$$
$$\partial C/\partial w \equiv C_w(Y,r,w,m) = \hat{L}(Y,r,w,m),$$
$$\partial C/\partial m \equiv C_m(Y,r,w,m) = \hat{M}(Y,r,w,m).$$

(IV) Distinct classical production possibilities yield distinct cost functions, and the production function can be recovered from the cost function (for positive K,L,M) via the relation

$$F(K,L,M) = \text{Max}\{Y | rK + wL + mM \geqq C(Y,r,w,m) \text{ for all } (r,w,m) > 0\}. \tag{2.2}$$

Further, any function $C(Y,r,w,m)$ satisfying (I) and (II) is the cost function for a classical production possibility set defined by (2.2).

(V) The second partial derivatives of C in input prices exist almost everywhere (and are independent of the order of differentiation) for each level of output.

If a function $C(Y,r,w,m)$ satisfies property (I), it will be termed a *quasi-cost function*, and if it satisfies both properties (I) and (II), it will be termed a *classical cost function*. Any function satisfying (I) and (II) can be shown to also satisfy (III)–(V) for the set of classical production possibilities defined by (2.2).

In further analysis, we assume that the second derivatives of C in input prices exist everywhere, and that marginal cost $\partial C/\partial Y = C_Y(Y,r,w,m)$ exists.

[4]These properties of the cost function are demonstrated by Uzawa (1964). See also Chapter I.1.

A set of classical production possibilities $Y = F(K,L,M)$ is said to be homothetic if there exists a strictly increasing transformation ϕ of the non-negative real line onto itself such that $\phi(F(K,L,M)) \equiv f(K,L,M)$ is positive linear homogeneous in inputs. Homothetic production possibilities have marginal rates of technical substitution which depend only on input proportions and not on the scale of production. The following lemma, due to Uzawa (1964), establishes a separability property of the cost functions of homothetic production possibilities.

Lemma. A set of classical production possibilities $Y = F(K,L,M)$ is homothetic if and only if its cost function has the separable form $C(Y,r,w,m) = \phi(Y)\gamma(r,w,m)$.

Proof: Suppose first that production possibilities are homothetic, with a transformation ϕ yielding $\phi(F(K,L,M)) = f(K,L,M)$ homogeneous of degree one. Then,

$$\begin{aligned} C(Y,r,w,m) &= \operatorname*{Min}_{K,L,M}\{rK + wL + mM | F(K,L,M) \geqq Y\} \\ &= \operatorname*{Min}_{K,L,M}\{rK + wL + mM | f(K,L,M) \geqq \phi(Y)\} \\ &= \phi(Y) \operatorname*{Min}_{K',L',M'}\{rK' + wL' + mM' | f(K',L',M') \geqq 1\} \\ &= \phi(Y)C(1,r,w,m), \end{aligned}$$

where $K' = K/\phi(Y)$, etc. Defining $\gamma(r,w,m) = C(1,r,w,m)$, the separable form is established.

Suppose next that the cost function has the separable form. From the relation

$$\begin{aligned} F(K,L,M) = \operatorname{Max}\{Y | &rK + wL + mM \\ &\geqq \phi(Y)\gamma(r,w,m) \text{ for all } (r,w,m) > 0\}, \end{aligned}$$

we have

$$\begin{aligned} f(K,L,M) &= \phi(F(K,L,M)) \\ &= \operatorname{Max}\{Y' | rK + wL + mM \\ &\quad \geqq Y'\gamma(r,w,m) \text{ for all } (r,w,m) > 0\}, \end{aligned}$$

where $Y' = \phi(Y)$. But $f(K,L,M)$ defined by this condition is clearly

homogeneous of degree one, establishing that $F(K,L,M)$ is homothetic. Q.E.D.

3. The Elasticity of Substitution

The elasticity of substitution (E.S.) between two factors of production is defined as the elasticity of the ratio of the factors with respect to the marginal rate of technical substitution between them.[5] When output Y is given by a twice continuously differentiable production function $Y = F(K,L)$ of capital services and labor alone, the E.S. between these factors is given by the formula

$$\sigma = -\frac{\mathrm{d}\ln(K/L)}{\mathrm{d}\ln(r/w)}\bigg|_{Y \text{ fixed}}, \tag{3.1}$$

where $r/w = F_K(K,L)/F_L(K,L) = \mathrm{MRS}_{K\to L}$ is the marginal rate of technical substitution (subscripts denote partial derivatives). Written out fully in terms of partial derivatives, (3.1) becomes

$$\sigma = \frac{1/KF_K + 1/LF_L}{-F_{KK}/F_K^2 + 2F_{KL}/F_KF_L - F_{LL}/F_L^2}, \tag{3.2}$$

where all derivatives are evaluated at the argument (K,L). When the two-factor production function is linear homogeneous, (3.2) reduces to

$$\sigma = \frac{F_K(K,L)F_L(K,L)}{F(K,L)F_{KL}(K,L)}.^{6} \tag{3.3}$$

The expression (3.3) can be rewritten in the suggestive forms,

$$\sigma = \frac{\partial \log F(K/L,1)/\partial(K/L)}{\partial \log F_L(K/L,1)/\partial(K/L)} = \frac{\mathrm{d}\log(Y/L)}{\mathrm{d}\log(w^*)}, \tag{3.3a}$$

and

$$\sigma = \frac{KF_K(K,L)}{F(K,L)} \frac{1}{\partial \log F_L(K,L)/\partial \log K} = s_K/\epsilon(w^*,K;L), \tag{3.3b}$$

[5]Hence, this elasticity is an index of the sensitivity of cost-minimizing factor input proportions to changes in relative factor prices. This definition is meaningful only when the independent variables upon which the input proportions depend are carefully specified.

[6]For a discussion of the concept of the E.S. and derivation of the formulae (3.1)–(3.3), see Allen (1938, pp. 340–343) and Hicks (1932).

where w^* is the real wage (measured in units of output), s_K is capital's share of output, and $\epsilon(w^*,K;L)$ is the cross-elasticity of the real demand price for labor with respect to the quantity of capital services (labor quantity being held constant). Under the assumption of a constant E.S., (3.3a) integrates immediately to the ACMS formula (1.1). The form (3.3b) gives a relation, first utilized by Hicks, between the E.S. and factor demand elasticities.

The E.S. can also be defined in terms of derivatives of the cost function. For many econometric purposes, the cost function can be viewed as a reduced form equation, and such a formulation is more convenient than the ones given above. For the production function depending only on capital services and labor (but not necessarily homogeneous), with a total cost function $c = C(Y,r,w)$, the E.S. definition (3.1) can be written as

$$\sigma = \frac{-C_{rr}/C_r^2 + 2(C_{rw}/C_rC_w) - C_{ww}/C_w^2}{1/rC_r + 1/wC_w}, \tag{3.4}$$

where the notation $C_r = \partial C/\partial r$, $C_{rw} = \partial^2 C/\partial r\partial w$, etc. is used for price derivatives, and all derivatives are evaluated at the argument (Y,r,w). For the two-factor production process, the linear homogeneity of the cost function in prices can be exploited to simplify (3.4) to

$$\sigma = CC_{rw}/C_rC_w, \tag{3.5}$$

which can be rewritten in the forms

$$\sigma = \frac{\partial \log C_w(Y,r/w,1)/\partial(r/w)}{\partial \log C(Y,r/w,1)/\partial(r/w)} = \left.\frac{\mathrm{d} \log L}{\mathrm{d} \log(c/w)}\right|_{Y \text{ fixed}}, \tag{3.5a}$$

and

$$\sigma = \frac{C}{wC_w}\frac{\partial \log C_r}{\partial \log w} = \frac{\epsilon(K,w;Y,r)}{s_L}, \tag{3.5b}$$

where c is the value of total cost, s_L is labor's share of total cost, and $\epsilon(K,w;Y,r)$ is the cross-elasticity of the demand for capital with respect to the price of labor (output quantity and capital price being held constant). The form (3.5) holds without any imposition of linear homogeneity on the production function.

In the case that the production process has three or more inputs, a number of alternative definitions for the E.S. have been suggested in the literature which differ in economic interpretation and implications. We

shall examine three of the most common of these forms.[7] We consider a classical, differentiable production function $Y = F(K,L,M)$ and its cost function $C = C(Y,r,w,m)$.

The Direct Elasticity of Substitution between capital and labor is given by the basic formula (3.1), holding fixed the quantity of the third factor. In terms of partial derivatives, the formula for this *E.S.* is given by (3.2), evaluated now at the argument (K,L,M). The direct E.S. can be interpreted as a "short-run" elasticity in which the supply of the third factor is fixed, and provides information on the behavior of the relative shares of capital and labor.

The Shadow Elasticity of Substitution between capital and labor is given by the basic formula (3.1), holding fixed average cost and the price of the third factor.[8] The formula (3.4), evaluated at the argument (Y,r,w,m), defines this elasticity in terms of partial derivatives. The shadow E.S. can be interpreted as a "long-run" elasticity in which the third factor can be traded freely at a fixed price, and again provides information on the behavior of relative shares.

The Allen–Uzawa Elasticity of Substitution between capital and labor is given by the formula (3.5), evaluated now at the argument (Y,r,w,m), and provides information [via equation (3.5b)] on the cross demand elasticities for inputs. This definition does not provide direct information on the behavior of relative shares.

4. The Specification of the ACMS Model

The formulae (1.1) and (1.2) above can be used to obtain econometric estimates of the elasticity of substitution for a suitable specification of the underlying model of the firm. The following list of assumptions gives one possible specification under which such estimates are valid:

(1) Production possibilities are non-stochastic and known with

[7]See McFadden (1963). When only two factors are productive, all these definitions reduce to the equivalent formulae (3.2), (3.4), and (3.5).

[8]When the production function is homothetic, this definition is unchanged if marginal cost, rather than average cost, is held fixed. For heterothetic production functions, an E.S. definition distinct from the shadow E.S. can be obtained – when capital and labor are non-retrogressive inputs – by holding marginal cost and the price of the third factor fixed. This definition can be made most simply in terms of the maximum profit function of the firm.

certainty by the firm, and can be represented by a production function with classical properties depending only on observed variables. Variations in the technology available to the sampled firms and variations in input quality are ruled out unless they can be observed and included in the production function.

(2) Input and output prices are known with certainty and treated as parameters by the firm, which minimizes input cost and determines output without error. All observed variables are measured without error except inputs and total cost. The distributions of the measurement errors are independent of the values of economic variables.

(3) The production function is linear homogeneous in the inputs, ruling out non-constant returns and heterotheticity.

(4) Capital services and labor are the only inputs to production, with raw material and intermediate good inputs excluded.

(5) The elasticity of substitution between capital services and labor is constant.

(6) Output price does not vary systematically with the wage rate.

Assumptions 1–6 comprise a stronger specification than is necessary for least-squares estimation of the E.S. from equations (1.1) or (1.2) (e.g., under Assumption 3, it is unnecessary to make assumptions on the statistical properties of measured output), but are a useful starting point for an analysis of specification bias.

A number of papers have investigated the validity of these assumptions in industry studies. Arrow–Chenery–Minhas–Solow (1961) point out Assumptions 1, 3, and 6 as particular sources of potential specification error in international cross-sectional samples. They find significant efficiency variations, casting doubt on Assumption 1, but find Assumptions 3 and 6 to be supported in partial tests. Dhrymes (1963) has carried out some tests on Assumption 2, concluding that deviations from competitive cost-minimizing behavior significantly affect E.S. estimates in some industries. Dhrymes and Kurz (1964) have tested Assumptions 3 and 5 for electric power data, using a modified form of the Arrow–Solow production function. They find evidence of increasing returns to scale and falling fuel-capital E.S. with scale. However, their results may depend in part on their choice of capacity as a capital index.

Other studies which give some indirect evidence on the assumptions are M. Brown and J. de Cani (1963), C. Ferguson (1963), J. Kendrick and R. Sato (1963), M. Kurz and A. Manne (1963), R. Lucas (1963), J. Minasian (1961), B. Minhas (1963), A. Harberger (1959), R. McKinnon (1963a), and R. Solow (1964).

5. Properties of the Elasticity of Substitution

Any cost function formula used to fit observed data will exhibit some elasticity of substitution (E.S.) between capital services and labor, which can be computed from the definitions of Section 3. Alternately, a function specifying this E.S. can be integrated to obtain a family of cost functions from which a fit to the observed data can be chosen. The form of this family will depend on the assumptions imposed on the analysis. In principle, Assumptions 1–6 may be tested by comparing the fits achieved by different families of cost functions.

(1) We first examine the implications of the constant E.S. Assumption 5. Assumptions 3 and 4, excluding third inputs and heterotheticity, are dropped, but Assumptions 1 and 2 are maintained. Following the notation of Section 2, we consider a production process utilizing inputs of capital services, labor, and raw materials with the total cost function $c = C(Y,r,w,m)$, where c is the level of true total cost without measurement error. Hereafter, a *cost* function will always be assumed to be twice continuously differentiable in prices and to satisfy condition (I) of Section 2.

Theorem 1. Suppose Assumptions 1 and 2 hold. Then, every classical cost function $c = C(Y,r,w,m)$ which has a constant Allen–Uzawa E.S. $(= \sigma)$ between capital services and labor must necessarily have the functional form

$$\begin{aligned} c &= \{[rA(Y,m/r)]^{1-\sigma} + [wB(Y,m/w)]^{1-\sigma}\}^{1/(1-\sigma)} && \text{for} \quad 0 \leqq \sigma < +\infty, \ \sigma \neq 1, \\ &= [rA(Y,m/r)]^{\theta}[wB(Y,m/w)]^{1-\theta} && \text{for} \quad \sigma = 1, \ 0 < \theta < 1, \\ &= \text{Min}\{[rA(Y,m/r)],[wB(Y,m/w)]\} && \text{for} \quad \sigma = +\infty, \end{aligned} \tag{5.1}$$

where A and B are positive functions. If a function of the form (5.1) is a classical cost function, then it has a constant Allen–Uzawa E.S. $(= \sigma)$ between capital and labor.

Proof: One can easily verify by computation that (5.1) exhibits a constant Allen–Uzawa E.S. between capital and labor. We now demonstrate that (5.1) is necessary in the case $1 \neq \sigma < +\infty$. Let C denote a general cost function with the constant Allen–Uzawa E.S. σ, and define

$g \equiv C^{1-\sigma}$. Applying the E.S. definition to C, one finds that g must satisfy $g_{rw} = 0$. Hence, g can be written as a function independent of w plus a function independent of r. Imposition of linear homogeneity in prices then yields the form (5.1).

For the case $\sigma = 1$, the result follows from the argument above applied to $g \equiv \log C$. For the case $\sigma = +\infty$, (5.1) follows by applying a limiting argument to the first case. Q.E.D.

Corollary 1.1. Suppose the cost function has the form (5.1) with $\sigma < +\infty$. Then the following relations hold:

$$\log(c/L) = (1-\sigma)\log(c/B(Y,m/w)) - \log(1-\beta) + \sigma \log w, \tag{5.2}$$

$$\log L = \sigma \log c + (1-\sigma)\log B(Y,m/w) + \log(1-\beta) - \sigma \log w, \tag{5.3}$$

$$\log(K/L) = (1-\sigma)\log[A(Y,m/r)/B(Y,m/w)] + \log(1-\alpha) - \log(1-\beta) + \sigma \log(w/r), \tag{5.4}$$

where $\alpha = \partial \log A(Y,m/r)/\partial \log(m/r)$ and $\beta = \partial \log B(Y,m/w)/\partial \log(m/w)$.

When Assumptions 3, 4, and 6 are imposed on (5.2) and (5.4), requiring constant returns, no third factor, and an output price which does not vary with the wage rate, formulae (5.2) and (5.4) reduce to (1.1) and (1.2), respectively. Hence, econometric techniques based on (5.2) and (5.4) can be used to test these assumptions.

Corollary 1.2. A classical cost function of the functional form $c = C(Y,r,w,m) = H(Y,m,G(Y,r,w))$, where $G(Y,r,w)$ is a quasi-cost function, has a constant A.U.E.S. σ, $\sigma \neq 1$, between capital and labor if and only if

$$c = \{(mD(Y))^{1-\sigma} + m^{\sigma^*(Y)-\sigma}[(rA(Y))^{1-\sigma^*(Y)} + (wB(Y))^{1-\sigma^*(Y)}]\}^{1/(1-\sigma)},$$

with A, B, and D positive functions, and $\sigma^*(Y)$ a non-negative function satisfying $\sigma \leqq \sigma^*(Y) < 1$ if $\sigma < 1$ and $1 < \sigma^*(Y) \leqq \sigma$ if $\sigma > 1$.

Proof: Computation verifies the "if" implication. To prove the "only if" statement, note first that $c = H(Y,m,v)$ with $v = G(Y,r,w)$ implies $C_r = H_v G_r$, $C_w = H_v G_w$, and $C_{rw} = H_{vv} G_r G_w + H_v G_{rw}$. Hence,

$$\sigma = \frac{CC_{rw}}{C_r C_w} = \frac{HH_{vv}}{H_v^2} + \frac{H}{vH_v}\sigma^*,$$

with $\sigma^* = GG_{rw}/G_rG_w$. Consider variations in r and w which keep $v = G(Y,r,w)$ fixed. Then, all the terms in the expression for σ are fixed except σ^*, implying that $\sigma^*(Y,r/w)$ can depend only on Y. Then, this equation can be written as a partial differential equation in v,

$$\sigma \frac{\partial \ln H}{\partial v} = \frac{\partial \ln H_v}{\partial v} + \frac{\sigma^*}{v},$$

which has a general solution

$$H(Y,m,v) = [D_1(Y,m) + A_1(Y,m)v^{1-\sigma^*(Y)}]^{1/(1-\sigma)},$$

in the case $\sigma^*(Y) \neq 1$. Homogeneity implies the form

$$H(Y,m,v) = [D(Y)m^{1-\sigma} + A(Y)m^{\sigma^*(Y)-\sigma}v^{1-\sigma^*(Y)}]^{1/(1-\sigma)}.$$

[If $\sigma^*(Y) = 1$, homogeneity cannot be imposed; hence, this case is ruled out.] The condition that σ^* be independent of r/w implies by Theorem 1 that $G(Y,r,w) = [(rA_1(Y))^{1-\sigma^*(Y)} + (wB_1(Y))^{1-\sigma^*(Y)}]^{1/(1-\sigma^*(Y))}$. Then, $C(Y,r,w,m) = H(Y,m,G(Y,r,w))$ has the functional form claimed by the corollary. The condition that the first derivatives of C be positive implies $(1-\sigma^*(Y))/(1-\sigma) > 0$ and $(\sigma^*(Y)-\sigma)/(1-\sigma) \geqq 0$. Hence, $\sigma < 1$ implies $\sigma \leqq \sigma^*(Y) < 1$ and $\sigma > 1$ implies $1 < \sigma^*(Y) \leqq \sigma$. Q.E.D.

The quasi-cost function $v = G(Y,r,w)$ can be interpreted as the "nominal value-added" by capital and labor to the production of Y units of output. Dual to this function is a distance function

$$g(Y,K,L) = \sup\{\lambda | rK + wL \geqq \lambda G(Y,r,w) \text{ for all } r,w > 0\},$$

homogeneous of degree one in (K,L), with $V = g(Y,K,L)$ interpretable as the production of "real value-added". Similarly, the cost function $c = H(Y,m,v)$ has a dual distance function

$$h(Y,M,V) = \sup\{\lambda | mM + vV \geqq \lambda H(Y,m,v) \text{ for all } m,v > 0\},$$

homogeneous of degree one in (M,V), with $1 = h(Y,M,V)$ interpretable as specifying the isoquant in materials and real value-added necessary to produce Y. The composition rules for cost functions in Chapter I.1 (Table 2) imply that the cost function $c = H(Y,m,G(Y,r,w))$ is dual to a technology with the distance function $h(Y,M,g(Y,K,L))$.

Consider the classical case with $G(Y,r,w)$ homothetic in Y (or independent of Y), so that $G(Y,r,w) = \phi(Y)\gamma(r,w)$. Then $g(Y,K,L)$ has the form $g(Y,K,L) = f(K,L)/\phi(Y)$, where $V = f(K,L)$ is linear homogeneous in (K,L). The technology then satisfies $1 = h(Y,M,f(K,L)/\phi(Y))$, implying

the existence of a production function $Y = F(M,f(K,L))$. The function $V = f(K,L)$ is then an index of real value-added, and $\gamma(r,w)$ is an index of the price of real value-added, defined independently of output. In Corollary 1.2, this case occurs if $\sigma^*(Y)$ and $A(Y)/B(Y) \equiv \alpha$ are independent of Y, implying that

$$f(K,L) = [(\alpha K)^{1-1/\sigma^*} + L^{1-1/\sigma^*}]^{\sigma^*/(\sigma^*-1)}.$$

From the cost function in Corollary 1.2,

$$c = \{(mD(Y))^{1-\sigma} + m^{\sigma^*(Y)-\sigma} v^{1-\sigma^*(Y)}\}^{1/(1-\sigma)},$$

we can define an A.U.E.S. between materials and real value-added,

$$\sigma_{MV} = HH_{mv}/H_mH_v$$
$$= \left(\sigma^*(Y) + \frac{\sigma^*(Y)-\sigma}{1-\sigma} A(Y)\left(\frac{v}{m}\right)^{1-\sigma^*(Y)}\right)$$
$$\Big/\left(1 + \frac{\sigma^*(Y)-\sigma}{1-\sigma} A(Y)\left(\frac{v}{m}\right)^{1-\sigma^*(Y)}\right).$$

If $\sigma < \sigma^*(Y) < 1$, then this formula implies $\sigma^*(Y) < \sigma_{MV} < 1$. When v/m is small, σ_{MV} is close to $\sigma^*(Y)$, and when v/m is large, σ_{MV} is close to one. In the second case, $1 < \sigma^*(Y) < \sigma$, one has $1 < \sigma_{MV} < \sigma^*(Y)$, with σ_{MV} increasing from 1 to $\sigma^*(Y)$ as v/m increases. This result implies in particular that non-substitutability of raw materials for capital and labor is inconsistent with the existence of a positive constant non-unitary A.U.E.S. between capital and labor.

Theorem 2. Suppose Assumptions 1 and 2 hold. Then, every classical cost function $c = C(Y,r,w,m)$ which has a constant shadow E.S. $(= \sigma)$ between capital services and labor must necessarily have the implicitly defined functional form

$$c = \{[rA(Y,m/c)]^{1-\sigma} + [wB(Y,m/c)]^{1-\sigma}\}^{1/1-\sigma} \quad \text{for} \quad 0 \leqq \sigma < +\infty, \ \sigma \neq 1,$$
$$= A(Y,m/c) r^{\theta} w^{1-\theta} \quad \text{for} \quad \sigma = 1, \ 0 < \theta < 1,$$
$$= \text{Min}\{[rA(Y,m/c)],[wB(Y,m/c)]\} \quad \text{for} \quad \sigma = +\infty, \qquad (5.5)$$

where A and B are positive functions.

Proof: If a function of the form (5.5) is a classical cost function, then computation shows that it has a constant shadow E.S. between capital

and labor. We now show that (5.5) is necessary in the case $1 \neq \sigma < +\infty$.

For each (Y,r,m), the classical cost function $c = C(Y,r,w,m)$ is a strictly monotone continuously differentiable transformation from w in $[0,+\infty]$, or equivalently from $w^{1-\sigma}$ in $[0,+\infty]$, onto c in $[C(Y,r,0,m), C(Y,r,+\infty,m)]$. Hence, there exists a continuously differentiable inverse transformation $w^{1-\sigma} = h(c,Y,r,m)$. Implicit differentiation of the identity $w^{1-\sigma} \equiv h(C(Y,r,w,m),Y,r,m)$ yields $C_r = -h_r/h_c$ and $C_w = (1-\sigma)w^{-\sigma}/h_c$. Hence, $C_r/C_w = (w/r)^{\sigma}A_0(c,Y,r,m)$, where $A_0(c,Y,r,m) \equiv -h_r(c,Y,r,m)r^{\sigma}/(1-\sigma)$. Formula (3.4) defining the shadow E.S. can be written as

$$\partial \log(C_r/C_w)/\partial r + \sigma/r = (C_r/C_w)\{\partial \log(C_r/C_w)/\partial w - \sigma/w\}.$$

Substituting into this expression the form obtained above for C_r/C_w, one obtains the condition $\partial A_0/\partial r \equiv 0$. Hence, h satisfies the partial differential equation

$$h_r(c,Y,r,m) = -(1-\sigma)A_0(c,Y,m)/r^{\sigma},$$

which has the general solution

$$w^{1-\sigma} \equiv h(c,Y,r,m) = -A_0(c,Y,m)r^{1-\sigma} + A_1(c,Y,m),$$

where $A_1(c,Y,m)$ is an arbitrary function. Imposition of the price homogeneity condition then yields (5.5).

In the case $\sigma = 1$, consider the identity $h(c,Y,r,m) \equiv w$. The procedure above leads to a function

$$w = h(c,Y,r,m) = A_1(c,Y,m)c^{1+A_0(c,Y,m)}r^{-A_0(c,Y,m)}.$$

Differentiating implicitly, we obtain

$$C_w = 1/wQ, \qquad C_r = A_0/rQ,$$

$$C_m = \left[(\log r - \log c)\frac{\partial A_0}{\partial m} - \frac{1}{A_1}\frac{\partial A_1}{\partial m}\right]\Big/ Q,$$

and

$$C_Y = \left[(\log r - \log c)\frac{\partial A_0}{\partial Y} - \frac{1}{A_1}\frac{\partial A_1}{\partial Y}\right]\Big/ Q,$$

where

$$Q = \frac{1}{A_1}\frac{\partial A_1}{\partial c} + \frac{1+A_0}{c} + (\log c - \log r)\frac{\partial A_0}{\partial c}.$$

Consider any solution (Y',r',w',m',c') of this system, $c' = C(Y',r',w',m')$. Then, (Y',r,w,m',c') with $w = w'(r/r')^{-A_0(c',Y',m')}$ is also a

solution for any $r>0$. Since C_w is positive, Q must be positive at this last argument for any value of $r>0$. For this to be true, $\partial A_0/\partial c$ must be zero. Applying the same argument to C_m and C_Y establishes $\partial A_0/\partial m$ and $\partial A_0/\partial Y$ both zero. Then, A_0 is a positive constant. Defining $\theta = A_0/(1+A_0)$ and using homogeneity one obtains (5.5). The case $\sigma = +\infty$ is again handled by applying a limiting argument to the first case. Q.E.D.

Corollary 2.1. Suppose the cost function has the form (5.5) with $\sigma < +\infty$, $\sigma \neq 1$. Let $V = rK + wL$ denote value-added. The following relations hold:

$$\log(V/L) = (1-\sigma)\log(c/B(Y,m/c)) + \sigma \log w, \tag{5.6}$$

$$\log(K/L) = (1-\sigma)\log[A(Y,m/c)/B(Y,m/c)] + \sigma \log(w/r). \tag{5.7}$$

The formulae (5.6) and (5.7) again reduce to (1.1) and (1.2) under Assumptions 3, 4, and 6, so that they can provide the basis for further econometric tests.

Corollary 2.2. A classical cost function of the functional form $c = C(Y,r,w,m) = H(Y,m,G(Y,r,w))$, where $G(Y,r,w)$ is also a classical cost function, has a constant shadow E.S. σ, $\sigma \neq 1$, between capital and labor if and only if

$$c = H(Y,m,[(rA(Y))^{1-\sigma} + (wB(Y))^{1-\sigma}]^{1/(1-\sigma)}).$$

Proof: The "only if" conclusion can be verified by computation. The "if" implication follows by noting in (5.7) that $A(Y,m/c)/B(Y,m/c)$ must be independent of m, and hence must have a common factor depending on m/c. Q.E.D.

The functional form in this corollary is a special case of the formula for a non-constant E.S. given in Lemma 6. Note that this form places no restrictions on the E.S. between materials and "value-added".

Applying the arguments of Theorem 2 directly to the production function in the case of a constant direct E.S. between capital and labor, one obtains the following theorem:

Theorem 3. Suppose Assumptions 1 and 2 hold. Then, every twice continuously differentiable classical production function $Y = F(K,L,M)$ which has a constant direct E.S. $(=\sigma)$ between capital and labor must necessarily have the implicitly defined functional

form

$$A(Y,M) = \{[KB(Y,M)]^{1-1/\sigma} + L^{1-1/\sigma}\}^{\sigma/(\sigma-1)} \quad \text{for} \quad 0 < \sigma \leqq +\infty, \ \sigma \neq 1,$$
$$= K^{\theta}L^{1-\theta} \quad \text{for} \quad \sigma = 1,$$
$$= \text{Min}\{[KB(Y,M)],L\} \quad \text{for} \quad \sigma = 0, \tag{5.8}$$

where A and B are positive functions. (The choice of L as "numeraire" is arbitrary.)

If equation (5.8) defines a classical, twice continuously differentiable production function, then it has a constant direct E.S. between capital and labor, and the following relations hold for $0 < \sigma < +\infty$, $\sigma \neq 1$,

$$\log(V/L) = (1-\sigma)\log[V/A(Y,M)] + \sigma \log w, \tag{5.9}$$

$$\log(K/L) = (\sigma - 1)\log B(Y,M) + \sigma \log(w/r), \tag{5.10}$$

where $V/w = (KF_K + LF_L)/F_L$ and $w/r = F_L/F_K$.

Again under Assumptions 3, 4, and 6, the formulae (5.9) and (5.10) reduce to (1.1) and (1.2). The condition for the production function to have a separable form $F(K,L,M) = F^1(M,F^2(K,L))$ is, from (5.10), that $B(Y,M)$ be independent of M.

(2) We now consider classes of cost functions which do *not* have a constant E.S. between capital and labor. Such cost functions will generate E.S. functions $\sigma = \sigma(Y,r,w,m)$ for each of the definitions in Section 4. As a test of the constancy of the E.S., one could fit observed data to members of an arbitrary chosen class of cost functions, and compute the σ function for the resulting best fit. This procedure was followed by Dhrymes and Kurz (1964). An alternative approach is to generate a class of cost functions from a suitable family of σ functions. In this way, the hypothesis of constancy can be tested against specific alternatives of interest. We take this second approach.

We analyze initially production processes which use *only* the factors of capital and labor. Let $\sigma = \sigma(Y,r/w)$ denote an E.S. candidate for this process. A preliminary question is whether this candidate could actually be obtained from some classical production function under the definition (3.1). The following result holds:

Theorem 4. If $\sigma = \sigma(Y,w/r)$ is an arbitrary positive continuous function, then there exists a quasi-cost function $c = C(Y,r,w)$ for which the E.S. given by (3.4) is defined and is equal to $\sigma(Y,w/r)$. This function has the form

$$\log C(Y,1,w/r) = \log A(Y) + \int_1^{w/r} [p + x(Y,p)]^{-1}\, dp, \tag{5.11}$$

where

$$\log x(Y,p) = \log B(Y) + \int_1^p \sigma(Y,q)\, d\log q, \tag{5.12}$$

with $p = w/r$, $x = K/L = C_r(Y,r,w)/C_w(Y,r,w)$, and where A and B are positive functions. Suppose further that $\sigma(Y,w/r)$ is bounded away from one as $w/r \to 0$ and $w/r \to +\infty$, and that $\sigma_Y(Y,w/r)$ exists and is uniformly bounded in w/r for each Y. Then, for any continuously differentiable positive function $B(Y)$, there exists a continuously differentiable positive function $A(Y)$ such that (5.11) is a classical cost function.

Proof: The differential equation determined by substituting the given function $\sigma(Y,w/r)$ in the left-hand side of (3.1) has (5.12) as its first integral. By price homogeneity of the cost function, Euler's law implies the condition $C(Y,1,p)/C_w(Y,1,p) = p + x(Y,p)$. The solution of this differential equation is (5.11).

Direct computation verifies that $C(Y,r,w)$ defined by (5.11) is concave and increasing in prices and has an E.S. equal to $\sigma(Y,w/r)$. Then, C has all the properties of a cost function, provided that $C_Y(Y,r,w)$ is positive. We now show that the last hypothesis of the theorem is sufficient for the satisfaction of this condition. From (5.12),

$$\log(rK/wL) = \log(x(Y,p)/p) = \log B(Y) + \int_1^p [\sigma(Y,q) - 1]\, d\log q.$$

By the hypothesis that $[\sigma(Y,q) - 1]$ is bounded away from zero for extreme q, the relative share $x(Y,p)/p$ approaches zero or infinity for extreme p at least as rapidly as p^{ϵ} or $p^{-\epsilon}$, for some $\epsilon > 0$. For example, in the case $p \to \infty$ and $\sigma(Y,p) - 1 \leqq -\epsilon < 0$ for $p \geqq p'$, one has

$$\begin{aligned}\log(x(Y,p)/p) &\leqq \log B(Y) \\ &\quad + \int_1^{p'} [\sigma(Y,q) - 1]\, d\log q - \epsilon \log p + \epsilon \log p' \\ &\leqq a_0 - \epsilon \log p,\end{aligned}$$

where a_0 is a positive constant for each Y. The remaining cases follow similarly.

From the bounds above, we obtain the inequality

$$[1+x(Y,p)/p]^{-2}x(Y,p)/p \leqq \text{Min}\{x(Y,p)/p, p/x(Y,p)\} \leqq \text{Min}\{a_0 p^{\epsilon}, a_0 p^{-\epsilon}\},$$

with a_0 a positive constant.

Differentiating (5.12) with respect to Y,

$$\partial \log(x(Y,p)/p)/\partial Y = B_Y(Y) + \int_1^p \sigma_Y(Y,q)\, \mathrm{d} \log q.$$

Since σ_Y is bounded for each Y, we obtain an inequality

$$|\partial \log(x(Y,p)/p)/\partial Y| \leqq a_1 + a_2 |\log p|,$$

where a_1 and a_2 are positive constants.

Differentiating (5.11) with respect to Y,

$$\begin{aligned} C_Y/C &= A_Y/A - \int_1^{w/r} [1+x(Y,p)/p]^{-2}(x(Y,p)/p) \\ &\quad \times [\partial \log(x(y,p)/p)/\partial Y]\, \mathrm{d} \log p \\ &\geqq A_Y/A - \left| \int_1^{w/r} a_0 \,\text{Min}(p^{\epsilon}, p^{-\epsilon})[a_1 + a_2|\log p|]\, \mathrm{d} \log p \right| \\ &\geqq A_Y/A - \left\{ \frac{a_0 a_1}{1+\epsilon} + \frac{a_0 a_2}{\epsilon}\left(1+\frac{1}{\epsilon}\right)\right\}. \end{aligned}$$

Since the term in brackets is bounded, we can choose A_Y/A large enough to make $C_Y(Y,r,w)$ positive. Q.E.D.

It should be noted that the "canonical" forms (5.11) and (5.12) for a two-factor cost function allow a convenient breakdown of returns-to-scale into: (1) a *neutral scale effect* through $A(Y)$, (2) a *scale bias* or heterotheticity through $B(Y)$, and (3) a *substitutability-scale effect* through $\sigma(Y,w/r)$. This classification is also useful in considering technological change effects.

One form of the E.S. definition in the two-factor case, (3.5a), allows definition of the E.S. as a function of output Y and a deflated wage rate w/P, where $P = c/Y$ is the average cost of output. This formulation has the potential empirical advantage of avoiding explicit dependence on the price of capital services. As in Theorem 4, we ask under what conditions an arbitrary positive function $\sigma = \sigma(Y,w/P)$ is the E.S. for some classical production function.

Theorem 5. Suppose that $a(Y)$ is an arbitrary positive continuously differentiable function of $Y>0$ and $\sigma=\sigma(Y,w/P)$ is an arbitrary positive continuous function defined for $Y>0$ and $w/P \geqq a(Y)$. Then, any labor demand function of the form

$$\log L(Y,w/P)=\log[Y\theta(Y)/a(Y)]-\int_{a(Y)}^{w/P}\sigma(Y,z)\,\mathrm{d}\log z, \tag{5.13}$$

where $w/P \geqq a(Y)$ and $\theta(Y)$ is an arbitrary continuously differentiable function satisfying $0<\theta(Y)<1$, has an E.S. from the formula (3.5a) equal to $\sigma(Y,w/P)$. Further, there exists a quasi-cost function $c=C(Y,r,w)$ which has $C_w(Y,r,w)\equiv L(Y,wY/c(Y,r,w))$ for $wY/C(Y,r,w)\geqq a(Y)$.

Proof: The differential equation defined by inserting $\sigma(Y,w/P)$ in (3.5a) has the general solution (5.13), where the restriction $\theta(Y)<1$ follows from the condition that at $w/P=a(Y)$, $1>wL/C=a(Y)L(Y,a(Y))/Y\equiv\theta(Y)$.

Define $p=w/r$. If a cost function $C(Y,r,w)$ exists yielding the labor demand (5.13), then it must satisfy $C_w(Y,1,p)=L(Y,pY/C(Y,1,p))$. Hence, we consider possible solutions $c=\gamma(Y,p)$ of the differential equation

$$\partial\gamma/\partial p=L(Y,pY/\gamma). \tag{5.14}$$

Let $\alpha(Y)$ be an arbitrary positive continuously differentiable function of Y. We first demonstrate that (5.14) has a solution for $p\geqq\alpha(Y)$ satisfying $Yp/\gamma(Y,p)\geqq Y\alpha(Y)/\gamma(Y,\alpha(Y))=a(Y)$.

For $z\geqq a(Y)$, we have $0<L(Y,z)\leqq L(Y,a(Y))$. Define the extension $L(Y,z)=L(Y,a(Y))$ for $z<a(Y)$, and consider solutions to (5.14) through the point $p=\alpha(Y)$, $\gamma=Y\alpha(Y)/a(Y)$. Since L is continuous and uniformly bounded for all (p,γ), the Cauchy–Peano theorem establishes the existence of a solution $c=\gamma(Y,p)$ with $\gamma(Y,\alpha(Y))=Y\alpha(Y)/a(Y)$. Further, for $p>\alpha(Y)$, $(1-\theta(Y))Y\alpha(Y)/a(Y)+\theta(Y)Yp/a(Y)\leqq Yp/a(Y)$, and $a(Y)\leqq Yp/\gamma(Y,P)$. This completes the desired demonstration and shows that $\gamma(Y,p)$ is defined for $p\geqq\alpha(Y)$ independently of the extension of $L(Y,z)$ for $z<a(Y)$. We also observe that $\gamma(Y,p)$ is increasing in $p>0$, with $\gamma(Y,0)=[1-\theta(Y)]Y\alpha(Y)/a(Y)$.

Now define $C(Y,r,w)=r\gamma(Y,w/r)$. For $w\leqq\alpha(Y)r$, $C_r(Y,r,w)=[1-\theta(Y)]Y\alpha(Y)/a(Y)$. For $w\geqq\alpha(Y)r$, $C_r(Y,r,w)=\gamma(Y,p)-p\gamma_p(Y,p)$. Since $\gamma_{pp}(Y,p)=-\sigma(Y,pY/\gamma(Y,p))[1-p\gamma_p(Y,p)/\gamma(Y,p)]/p$, we have $\partial C_r(Y,1,p)/\partial p=\sigma(Y,pY/\gamma(Y,p))C_r(Y,1,p)/\gamma(Y,p)$. Since $C_r(Y,1,\alpha(Y))>$

0, this relation establishes that $C_r(Y,1,p)$ is increasing in p, and is consequently positive for $p>0$. Further, as a consequence of homogeneity, the condition $C_{rw}(Y,1,p)\geqq 0$ just established also shows that C is concave. Hence, provided $C_Y(Y,r,w)$ is positive, $C,(Y,r,w)$ satisfies all the properties of a classical cost function. Q.E.D.

The following corollary gives one set of restrictions on the functions in Theorem 5 which guarantee that the resulting $C(Y,r,w)$ is increasing in Y:

> *Corollary* 5.1. Suppose in the hypotheses of Theorem 5 that θ is independent of Y, $a(Y)$ satisfies $Ya'(Y)/a(Y)<1$, and $\sigma(Y,w/P)$ has the form $\sigma=\sigma(w/Pa(Y))$. Then, Theorem 5 is satisfied by a classical cost function of the (homothetic) form $C(Y,r,w)=r\beta(w/r)Y/a(Y)$ with C_Y positive.

Proof: Under the hypotheses of the corollary, (5.13) may be written

$$\log L(Y,z)=\log[Y\theta/a(Y)]-\int_1^{z/a(Y)}\sigma(z')\,\mathrm{d}\log z'.$$

Using the transformation $\beta(Y,p)=\gamma(Y,p)a(Y)/Y$ in the equation $\gamma_p(Y,p)=L(Y,pY/\gamma(Y,p))$, we obtain a differential equation $\beta_p=\theta\exp[-\int_1^{p/\beta}\sigma(z)\,\mathrm{d}\log z]$ which has a solution through the point $p=\beta=1$ independent of Y (using the same proof as in Theorem 5). Q.E.D.

We will now explore extensions of Theorems 4 and 5 when a third productive input M is present. The E.S. formula (3.4) on which the construction of Theorem 4 is based continues to hold provided the shadow E.S. definition is assumed, and (5.12) becomes

$$\log x(Y,p,m/P)=\log B(Y,m/P)+\int_1^p\sigma(Y,m/P,q)\,\mathrm{d}\log q, \qquad (5.12a)$$

where m is the price of the third input and P is the average cost of production. One cannot obtain from (5.12a) an explicit formula like (5.11) for the cost function, except in special cases. One useful case is given in the following extension. This result also provides a general condition for the separability of "value-added" and remaining inputs. Note that if m/P is constant, then this result holds trivially by the Hicks aggregation theorem [see Diewert (1974c)].

Lemma 6. If $\sigma = \sigma(Y,w/r)$ is an arbitrary positive continuous function of output and the relative price of labor and capital, and is independent of the deflated price m/P of a third input, then there exists a quasi-cost function $c = C(Y,r,w,m)$ for which the shadow E.S. between capital and labor is defined and is equal to $\sigma(Y,w/r)$. This function has the composite form

$$C(Y,r,w,m) = H(Y,m,G(Y,r,w)), \tag{5.15}$$

where $H(Y,m,v)$ is an arbitrary twice continuously differentiable strictly quasi-concave quasi-cost function and $G(Y,r,w)$ is a quasi-cost function satisfying

$$\log G(Y,r,w) = \log[rA(Y)] + \int_1^{w/r} [p + x(Y,p)]^{-1}\,dp, \tag{5.16}$$

with $x(Y,p)$ given by (5.12) and $A(Y),B(Y)$ arbitrary continuously differentiable positive functions.

If C is a classical cost function, then it is the cost function of a set of classical production possibilities of the implicitly defined form $F(Y,f(Y,K,L),M) = 1$.

If H and G are classical cost functions (as under the last hypothesis of Theorem 4, for example), then C is also a classical cost function. Alternately, if H is a classical cost function and $\sigma(Y,w/r)$ is independent of Y, then taking B to be a positive constant yields a classical cost function $C(Y,r,w,m) = H(Y,m,G(r,w))$ which is the cost function of a set of classical production possibilities of the form $Y = F(M,f(K,L))$, where $f(K,L)$ is linear homogeneous.

Proof: Direct computation verifies that $C(Y,r,w,m)$ has a shadow E.S. between capital and labor equal to $\sigma(Y,w/r)$. We next suppose that C is a classical cost function and show that the corresponding production possibilities have the stated forms. Define a distance function $f(Y,K,L) = \text{Max}\{\lambda \mid rK + wL \geqq \lambda G(Y,r,w) \text{ for all } r,w > 0\}$. One may verify that $f(Y,K,L)$ is linear homogeneous, concave, and non-decreasing in (K,L). Further, one has $G(Y,r,w)$ strictly quasi-concave in (r,w) from its definition, which can be shown to imply that $f(Y,K,L)$ is continuously differentiable in (K,L). From the definition of f, $1 \equiv f(Y,G_r(Y,r,w),G_w(Y,r,w))$, implying $0 \equiv f_K G_{rr} + f_L G_{wr}$ with the derivatives evaluated at $K = G_r(Y,r,w), L = G_w(Y,r,w)$. Since $G_{rr} < 0$ and $G_{wr} > 0$ from the definition of G and $Kf_K + Lf_L > 0$ by linear homogeneity, we

have f_K, f_L strictly positive on the set of points

$$S = \{(Y,K,L) > 0 | K = \lambda G_r(Y,r,w), \quad L = \lambda G_w(Y,r,w) \text{ for some } \lambda,r,w > 0\}.$$

Define the sets $R = \{(Y,K,L,M) | (Y,K,L) \in S, M > 0\}$ and $T = \{(Y,X,L,M) | \exists K \ni (Y,K,L) \in S$ and $X = f(Y,K,L), M > 0\}$. Then, $X = f(Y,K,L)$ defines a one-to-one transformation of R onto T. Let $K = \varphi(X,Y,L)$ denote the inverse transformation of $X = f(Y,K,L)$.

The classical cost function $C(Y,r,w,m)$ defines via (2.2) a set of classical production possibilities $Y = E(K,L,M)$. The functions G and H are strictly quasi-concave in their respective price arguments, by construction for G and by assumption for H. Hence, C is strictly quasi-concave in (r,w,m), which can be shown to imply that E is continuously differentiable. Define $F(Y,X,L,M) \equiv E(\varphi(X,Y,L),L,M) - Y$ on T. Then, $F(Y,f(Y,K,L),L,M) \equiv E(K,L,M) - Y$ on R. Differentiating with respect to K and L, we obtain

$$F_X(Y,f(Y,K,L),L,M)f_K(Y,K,L) \equiv E_K(K,L,M),$$
$$F_X(Y,f(Y,K,L),L,M)f_L(Y,K,L) + F_L(Y,f(Y,K,L),L,M) \equiv E_L(K,L,M).$$

Consider any point (Y,K,L,M) in R with $Y = E(K,L,M)$ and let $(r,w) > 0$ be a price vector such that the condition $K/L = G_r(Y,r,w)/G_w(Y,r,w)$ defining the set S is satisfied. By the cost minimization conditions, there exists a non-negative scalar μ such that $E_K(K,L,M) = \mu r$ and $E_L(K,L,M) = \mu w$ for the (r,w) vector above. If $F_X = 0$ at this point, then $\mu = 0$, implying $F_L = 0$. If $F_X > 0$, then since $f_L/f_K = w/r$,

$$w/r = E_L/E_K = f_L/f_K + F_L/F_X f_K$$
$$= w/r + F_L/F_X f_K$$

implies $F_L = 0$. Hence, $F_L \equiv 0$ on R, and the production possibilities satisfy $F(Y,f(Y,K,L),M) = 0$.

Finally, we note that when $\sigma(Y,w/r)$ and B are independent of Y, then by Lemma 1, $f(Y,K,L) = h(K,L)/A^{-1}(Y)$, where A^{-1} is the inverse of $A(Y)$ and h is linear homogeneous. Then the equation $F(Y,h(K,L)/A^{-1}(Y),M) = 0$ can be solved to give the formula $Y = X(M,h(K,L))$. With a change of notation, this is the final formula of the lemma. Q.E.D.

6. An Econometric Model of the Electricity Generation Industry

(1) To obtain some preliminary evidence on the validity of the assumptions presented in Section 2, we have carried out a pilot study using cross-sectional, plant-level data in the electricity generation industry for 1957–1960. This industry was chosen to take advantage of detailed cost and physical operating data available for steam generating plants, allowing us to test a wide range of hypotheses. Our efforts are directed primarily to tests of A.3 (homogeneity, homotheticity), A.4 (constant elasticity of substitution), and A.7 (output price unrelated to wage rate). The results of these tests should be viewed as indicative – rather than conclusive – for this industry, and obviously have no implications for time-series or international cross-sectional analysis, or studies of other industries.

(2) Our analysis is confined to privately owned, conventionally fueled, steam-electric generating plants. Approximately 80% of all electricity generated in the period studied came from private utilities. Further, approximately 90% of the electricity generated by private utilities came from steam-electric plants. Hence, these plants represented a substantial segment of the electricity generating industry.

Inputs

Electricity generation requires three primary inputs – fuel, capital equipment, and labor. In a typical plant, these inputs will account for about 50%, 45%, and 5% of total costs, respectively.

Coal, oil, and gas are the principal fuels used. Converted to B.T.U. equivalents, they are virtually perfect substitutes in production, and will not be distinguished in our analysis.[9] The power engineering literature and examination of published heat rates (B.T.U.'s consumed per kilowatt-hour produced) suggest strongly that the relation between fuel consumption and output is primarily technological: Plants of ap-

[9]Although these fuels are perfect substitutes *ex ante*, the plant will be confined, *ex post*, to use of one of them, unless an additional capital investment is made to allow convertibility. About one-third of the plants constructed in the decade 1954–1964 have convertibility between two or more fuels.

proximately equal capacity constructed in the same year have approximately equal heat rates. Further, the heat rate for new plants falls fairly uniformly from year to year, suggesting that plant capacity and the state of technology are the primary determinants of heat rates.[10]

Capital investment in generating plants has three major components: land, structures, and equipment (fuel-handling equipment, the boiler-turbine-condenser-generator complex, control equipment). Of these components, land is the least important, accounting for less then one per cent of the total investment in most plants. The percentage of investment in structures shows relatively little variation between plants of a given type: 20% in a typical conventional plant, 15% with outdoor boilers, and 10% in a full outdoor plant. Further, this percentage seems to vary little with plant capacity. The choice of plant type is generally governed by the climate of a region, and does not normally reflect a substitution of maintenance labor for capital.

Equipment represents the major portion of capital costs, with operating units being predominant. Engineering data indicates that choice of operating units does not depend on *explicit* cost computations. Firms generally choose the most efficient operating unit available, consistent with the capacity restrictions on the planned plant and the willingness of the firm to incur the increased "shake-down time" and risk attached to more advanced, but experimental, equipment. Plants under construction exhibit a wide variety of fuel-handling and control equipment. The engineering literature discusses this type of equipment primarily in terms of its possible substitution for labor. It would appear that most of the observed capital–labor substitution in plants involves this component of capital.

Because of the relatively stable proportions in the observed income shares of the major capital components, comparisons of capital investment in new plants can be made with reasonable accuracy with a single capital index.

Production and maintenance labor is a substantially smaller

[10]There is some variation among contemporary plants in the degree of adaptation of recent innovations, which might be attributed to technological substitutability between capital and fuel. The results of Dhrymes and Kurz (1964) suggest such an effect. However, the power engineering literature suggests that the primary effect of experimental innovations is to increase "shake-down" time and introduce some risk into the determination of the resulting heat rate, rather than to increase direct (measured) capital costs.

component of total cost than fuel and capital. Skilled maintenance and operation laborers and technicians account for most of the wage bill in electric plants.[11] Except for a few automated plants, the mix of workers of various types is fairly uniform across plants, so that a single labor input index, unadjusted for "quality", can be used.

Input Prices

Electric utilities purchase fuels under long-term contracts in competition with other users. The cost of capital services is influenced by the initial price of a unit of physical capital, the depreciation rate, and the market interest rate. The major components of physical capital are operating units, for which prices are fixed nationally by the major electrical equipment manufacturers. Variations in transportation cost for operating units are likely to have a relatively small effect on initial capital price. Similarly, the effects of variations in construction costs on the initial price of structures will probably result in relatively small variations in the initial capital price. Hence, for comparison of new, contemporary plants, the initial "book value" of capital will be an approximate index of physical capital.

The market for capital funds which electrical utilities face has two important institutional features: (1) Most capital for construction is obtained through long-term (usually 30-year) mortgages. These mortgages appear to trade in broad and actively competitive markets, with only weak variations in interest rates over large regions. With the exception of very small utilities, mortgage interest rates do not vary systematically with firm size, indicating that utilities have relatively little market power. (2) Rate commissions maintain rather stringent control on proposed investments of utilities. Normally, a new investment is expected to yield the current rate of return on capital at established output prices. The "fair" rates of return established by rate commissions are usually one to three per cent higher than mortgage interest rates. Hence, the imputed, or shadow, interest rate on which investment decisions will be based will equal approximately the (higher) "fair" rate of return prevailing in the construction area. The imputed price of capital services

[11]Most administrative, accounting, and engineering expenses are incurred at the firm level, and not imputed to individual plants.

(measured in "book value" interest units) will then equal the imputed interest rate plus the actual depreciation rate.[12]

The wage rate for production workers appears to be relatively stable over time for generating plants. Since utilities are small employers in local markets for skilled laborers, and compete in broad markets for technicians and engineers, they take wage rates as given.

Output Price and Quantity

Utilities are required to meet current electricity demand at prevailing prices, either by production, or by purchases from other utilities. The allocation of produced output among the utility's plants is done on the basis of marginal cost calculations, which are, in turn, based on historical operating experience in each plant. Further, the net amount of purchased power, which is significant in some utilities, will be influenced by demand and production costs over wide regions. Hence, the output of a given plant, although determined primarily by the external demand, will be influenced by decisions of the utility. Despite this influence, the impact of utility decisions on the output of relatively new plants will be small, since most variation in total production in the utility will be absorbed by older plants operated on a standby or peak-service basis. As a first approximation, the output of a new plant can then be taken as an exogenous variable.

The price of delivered electricity faced by a utility is determined by a rate commission, and is set to give a "fair" rate of return on the utility's capital, in light of historical data on operating costs and output. Again as a first approximation, one can assume that the utility will treat output price and demand as pre-determined variables in new-plant design decisions, and will attempt to choose a cost-minimizing input combination for such plants.[13]

[12]The empirical results reported here use as a measure of capital service price the rate r_0 in the data appendix. This rate is determined by dividing the "economic" capital service costs of the plant by the book value of capital in the plant. Since plants in the sample are new, the book value of capital is a reasonable approximation to real economic capital. Economic capital service cost is defined for the electric utility plant of the parent utility by summing depreciation and amortization, net operating income (profit), and interest charges. This cost is then allocated to electric plants in proportion to net (depreciated) book value of plant. Several alternative capital service cost measures (r_1,r_2,r_3) are defined in the Appendix. Substitution of these measures of capital cost in the regressions reported below did not yield substantially different results; we do not report these alternative regressions.

[13]The claim is sometimes made that an electric utility has no incentive to minimize costs, given the present criteria of rate commissions in which cost reductions are offset by rate reductions. However, two effects make cost minimization rational for the utility: (1) since

There will normally be a systematic relationship between the true unit costs in a new plant and historically observed unit costs for the utility, due to input prices which are stable over time and a smoothly improving technology. Hence, historical unit costs may be used as an instrumental variable in the place of observed current unit costs.

(3) We now develop an econometric model describing utility behavior in the design of new plants. On the basis of this model, we will make inferences about the assumptions in Section 2 on production possibilities. The model is designed for application to cross-section data on new plants whose construction incorporates a uniform technology.

Variables

r,w,m	the observed prices of capital services, labor, and fuel, respectively, in the first year of operation;
R	the observed plant capacity in the first year of operation;
K,L,M	the exact cost-minimizing inputs of capital, labor, and fuel, respectively, for a given designed output at specified input prices;
Y,l	the observed output and load factor (observed output/potential output), respectively, in the first year of operation;
$\bar{L},\bar{M}$	the observed labor and fuel inputs in the first year of operation.

Assumptions

1. The observed prices r,w,m are exogenous, and are instrumental variables reflecting the future prices expected by the utility at the time of the design of the plant.

2. The observed plant capacity R is, as a first approximation, predetermined, and is an instrumental variable reflecting the designed plant output level. (Designed output level, in turn, is influenced primarily by external demand, as we noted earlier.)

3. The design possibilities available to the utility are representable by

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rates for a given utility are influenced by the rates of other utilities, short-run rate reductions are unlikely to completely offset a cost reduction, and (2) because of the lag with which rate commissions operate, substantial short-run profits can be gained by a cost reduction. The rate criteria do have distorting effects, making disguised profits attractive, as well as the carrying of quantities of excess capacity to inflate the rate base.

an exact classical cost function $C = G(R,r,w,m) = rK + wL + mM$, where K,L,M,C correspond to exact cost minimization at the values (R,r,w,m) of the exogenous variables. There are no variations in available technology over the period or regions sampled.[14]

4. Actual ex ante design inputs K^*,L^*,M^* are endogenous, with non-stochastic components equal to the respective exact cost-maximizing inputs.

5. Observed output in the first year of operation (Y) is, as a first approximation, exogenous. Observed capital input in the first year of operation is equal to K^*. Observed labor and fuel inputs in the first year are functions of their respective actual ex ante designed inputs and the observed load factor in the first year. As a first approximation, these functions do not depend on other characteristics of the plant design or the observed relative prices.

Variables

$C = rK + wL + mM$	exact total cost;
$P = C/Y$	exact unit cost;
$V = rK + wL$	exact value-added;[15]
$C^* = rK^* + wL^* + mM^*$; $P^* = C^*/Y^*$	actual ex ante designed total cost;
$P^* = C^*/Y$	actual ex ante designed unit cost;
$V^* = rK^* + wL^*$	actual ex ante designed value-added;
$\bar{C} = rK^* + w\bar{L} + m\bar{M}^*$	reported "total" cost;
$\bar{P} = \bar{C}/Y$	reported "unit" cost;
$\bar{V} = rK^* + w\bar{L}$	value-added in first year of operation;
P'	historical unit cost in the utility, an approximate instrumental variable for P;
$\bar{L} = \lambda(l)L^*$; $\bar{M} = \mu(l)M^*$	the relations between designed and reported labor and fuel inputs, determined by the load factor l.

[14]In a more careful study, one might wish to distinguish regions of the country on the grounds that structure technology (conventional, outdoor boilers, etc.) varies with climate.

[15]Throughout, we shall define value-added at factor costs, noting that $V = rK + wL = PY - mM$.

Definitions

$\pi_L = wL/C$	the exact cost share of labor; define similarly π_K, π_M as the exact cost shares of capital and fuel;
$\epsilon_L = L^*/L$	the stochastic component of actual designed labor input; define similarly ϵ_K, ϵ_M as the stochastic components of actual designed capital and fuel inputs;
$\alpha = P'/P$	the (non-stochastic) relation between exact unit cost and historical unit cost;
$\epsilon_C = C^*/C$; $\epsilon_v = V^*/V$	the stochastic components of actual designed cost;
$\eta_C = \bar{C}/E(\bar{C})$; $\eta_V = \bar{V}/E(\bar{V})$	the stochastic components of ex post reported cost;
$b_C = E(\bar{C})/C$; $b_V = E(\bar{V})/V$	the non-stochastic component of the relation between exact ex ante cost and reported ex post cost.[16]

To establish econometric relationships between observed variables, we can substitute in the exact relations of Section 4 the following terms:

$$C = \bar{C}/b_C\eta_C,$$
$$V = \bar{V}/b_V\eta_V,$$
$$K = K^*/\epsilon_K,$$
$$L = \bar{L}/\lambda(l)\epsilon_L,$$
$$M = \bar{M}/\mu(l)\epsilon_M,$$
$$P = P'/\alpha.$$

We now present the hypotheses to be tested, and the results of the tests. The sample consists of 36 steam-electric plants in the continental United States which began operation in the period 1957–1960. The

[16]Computation establishes the following relations:

$$\epsilon_C = \pi_K\epsilon_K + \pi_L\epsilon_L + \pi_M\epsilon_M, \qquad \epsilon_V = (\pi_K\epsilon_K + \pi_L\epsilon_L)/(\pi_K + \pi_L).$$

When $E(\epsilon_K) = E(\epsilon_M) = 1$,

$$b_C = \pi_K + \lambda(l)\pi_L + \mu(l)\pi_M, \qquad b_V = \pi_K + \lambda(l)\pi_L;$$
$$\eta_C = (\pi_K\epsilon_K + \lambda(l)\pi_L\epsilon_L + \mu(l)\pi_M\epsilon_M)/b_C, \qquad \eta_V = (\pi_K\epsilon_K + \lambda(l)\pi_L\epsilon_L)/b_V.$$

observed data for these plants and a discussion of their construction are given in the appendix.

(4) *Hypothesis I* – Fuel is not substitutable for any combination of capital and labor.

Under Assumptions A.1 and A.6 in Section 4 and the econometric model specified above,

$$\bar{M} = \mu(l)G_m(R,1,w/r,m/r)\epsilon_M,$$

$$\frac{\bar{M}}{\bar{L}} = \frac{\mu(l)G_m(R,r/w,1,m/w)\epsilon_M}{\lambda(l)G_w(R,r/w,1,m/w)\epsilon_L},$$

$$\frac{\bar{M}}{K^*} = \frac{\mu(l)G_m(R,1,w/r,m/r)\epsilon_M}{G_r(R,1,w/r,m/r)\epsilon_K}.$$

Under the null hypothesis, G_m must be independent of w/r and m/r and G_w and G_r must be independent of m. Under the alternative hypothesis, G_m must be decreasing in m, $\partial \log(\bar{M}/\bar{L})/\partial \log(w/m)|_{l,r/w} > 0$ is the shadow E.S. between fuel and labor, and $\partial \log(\bar{M}/K^*)/\partial \log(r/m)|_{l,r/w} > 0$ is the shadow E.S. between fuel and capital.

A fairly stringent test of the hypothesis can be obtained starting from first-order Taylor's expansions of logarithms of these equations in the variables $\log R$, l, $\log r$, $\log w$, $\log m$ about their sample means, which yield the equations

$$\begin{aligned}
\log \bar{M} &= a_{10} + a_{11}\log R + a_{12}l + a_{13}\log r + a_{14}\log w + a_{15}\log m + \epsilon_1', \\
\log(\bar{M}/\bar{L}) &= a_{20} + a_{21}\log R + a_{22}l + a_{23}\log r + a_{24}w + a_{25}\log m + \epsilon_2', \\
\log(\bar{M}/K^*) &= a_{30} + a_{31}\log R + a_{32}l + a_{33}\log r \\
&\quad + a_{34}\log w + a_{35}\log m + \epsilon_3'. \qquad (6.1)
\end{aligned}$$

Price homogeneity imposes the linear constraint $a_{i5} = -a_{i3} - a_{i4}$ for $i = 1,2,3$. Under the null hypothesis, $a_{13} = a_{14} = a_{15} = a_{25} = a_{35} = 0$, while under the alternative hypothesis $a_{15} > 0$, $a_{25} < 0$, $a_{35} < 0$. Assume as a first approximation that the shock ϵ_i' is independently, identically normally distributed over the cross-section for each i.[17]

[17] I am indebted to a referee for pointing out that this parameterization reduces the stringency of the test of hypothesis, and that more generally the critical level of this test is not independent of the true functional form of G and the choice of expansion. In principal, a non-parametric test of association would be most satisfactory. Nevertheless, it is reasonable to suppose on the basis of the general robustness of the normal linear model that the parametric test is valid as a first approximation.

Estimates based on these equations and associated tests are given in Tables 1–3. We shall accept Hypothesis I conditionally on the basis of the tests in Tables 1 and 2. Note however that the power of these tests are low against the alternative of a fuel–capital E.S. in the 0.15–0.30 range, and that the tests in Table 3 suggest the hypothesis may be rejected in larger samples. Regression 6 can be interpreted as providing an estimate of the fuel–labor E.S. = 0.123 (standard error = 0.127); and Regression 11 as providing an estimate of the fuel–capital E.S. = 0.322 (standard error = 0.163). These estimates are in approximate agreement with the results of Dhrymes and Kurz (1964) which use a "capacity" index of capital stock.

The coefficient of $\log R$ in Regression 4 is an elasticity of fuel input with respect to plant capacity, with relative prices and load factor fixed, and the estimate 0.772 suggests substantial fuel-saving with increased scale. The estimated coefficient 0.445 of $\log R$ in Regression 8 suggests even larger labor-saving with increased scale, while the estimated coefficient -0.012 of $\log R$ in Regression 12 suggests that increased scale results in capital-saving to about the same degree as fuel-saving. We can anticipate from these regressions that the technology will exhibit strongly increasing returns to plant capacity, biased in favor of labor-saving so that larger plants will be more capital intensive.

The coefficients of l in Regressions 4, 8, and 12 provide an indication of short-run returns to scale at outputs below capacity. For example, $\partial \log \bar{M} / \partial \log Y = a_{12} l$.[18] At the sample mean load factor of 0.664, these estimates yield a short-run elasticity of fuel input with respect to output of 1.33 from Regression 4 and of 1.22 from Regression 12, and of labor input with respect to output of $1.33 - (0.566)(0.664) = 0.96$ from Regression 8. This suggests approximately "constant returns" for labor input and substantial "decreasing returns" to fuel in the short-run. The last result is not consistent with engineering experience, which suggests short-run constant returns to fuel.

Hypothesis II – The elasticity of substitution between capital and labor is constant. (We use here the shadow E.S. definition; by Corollary 1.2, the Allen–Uzawa E.S. cannot be positive and constant when there is no substitutability of fuel for capital or labor.)

Under Assumptions A.1 and A.6 of Section 4, the econometric model

[18]The regressions in Tables 1–3 were also run with l replaced by $\log l$. This alternative yielded essentially the same parameter estimates, but marginally worse overall fits.

TABLE 1
Test of substitutability of fuel for capital and labor (fuel demand equation); dependent variable: log $\bar{M}$.

Regression	Independent variables						Error		Mult. corr. coeff.
	Const. a_{10}	log R a_{11}	l a_{12}	log r a_{13}	log w a_{14}	log m a_{15}	Sum of squares	DF	
1	0.646 (0.126)	0.772 (0.031)	1.925 (0.191)				0.48186	33	0.975
2	0.807 (0.195)	0.772 (0.031)	1.936 (0.191)		−0.058 (0.054)	0.058 (0.054)	0.46512	32	0.976
3	0.668 (0.134)	0.773 (0.032)	1.939 (0.195)	0.033 (0.063)		−0.033 (0.063)	0.47789	32	0.975
4	1.061 (0.254)	0.772 (0.031)	1.997 (0.191)	0.119 (0.078)	−0.120 (0.067)	0.001 (0.130)	0.43249	31	0.978

Test $a_{13} = a_{14} = 0$: $F = \dfrac{(0.48186 - 0.432492)/2}{0.432492/31} = 1.77 < F_{0.90}(2,31) = 3.31$, and the null hypothesis is accepted at the 10% significance level.

Note: All logarithms are to base e, standard errors are given in parentheses below estimates. Regressions 2–4 are estimated subject to the homogeneity restrictions $a_{14} + a_{15} = 0$, $a_{13} + a_{15} = 0$, and $a_{13} + a_{14} + a_{15} = 0$, respectively.

TABLE 2

Test of substitutability of fuel for capital and labor (relative fuel–labor demand equation); dependent variable: $\log(\bar{M}/\bar{L})$.

Regression	Independent variables						Error		Mult. corr. coeff.
	Const. a_{20}	$\log R$ a_{21}	l a_{22}	$\log r$ a_{23}	$\log w$ a_{24}	$\log m$ a_{25}	Sum of squares	DF	
5	−2.254 (0.296)	0.444 (0.074)	0.676 (0.449)				2.66501	33	0.660
6	−2.956 (0.461)	0.446 (0.074)	0.653 (0.450)		0.123 (0.127)	−0.123 (0.127)	2.58938	32	0.670
7	−2.266 (0.317)	0.444 (0.075)	0.668 (0.460)	−0.017 (0.150)		0.017 (0.150)	2.66390	32	0.660
8	−2.957 (0.613)	0.445 (0.074)	0.566 (0.462)	−0.169 (0.188)	0.212 (0.162)	−0.043 (0.314)	2.52362	31	0.678

(a) Test $a_{24}(=-a_{25})=0$, Regression 6: $F=\dfrac{(2.66501-2.58938)/1}{2.58938/32}=0.935<F_{0.90}(1,32)=2.88.$

(b) Test $a_{23}(=-a_{25})=0$, Regression 7: $F=\dfrac{(2.66501-2.66390)/1}{2.66390/32}=0.013<F_{0.90}(1,32)=2.88.$

(c) Test $a_{25}=a_{23}=a_{24}=0$, Regression 8: $F=\dfrac{(2.66501-2.52362)/2}{2.52362/31}=0.868<F_{0.90}(2,31)=2.49.$

Regressions 6–8 are estimated subject to the homogeneity restrictions $a_{24}+a_{25}=0$, $a_{23}+a_{25}=0$, and $a_{23}+a_{24}+a_{25}=0$, respectively.

In tests (a) to (c), the null hypothesis is accepted at the 10% significance level. The power of test (a) against the alternative $a_{24}=-a_{25}=0.20$ is 0.44.

TABLE 3

Test of substitutability of fuel for capital and labor (relative fuel–capital demand equation); dependent variable: $\log(\bar{M}/K^*)$.

Regression	Independent variables						Error		Mult. corr. coeff.
	Const. a_{30}	log R a_{31}	l a_{32}	log r a_{33}	log w a_{34}	log m a_{35}	Sum of squares	DF	
9	−1.957 (0.340)	−0.014 (0.085)	1.544 (0.516)				3.52693	33	0.257
10	−1.610 (0.532)	−0.016 (0.085)	1.566 (0.519)		0.125 (0.147)	−0.125 (0.147)	3.44912	32	0.273
11	−1.741 (0.344)	−0.009 (0.082)	1.679 (0.500)	0.322 (0.163)		−0.322 (0.163)	3.14306	32	0.338
12	−0.198 (0.603)	−0.012 (0.073)	1.908 (0.454)	0.660 (0.185)	−0.474 (0.159)	−0.186 (0.309)	2.44399	31	0.485

(a) Test $a_{34}(=-a_{35})=0$, Regression 10: $F=\dfrac{(3.52693-3.44912)/1}{3.44912/32}=0.722<F_{0.90}(1,32)=2.88$.

(b) Test $a_{33}(=-a_{35})=0$, Regression 11: $F=\dfrac{(3.52693-3.14306)/1}{3.14306/32}=3.908$ and $F_{0.90}(1,32)=2.88<3.908<F_{0.95}(1,32)=4.15$.

(c) Test $a_{35}=a_{33}=a_{34}=0$, Regression 12: $F=\dfrac{(3.52693-2.44399)/2}{2.44399/31}=6.868>F_{0.99}(2,31)=5.37$.

In test (a), the null hypothesis is accepted at the 10% significance level.
In test (b), the null hypothesis is rejected at the 10% significance level, but accepted at the 5% significance level.
In test (c), the null hypothesis is rejected at the 1% significance level.

The power of test (b) at the 5% significance level against the alternative $a_{33}=-a_{35}=0.20$ is 0.33.
Regressions 10–12 are estimated subject to the homogeneity restrictions $a_{34}+a_{35}=0$, $a_{33}+a_{35}=0$, and $a_{33}+a_{34}+a_{35}=0$, respectively.

specified above, and the tentative conclusion that fuel is not substitutable for capital or labor, Theorem 4 can be used to derive estimation formulae under the hypothesis and alternatives. If one accepts the non-substitutability of fuel, then the exact cost function can be written $C = G(R,r,w) + mM$, where M is a function of R alone, and $V = G(R,r,w)$ is value-added. Given a family of (shadow) E.S. functions $\sigma(R,w/r)$ encompassing the hypothesis and alternatives of interest, equation (5.11) allows construction of cost functions which, in principle, could be adapted for direct estimation. Alternatively, the price derivatives of (5.11) yield the exact relation

$$K/L = B(R)\exp\left\{\int_1^{w/r} \sigma(R,z)\, \mathrm{d}\ln z\right\}. \tag{6.2}$$

From (6.2) we have the observable relation

$$\ln(K^*/\bar{L}) = \ln B(R) - \ln \lambda(l) + \int_1^{w/r} \sigma(R,z)\, \mathrm{d}\ln z + \ln(\epsilon_K/\epsilon_L).$$

We consider the following families of E.S. functions $\sigma(R,r/w)$, where γ,δ,ξ,η are parameters:

$$\sigma(R,r/w) = \gamma + \delta\xi(w/50r)^{\xi}, \tag{6.3}$$

$$\sigma(R,r/w) = \gamma + \delta(w/r)/(w/r+\xi), \tag{6.4}$$

$$\sigma(R,r/w) = \gamma + \delta(w/r)(\xi + w/r). \tag{6.5}$$

[The constant in (6.3) is introduced to scale the observations for computational purposes.] To test the dependence of the E.S. on designed capacity R, we consider

$$\sigma(R,r/w) = \gamma + \eta R + \delta(w/r)(\xi + w/r). \tag{6.6}$$

The family (6.3) covers a large class of E.S. functions which are monotone in w/r, and exhibit elasticity values approaching zero or infinity at one or both price-ratio extremes. The family (6.4) is also monotone in w/r, but has positive finite upper and lower bounds. The families (6.5) and (6.6) allow the E.S. value to obtain a minimum or maximum at some intermediate w/r ratio if ξ is negative. These families would appear to cover most interesting alternatives to the constant elasticity hypothesis. Tests based on these families should provide reasonable robustness against any alternative of a monotone or unimodal elasticity.

Estimation formulae for these families are given below. The constraint

that $\sigma(R,w/r)$ be non-negative is ignored in these formulae, and the resulting estimates must be checked to verify satisfaction of this constraint over the range of sample. (Failure of the condition would require a regression subject to non-linear constraints.) First-order Taylor's expansions of $B(R)$ and $\lambda(l)$ in $\log R$ and l, respectively, are utilized in these regressions:

$$\log(K^*/\bar{L}) = a_0 + a_1 \log R + a_2 l + f(R,w/r) + \ln(\epsilon_K/\epsilon_L), \tag{6.7}$$

where f has one of the following forms:

$$f(w/r) = \gamma \log(w/r), \qquad \text{Family (6.3), } \delta = 0 \text{ (constant E.S. case),} \tag{6.8}$$

$$f(w/r) = \gamma \ln(w/r) + \delta(w/50r)^{\xi}, \qquad \text{Family (6.3), } \delta \neq 0, \ \xi \neq 0, \tag{6.9}$$

$$f(R,w/r) = \gamma \ln(w/r) + \delta \ln(\xi + w/r), \qquad \text{Family (6.4), } \delta \neq 0, \tag{6.10}$$

$$f(R,w/r) = \gamma \ln(w/r) + (\delta/2)(\xi + w/r)^2, \qquad \text{Family (6.5), } \delta \neq 0, \tag{6.11}$$

$$f(R,w/r) = (\gamma + \eta R) \ln(w/r) + (\delta/2)(\xi + w/r)^2, \qquad \text{Family (6.6), } \delta \neq 0, \ \eta \neq 0. \tag{6.12}$$

Assume the stochastic elements ϵ_K/ϵ_L are independently, identically distributed log-normal, with $E(\ln(\epsilon_K/\epsilon_L)) = 0$. The regression results and associated tests are given in Table 4.

We conclude from the tests in Table 4 that the null hypothesis of a constant E.S. between capital and labor is accepted at the 10 percent significance level. Several reservations should be noted however about this conclusion. First, while the alternatives of a monotone or unimodal dependence on relative prices and a monotone dependence on plant capacity should capture first-order effects, the tests have not been performed for a comprehensive set of models. Second, the range of relative prices in the sample is fairly small, and variations from constancy of the E.S. may be significant only at extreme price ratios. Third, the power of these tests is very low against alternatives of

TABLE 4

Test of constancy of the elasticity of substitution between capital and labor; dependent variable: $\log(K^*/\bar{L})$.

Regression	Estimated coefficients								Error		Mult. corr. coeff.
	$f(w/r)$ from equation	Const. a_0	$\log R$ a_1	l a_l	γ	δ	ξ	η	Sum of squares	DF	
13	(6.8)	−2.817 (0.471)	0.459 (0.057)	−1.308 (0.356)	0.731 (0.119)				1.55805	32	0.765
14	(6.9)	−3.622 (1.029)	0.442 (0.061)	−1.160 (0.395)	0.895 (0.221)	0.047 (0.054)	−2.8 (n.c.)		1.52001	30	0.771
15	(6.10)	−5.691 (3.245)	0.435 (0.063)	−1.119 (0.415)	−7.395 (9.078)	8.674 (9.690)	3.45 (n.c.)		1.51879	30	0.771
16	(6.11)	−2.692 (8.744)	0.459 (0.059)	−1.308 (0.362)	0.705 (1.825)	-7×10^{-6} (4.6×10^{-4})	−134 (n.c.)		1.55804	30	0.765
17	(6.12)	−2.447 (6.420)	0.293 (0.148)	−1.029 (0.428)	0.556 (1.410)	0.0 (0.0005)	−112 (n.c.)	0.024 (0.023)	1.47451	29	0.778

(a) Test $\delta = 0$, Regression 14: $F = \dfrac{(1.55805) - 1.52001)/2}{1.52011/30} = 0.375 < F_{0.90}(2,30) = 2.49.$

(b) Test $\delta = 0$ (with restriction $\xi \neq 0$), Regression 15: $F = \dfrac{(1.55805 - 1.51879)/2}{1.51879/30} = 0.388 < F_{0.90}(2,30) = 2.49.$

(c) Test $\delta = 0$, Regression 16: $F = \dfrac{(1.55805) - 1.55804)/2}{1.55804/30} = 0.00009 < F_{0.90}(2,30) = 2.49.$

(d) Test $\delta = 0$, $\eta = 0$, Regression 17: $F = \dfrac{(1.55805 - 1.47451)/3}{1.47451/29} = 0.548 < F_{0.90}(3,29) = 2.28.$

In each test, the null hypothesis is accepted at the 10% significance level.

The power of these tests against the alternatives graphed in Figure 1 are all less than 0.20.

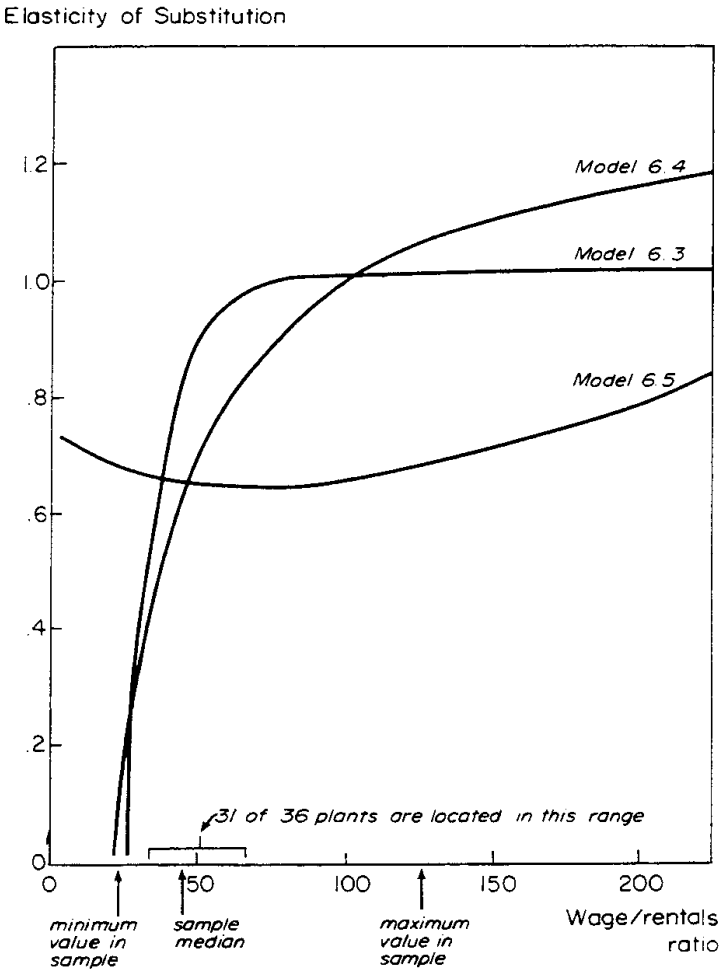

FIGURE 1. Estimates of the elasticity of substitution between capital and labor as a function of the relative price w/r.

economic interest in which the E.S. varies, say, from 0.6 to 1.1 over the sampled price ratios.[19]

Figure 1 graphs the estimates of the E.S. from Regressions 13–16 against the relative price ratio. We note that the non-negativity constraint on the E.S. is satisfied over the sample in each case. The interval [31,65] contains 31 of the 36 sampled values of w/r, and on this interval all three of the models (6.3) to (6.5) yield values of the estimated E.S. in the range [0.40,0.95]. Specifically, (6.5) gives an essentially constant E.S.

[19]This point is related to the observation that a set of points on a "true" isoquant can often be fitted closely by curves of significantly different curvatures (and thus significantly different E.S. values), and hence that a "second derivative" measure such as the E.S. is difficult to estimate accurately.

of 0.65 on this interval; (6.4) gives an elasticity which increases monotonically in w/r from 0.40 to 0.85; and (6.3) gives an elasticity which increases monotonically in w/r from 0.45 to 0.95. Models (6.4) and (6.5) exhibit an increasing E.S. at high values of w/r, particularly for observation 6, the Eddystone plant, with $w/r = 123$. It is worth noting that this is the largest plant in the sample and has an automated control system. It may, by virtue of its innovative technology, allow a degree of substitutability between capital and labor that could not be achieved by direct extrapolation of the common technology of the remaining plants. To the extent that this is a real phenomenon and "unconventional technologies" can be activated in response to extreme relative prices, it is reasonable to expect a non-constant E.S. of the form in (6.5).[20]

Regressions 13–17 again suggest increasing capital intensity (relative labor-saving) with increasing plant capacity. The next hypothesis provides a formal test of this tendency.

Hypothesis III – The ex ante production function is homothetic.

The hypothesis of homotheticity requires that cost-minimizing factor input proportions be determined by input price ratios, and be independent of scale. Since Hypotheses I and II have been accepted, the exact relation (5.10) holds, with

$$\ln(K/L) = \sigma \ln(w/r) + \ln D(R),$$

where D is some positive function independent of m. Then,

$$\ln(K^*/\bar{L}) = \sigma \ln(w/r) + \ln D(R) - \ln \lambda(l) + \ln(\epsilon_K/\epsilon_L).$$

Taking first-order Taylor's expansions of $D(R)$ and $\lambda(l)$ in log R and l, respectively, the results in Table 5 allow a test of the hypothesis. We conclude that the hypothesis of homotheticity is strongly rejected. We note further that the omission of the plant capacity variable in Regressions 18 and 19 has little effect on the estimates of the E.S. An auxiliary test for a unitary E.S. in Regression 20 is rejected.

In the absence of homotheticity, returns to scale cannot be defined independently of input proportions. However, from estimates of the elasticity of input with respect to plant capacity one can estimate an index μ of the degree of returns to scale at a point, equal to the

[20]It should be emphasized that these conclusions are based on the assumption that labor input is homogeneous in quality over the sample. A more plausible alternative is that the mix of technicians and supervisory personnel is higher in more automated plants, so that the observed wage overestimates the wage per standard worker facing the firm. The result is an upward bias in the E.S. estimate at high w/r values.

TABLE 5
Test of homotheticity (relative capital–fuel demand equation); dependent variable: $\log(K^*/\bar{L})$.

	Independent variables				Error		
Regression	Const. a_0	$\log R$ a_1	l a_2	$\log(w/r)$ a_3	Sum of squares	DF	Mult. corr. coeff.
18	−3.634 (0.754)			0.738 (0.195)	4.67939	34	0.296
19	−3.669 (0.783)		0.108 (0.527)	0.728 (0.204)	4.67343	33	0.297
20	−2.817 (0.471)	0.459 (0.057)	−1.308 (0.356)	0.731 (0.119)	1.55805	32	0.765

(a) Test $a_1 = 0$, Regression 20: $t = 8.00 > t_{0.995}(32) = 2.74$.
The null hypothesis is rejected at the 1% significance level.

(b) Test $a_3 = 1$, Regression 20: $t = 2.252 > t_{0.975}(32) = 2.04$.
This hypothesis is rejected at the 5% significance level.

elasticity of capacity with respect to total cost, prices held constant. I.e.,

$$\frac{1}{\mu} = \frac{\partial \log C}{\partial \log R} = \left(\frac{wL}{C}\right)\frac{\partial \log L}{\partial \log R} + \left(\frac{rK}{C}\right)\frac{\partial \log K}{\partial \log R} + \left(\frac{mM}{C}\right)\frac{\partial \log M}{\partial \log R},$$

where the derivatives are evaluated with l,r,w,m constant. For the typical values $wL/C = 0.05$, $rK/C = 0.45$, $mM/C = 0.50$, and crude estimates $\partial \log L/\partial \log R = 0.327$, $\partial \log K/\partial \log R = 0.784$, $\partial \log M/\partial \log R = 0.772$ (obtained from Regressions 8, 12, and 4, respectively) we have the estimate $\mu = 1.32$, suggesting substantial increasing returns to scale. Estimates of this effect could also be obtained by direct non-linear estimation of the input demand functions.

(5) The remainder of our empirical analysis is devoted to a comparison of capital–labor E.S. estimates from equations (1.1), (1.2) and their generalizations under the assumptions that the E.S. is constant (Hypothesis II holds), that production is heterothetic (Hypothesis III fails), and that the substitutability of fuel for capital and labor is low, but not necessarily zero (Hypothesis I is ambiguous). Corollary 1.2 implies that the assumption of a constant A.U.E.S. between capital and labor is inconsistent with zero substitutability of fuel for these factors. Hence, we confine our attention to the shadow E.S. For this definition, the

assumptions above imply that the exact cost function has the form specified in Theorem 2,

$$c = \{(rA(Y,m/c))^{1-\sigma} + (wB(Y,m/c))^{1-\sigma}\}^{1/(1-\sigma)}, \quad 0 < \sigma \leqq +\infty, \quad \sigma \neq 1. \tag{6.13}$$

A further restriction on this functional form from Corollary 2.2 holds if the hypothesis that the cost function has the separable form $c = H(Y,m,[(rA(Y))^{1-\sigma} + (wB(Y))^{1-\sigma}]^{1/(1-\sigma)})$ is accepted. An implication of this hypothesis is that $\log(K/L)$ in (5.7) be independent of m/c, or alternately that $\log(V/L)$ in (5.6) be independent of m.

The following regression provides a test of this hypothesis:

$$\begin{aligned}\log(K^*/\bar{L}) = {} & 2.653 + 0.527 \log R - 1.236l + 0.737 \log(w/r) \\ & (0.643)\ (0.187) \qquad\quad (0.407)\ (0.122) \\ & + 0.088 \log(m/c); \\ & (0.231)\end{aligned}$$

(Regression 21, error sum of squares = 1.55079, multiple correlation coefficient = 0.767.)

A t-test of the hypothesis that the coefficient of $\log(m/c)$ is zero is accepted at the 10% significance level. Then the functions $A(Y,m/c)$ and $B(Y,m/c)$ in (6.13) can be assumed to have the form $A = A_1(Y)A_2(Y,m/c)$ and $B = B_1(Y)A_2(Y,m/c)$, implying

$$c = A_2(Y,m/c)[(A_1(Y)r)^{1-\sigma} + (B_1(Y)w)^{1-\sigma}]^{1/(1-\sigma)}.$$

Solving for c gives a general functional form

$$c = H(Y,m,[(A_1(Y)r)^{1-\sigma} + (B_1(Y)w)^{1-\sigma}]^{1/(1-\sigma)}). \tag{6.14}$$

Then the following cost relations hold:

$$\log(V/L) = (1-\sigma)\log B_1(Y) + \sigma \log w, \tag{6.15}$$

$$\log(K/L) = (1-\sigma)\log(A_1(Y)/B_1(Y)) + \sigma \log(w/r). \tag{6.16}$$

In terms of observable variables, (6.15) gives

$$\begin{aligned}\log(\bar{V}/\bar{L}) = {} & (1-\sigma)\log \bar{c} - (1-\sigma)\log B_1(R) + \sigma \log w - \log \lambda(l) \\ & + \log b_V - (1-\sigma)\log b_C + \log \eta_V \\ & - (1-\sigma)\log \eta_C - \log \epsilon_L;\end{aligned} \tag{6.17}$$

b_V and b_C are defined above. Equation (6.16) leads to the functional form (6.8) previously analyzed in Hypotheses II and III. Defining an average cost of output (approximately equal to output price for the

27[a] (OLSQ)	$\log(\overline{V}/\overline{P}K^*)$	−0.004 (0.088)	−0.045 (0.008)	0.136 (0.042)	−0.994 (0.019)		0.994 (0.019)	0.02145	32	0.992
27A[a,d] (INST)	$\log(\overline{V}/\overline{P}K^*)$	−1.228 (1.458)	0.013 (0.072)	0.257 (0.182)	−0.712 (0.335)		0.712 (0.335)	0.16314	32	0.941
28[a] (OLSQ)	$\log(\overline{V}/\overline{P}K^*)$	−0.151 (0.154)	−0.037 (0.011)	0.166 (0.049)	−0.986 (0.020)	−0.031 (0.027)	0.986 (0.020)	0.02056	31	0.993
28A[a,e] (INST)	$\log(\overline{V}/\overline{P}K^*)$	−0.903 (0.550)	0.005 (0.030)	0.297 (0.105)	−0.913 (0.070)	−0.149 (0.075)	0.913 (0.070)	0.03704	31	0.987
29 (OLSQ)	$\log(\overline{V}/\overline{P}K^*)$	−2.619 (0.929)	0.156 (0.058)	0.611 (0.355)			0.759 (0.400)	1.60535	32	0.421
30 (OLSQ)	$\log(\overline{V}/\overline{P}K^*)$	−0.114 (0.117)	−0.047 (0.008)	0.127 (0.042)	−1.003 (0.020)		0.935 (0.046)	0.02016	31	0.993
31 (OLSQ)	$\log(\overline{V}/K^*)$	−0.123 (0.104)	−0.046 (0.007)	0.129 (0.040)			0.935 (0.045)	0.02018	32	0.937

[a]Price coefficients are constrained to sum to zero.
[b]The instruments are Const., log R, l, log w, log $\tilde{P}$ (output price).
[c]The instruments are Const., log R, l, log w, log $\tilde{P}$, log m.
[d]The instruments are Const., log R, l, log r, log $\tilde{P}$.
[e]The instruments are Const., log R, l, log r, log $\tilde{P}$, log m.

TABLE 6
Estimates of the capital–labor E.S. from equation (1.2) and its generalization.

Regression	Dependent variable	Independent variables							Error		Mult. corr. coeff.
		Const.	log R	l	log w	log $\bar{P}$	log(m/P)	log r	Sum of squares	DF	
22[a] (OLSQ)	$\log(\overline{V}/\overline{PL})$	−3.989 (0.228)	0.474 (0.056)	−1.029 (0.330)	0.702 (0.117)	−0.702 (0.117)			1.20945	32	0.874
22A[a,b] (INST)	$\log(\overline{V}/\overline{PL})$	−3.888 (0.250)	0.451 (0.061)	−1.146 (0.352)	0.814 (0.159)	−0.814 (0.159)			1.24470	32	0.871
23[a] (OLSQ)	$\log(\overline{V}/\overline{PL})$	−4.183 (0.834)	0.488 (0.079)	−0.976 (0.402)	0.691 (0.127)	−0.691 (0.127)	−0.050 (0.207)		1.20718	31	0.874
23A[a,c] (INST)	$\log(\overline{V}/\overline{PL})$	−2.375 (2.062)	0.346 (0.168)	−1.569 (0.748)	0.905 (0.259)	−0.905 (0.259)	0.389 (0.503)		1.42188	31	0.852
24 (OLSQ)	$\log(\overline{V}/\overline{PL})$	−5.204 (0.299)	0.616 (0.062)	−0.567 (0.386)	0.485 (0.137)				1.84316	32	0.808
25 (OLSQ)	$\log(\overline{V}/\overline{PL})$	−3.991 (0.389)	0.474 (0.062)	−1.029 (0.338)	0.702 (0.125)	−0.701 (0.174)			1.20945	31	0.874
26 (OLSQ)	$\log(\overline{V}/\overline{L})$	−3.473 (0.253)	0.414 (0.053)	−1.226 (0.327)	0.794 (0.116)				1.32474	32	0.779

profit-regulated utility) $\bar{P} = \bar{c}/Y = \bar{c}/lR$, (6.17) can be written

$$\log(\bar{V}/\bar{P}\bar{L}) = (1-\sigma)\log[R/B_1(R)] + \sigma\log(w/\bar{P}) - \log[\lambda(l)/l^{1-\sigma}] + b + \epsilon,$$

where $b = \log b_V - (1-\sigma)\log b_C$ and $\epsilon = \log \eta_V - (1-\sigma)\log \eta_C - \log \epsilon_L$. We now assume that b is constant and ϵ is independently identically normally distributed over the sample. In fact, $b = \log[(\pi_K + \lambda(l)\pi_L)/(\pi_K + \lambda(l)\pi_L + \mu(l)\pi_M)]$ varies from -0.69 to -0.37 in our sample when the true parameters are $\sigma = 0.75$, $\log\lambda(l) = 1.04l$, $\log\mu(l) = 1.54l$, and the assumption of a constant b is a poor approximation. Since the variable b has a low correlation with $\log w$, and since the variable l with which b is highly correlated has in turn a low correlation with w, we expect this misspecification to bias the estimates of $\lambda(l)$ in (6.18), but to have little effect on the estimate of σ. It should also be noted that the stochastic specification for ϵ is inconsistent with the specification in previous tests. Because output price $\bar{P}$ measured by average cost is an endogenous variable, ordinary least squares (OLSQ) estimates based on (6.18) will contain some bias. Therefore, we have also fitted the equations in which this appears as a right-hand variable using instrumental variables (INST), with firm output price ($\tilde{P}$) used as an instrument for plant average cost ($\bar{P}$). Table 6 gives estimates for (6.18), and for variants which replace labor with capital or modify right-hand side variables. The terms $\log(R/B_1(R))$ and $\log(\lambda(l)l^{1-\sigma})$ are replaced by first-order Taylor's expansions, yielding terms in $\log R$ and l.

The estimates in this table of the capital–labor E.S. are in general agreement with values obtained in earlier estimates. Regressions 22A and 27A correspond to (6.18) for labor and capital, respectively, fitted by the preferred instrumental variables method. They yield respective E.S. estimates of 0.814 and 0.712, which compare with the estimate of 0.731 from Regression 20. The OLSQ versions of Regressions 22 and 27 differ somewhat from the INST estimates, suggesting the presence of some bias due to the endogenous variable $\bar{P}$. Regressions 23 (23A) and 28 (28A) provide additional tests of the separability of fuel in the cost function. In Regressions 23, 23A, and 28, the hypothesis of separability is accepted at the 10% confidence level, whereas in Regression 28A, it is accepted at the 5% confidence level, but not at the 10% confidence level. Regressions 25 and 30 omit the restriction of zero degree homogeneity in prices. An F test for price homogeneity accepts this hypothesis at the 10% level in Regression 25 (in comparison with Regression 22), and rejects it at the 10% level, but accepts it at the 1% level, in Regression 30

(in comparison with Regression 27). We conclude that there is no convincing evidence against price homogeneity. Regressions 24 and 26 correspond to forms often fitted in the literature when explicit price information is unavailable and value-added is available in either deflated (Regression 24) or undeflated (Regression 25) terms. The results suggest some downward bias in the E.S. estimate (compared with Regression 22 or 22A) when value-added is deflated, but wage is not deflated; and relatively little bias when neither wage or value-added are deflated. With capital in place of labor, Regressions 29 and 31 give analogous results, with deflated value-added giving a lower E.S. estimate than undeflated value-added. In this case, the estimates from Regressions 29 and 31 both lie above the E.S. estimate in Regression 27A.

We draw from Table 6 the overall conclusions that the use of value-added per unit of factor input as an estimating equation for the E.S. yields results comparable to those from the factor shares Regression 20, and that failure to carry out a consistent deflation of value-added and input price introduces empirically significant biases in the E.S. estimates.

This pilot study of electricity generation has yielded the summary conclusions that as a first approximation fuel is non-substitutable for capital and labor, that production is heterothetic, with capital intensity increasing with plant scale, and that the capital–labor E.S. is constant, with a value of approximately 0.75. More generally, this analysis suggests that econometric analysis of substitutability can be carried out under much weaker maintained hypotheses than have traditionally been imposed.

7. Appendix

Data for steam-electric generating plants, one year after initial operation[1]

No.	Plant[2]	Company	Location	Year of observation	Type of construction[3]	Plant capacity (R)[4]
1	Ocotillo	Arizona Public Service Co.	Tempe, Ariz.	1961	FO	2.010
2	Norwalk Habor	Connecticut Light and Power Co.	Norwalk, Conn.	1961	C	1.384
3	E.D. Edwards	Central Illinois Light Co.	Bartonville, Ill.	1961	C	1.095
4	Willow Glen	Gulf States Utilities Co.	St. Gabriel, La.	1961	FO	1.419
5	J.H. Warden	Upper Peninsula Power Co.	L'Anse, Mich.	1961	C	0.175
6	Eddystone (critical heat plant, automated)	Philadelphia Electric Co.	Eddystone, Pa.	1961	C	6.360
7	H.B. Robinson	Carolina Power & Light Co.	Hartsville, S.C.	1961	FO	1.621
8	Newman	El Paso Electric Co.	El Paso, Tex.	1961	OB	0.756
9	Nichols	Southwestern Public Service Co.	Amarillo, Tex.	1961	OB	0.929
10	N. Dewey	Wisconsin Power & Light Co.	Cassville, Wis.	1961	C	0.876
11	Mandalay Beach	Southern California Edison Co.	Oxnard, Cal.	1960	FO	3.767

12	P.L. Bartow	Florida Power Corp.	St. Petersburg, Fla.	1960	C	1.095
13	Ft. Meyers	Florida Power & Light Co.	Ft. Meyers, Fla.	1960	FO	1.238
14	Lewis & Clark	Montana-Dakota Utilities Co.	Sidney, Mont.	1960	OB	0.447
15	Portland	Metropolitan Edison Co.	Portland, Pa.	1960	OB	1.349
16	Clinch River	Appalachian Power Co.	Lebanen, Va.	1960	OB	3.942
17	D. Johnson	Pacific Power & Light Co.	Glenrock, Wyo.	1960	OB	0.929
18	Huntington Beach	Southern California Edison Co.	Huntington Beach, Cal.	1959	FO	3.811
19	Bridgeport Harbor	United Illuminating Co.	Bridgeport, Conn.	1959	OB	0.718
20	Indian River	Delaware Power & Light Co.	Millsboro, Del.	1959	C	1.489
21	W.F. Wyman	Central Maine Power Co.	Cousins Island, Me.	1959	C	0.841
22	C. Boswell	Minnesota Power &, Light Co.	Grand Rapids, Mich.	1959	FO	0.613
32	Montrose	Kansas City Power & Light Co.	Montrose, Mo.	1959	FO	1.533
24	S. McMeekin	South Carolina Electric & Gas Co.	near Irmo, S.C.	1959	OB	2.409
25	Bates	Central Power & Light Co.	Mission, Tex.	1959	OB	0.657
26	W.A. Parrish	Houston Lighting & Power Co.	Richmond, Tex.	1959	FO	2.891
27	Cameo	Public Service Co. of Colorado	Cameo, Colo.	1958	OB	0.196

No.	Plant[2]	Company	Location	Year of observation	Type of construction	Plant capacity $(R)^4$
28	Cherokee	Public Service Co. of Colorado	Denver, Colo.	1958	OB	0.937
29	D. Mitchell	Northern Indiana Public Service Co.	Gary, Ind.	1958	C	1.139
30	E.M. Brown	Kentucky Utilities Co.	Burgin, Ky.	1958	C	0.920
31	Michoud	New Orleans Public Service, Inc.	New Orleans, La.	1958	FO	1.051
32	Gulf Coast	Mississippi Power Co.	Harrison Co., Miss.	1958	C	0.710
33	Cunningham	Southwestern Public Service Co.	Hobbs, N.M.	1958	OB	0.657
34	E.F. Barrett	Long Island Lighting Co.	Hempstead, N.Y.	1958	FO	1.489
35	G.G. Allen	Duke Power Co.	Belmont, N.C.	1958	C	2.886
36	Yorktown	Virginia Electric & Power Co.	Yorktown, Va.	1958	OB	1.559

No.	Plant[2]	Output $(Y)^5$	Load factor $(l)^6$	Labor force $(\bar{L})^7$	Cost of plant $(K^*)^8$	Wage rate $(w)^9$	Price of capital			Value-added $(\bar{V})^{12}$
							$(r_0)^{10}$	$(r_1)^{11}$	$(r_2)^{11}$	
1	Ocotillo	1.559	0.776	42	24.2	6.55	0.088	0.050	0.078	2.41
2	Norwalk Harbor	1.082	0.782	58	29.0	5.31	0.099	0.060	0.088	3.19
3	E.D. Edwards	0.811	0.741	39	27.4	5.95	0.101	0.054	0.091	3.00
4	Willow Glen	0.809	0.570	29	21.7	4.28	0.089	0.053	0.080	2.06
5	J.H. Warden	0.094	0.538	18	5.0	5.61	0.105	0.070	0.099	0.63
6	Eddystone	3.726	0.586	58[18]	149.7[18]	14.03[18]	0.114	0.064	0.099	17.82
7	H.B. Robinson	1.084	0.669	44	26.8	3.59	0.094	0.060	0.085	2.68
8	Newman	0.456	0.603	21	9.6	4.33	0.117	0.075	0.100	1.21

9	Nichols	0.392	0.422	33	13.6	2.58	0.107	0.063	0.089	1.54
10	N. Dewey	0.635	0.725	47	15.9	4.36	0.105	0.063	0.099	1.88
11	Mandalay Beach	3.061	0.812	55	47.4	6.07	0.096	0.057	0.084	5.00
12	P.L. Bartow	0.755	0.689	54	21.4	4.57	0.098	0.064	0.091	2.35
13	Ft. Meyers	0.779	0.629	48	15.3	4.22	0.120	0.073	0.107	2.06
14	Lewis & Clark	0.182	0.408	27	10.4	3.59	0.094	0.047	0.087	1.08
15	Portland	0.894	0.663	59	28.7	4.10	0.107	0.064	0.092	3.31
16	Clinch River	3.562	0.904	129	56.8	3.84	0.111	0.065	0.101	6.78
17	D. Johnson	0.667	0.718	54	18.7	4.94	0.090	0.059	0.078	1.95
18	Huntington Beach	2.812	0.738	51	52.3	5.53	0.086	0.052	0.079	4.77
19	Bridgeport Harbor	0.516	0.719	43	18.5	6.58	0.130	0.074	0.110	2.68
20	Indian River	0.768	0.515	64	27.6	3.86	0.107	0.060	0.099	3.20
21	W.F. Wyman	0.622	0.740	46	18.4	5.46	0.088	0.054	0.077	1.86
22	C. Boswell	0.406	0.661	35	12.3	4.74	0.091	0.061	0.082	1.29
23	Montrose	1.089	0.710	61	23.6	5.80	0.099	0.052	0.081	2.69
24	S. McMeekin	1.691	0.702	58	31.4	3.86	0.097	0.064	0.093	3.25
25	Bates	0.331	0.503	33	8.0	3.51	0.107	0.064	0.094	0.97
26	W.A. Parrish	2.149	0.743	58	26.0	3.69	0.106	0.068	0.098	2.98
27	Cameo	0.109	0.553	17	5.6	8.35	0.095	0.054	0.083	0.68
28	Cherokee	0.715	0.762	36	17.5	6.47	0.093	0.054	0.083	1.86
29	D. Mitchell	0.850	0.746	83	21.7	4.58	0.099	0.061	0.087	2.52
30	E.M. Brown	0.600	0.652	40	13.6	4.05	0.109	0.064	0.097	1.65
31	Michoud	0.690	0.656	37	11.8	5.03	0.107	0.056	0.097	1.45
32	Gulf Coast	0.453	0.639	25	11.3	4.36	0.111	0.067	0.103	1.37
33	Cunningham	0.217	0.330	27	8.9	3.33	0.095	0.059	0.084	0.93
34	E.F. Barrett	1.037	0.696	61	25.7	4.39	0.099	0.057	0.089	2.82
35	G.G. Allen	2.192	0.760	72	35.7	5.32	0.108	0.064	0.104	4.26
36	Yorktown	1.336	0.857	73	24.3	3.19	0.102	0.063	0.090	2.71

No.	Plant[2]	Type(s) of fuel used[13]	Amount of fuel used[14] ($\bar{M}$)	Fuel price[15] (m)	Output price[16] ($\bar{P}$)	Unit cost[17] ($\bar{P}_0$)
1	Ocotillo	G	15.50	0.334	16.39	4.87
2	Norwalk Harbor	C	10.28	0.339	22.36	6.17
3	E.D. Edwards	C	7.96	0.231	18.22	5.97
4	Willow Glen	G	8.45	0.232	14.03	4.98
5	J.H. Warden	C	1.69	0.329	19.81	12.57
6	Eddystone	C	16.07	0.347	17.42	6.28
7	H.B. Robinson	C	10.64	0.276	13.66	5.19
8	Newman	G	4.64	0.306	17.58	5.77
9	Nichols	G	3.70	0.181	14.85	5.63
10	Nelson Dewey	C	6.17	0.278	22.29	5.66
11	Mandalay Beach	G,O	28.23	0.331	17.48	4.68
12	P.L. Bartow	G,O	7.83	0.326	22.88	6.48
13	Ft. Meyers	O	7.77	0.353	24.80	6.17
14	Lewis & Clark	L,G	2.53	0.214	27.53	8.91
15	Portland	C	8.43	0.342	17.89	6.92
16	Clinch River	C	31.96	0.172	12.08	3.45
17	Dave Johnson	C,G	7.31	0.121[19]	12.36	4.24
18	Huntington Beach	G,O	26.54	0.338	17.79	4.88
19	Bridgeport Harbor	C,O	5.09	0.363	21.79	8.78
20	Indian River	C	7.86	0.350	19.89	7.76
21	Walter F. Wyman	O	6.73	0.353	20.91	6.81
22	Clay Boswell	C	4.09	0.416	19.97	7.37
23	Montrose	C	11.08	0.208	22.02	4.58
24	Silas McMeekin	C,G	15.43	0.300	13.93	4.66
25	Bates	G	3.93	0.173	17.69	4.99
26	W.A. Parrish	G	22.10	0.164	12.68	3.07
27	Cameo	C	1.40	0.229	18.12	9.17
28	Cherokee	G,C	7.67	0.210	18.12	4.85
29	Dean Mitchell	C,G,Ck	8.35	0.297	19.29	5.88
30	E.M. Brown	C	6.11	0.233	20.77	5.12
31	Michoud	G	7.42	0.122	21.08	3.41
32	Gulf Coast	G	4.68	0.234	17.34	5.43
33	Cunningham	G	2.54	0.192	14.79	6.56
34	E.F. Barrett	C,G	10.06	0.395	26.82	6.56
35	G.G. Allen	C	20.12	0.307	12.51	4.76
36	Yorktown	C,Ck	12.82	0.272	19.02	4.64

[1]Sources for the data were: (1) Federal Power Commission, *Statistics of Electric Utilities in the United States, 1958–61, Classes A and B Privately Owned Companies* (Washington, D.C.: U.S. Government Printing Office, 1959–62), and (2) Federal Power Commission, *Steam-Electric Plant Construction Cost and Annual Production Expenses, Annual Supplements, 1957–61* (Washington, D.C.: U.S. Government Printing Office, 1958–62). For most of the variables, the figures reported here are rounded from those actually used in the regressions.

[2]All plants satisfying the following criteria were included in the sample: 1) Initial operation 1957–1960. 2) No additions to plant capacity the following year. 3) Privately owned. 4) All data required for our purposes supplied.

[3]C stands for conventional, FO for full outdoor, and OB for outdoor boilers (or other semi-outdoor types of construction).

[4]Net continuous plant capability when not limited by condenser water, in millions of megawatt-hours. From (2).

[5]Net generation for the year in millions of megawatt-hours. From (2).

[6]The ratio of output to capacity. Sometimes referred to as power factor or plant factor.

[7]Average number of employees during the year. From (2).

[8]Net value of plant after depreciation, in millions of dollars. Depreciation was calculated at 2.5% per annum, the standard rate for this type of equipment, from the month the equipment went into service to the end of the year of observation. From (2).

[9]In thousands of dollars per year. Found by adding the plant's operation labor, supervision, and engineering costs [from (2)] to imputed plant maintenance labor costs, and dividing the sum by the labor force ($\bar{L}$). Plant maintenance labor costs were imputed by finding the ratio of maintenance labor costs to total maintenance costs for the company as a whole [from (1)], and then multiplying this ratio by the plant's total maintenance costs [from (2)]. Maintenance labor generally accounted for roughly 10% of the total wage bill.

[10]Found by dividing $(\bar{V} - w\bar{L} \times 10^{-3})$ by K^*. Includes profit, interest, and depreciation charges at the company rate and non-labor maintenance charges at the plant rate.

[11]The alternate price of capital series were constructed as follows: r_1 is the company rate on profit and interest charges [from (1)]; $r_2 = r_1$ plus the company depreciation and amortization rate [from (1)]; $r_3 = r_1 + 0.025$ (assumed plant depreciation rate); $r_4 = r_3 + (r_0 - r_2) = r_3$ plus maintenance charges at the plant rate.

[12]In millions of dollars. Found by adding the plant's production expenses exclusive of fuel [from (2)] to imputed plant "economic" capital costs. "Economic" capital service costs are here defined as depreciation and amortization plus net operating income (profit plus interest charges). These were found for the company's electric utility plant as a whole [from (1)], and imputed to the plant according to the ratio of the plant's net value of plant (K^*) to the company's net value of plant [from (1)].

[13]Listed in order of importance. C stands for coal, Ck for coke, G for gas, L for lignite, and O for oil. From (2).

[14]In trillions (10^{12}) of B.t.u.'s From (2).

[15]In dollars per million B.t.u.'s From (2).

[16]In dollars per megawatt-hour. Total revenues from sales of electricity divided by total megawatt-hour sales for the firm. From (1). When this measure is used, output price is assumed to be the same for each plant within a firm.

[17]In dollars per megawatt-hour. Total cost $(\bar{V} + m\bar{M})$ divided by output $(Y)\cdot\bar{P}$ is greater than $\bar{P}_0$ because the revenue to the firm is partly used to cover tax, transmission, distribution, administration, and other overhead expenses not allocated to the individual generating plants.

[18]The high w and high $K^*/\bar{L}$ ratio are undoubtedly due to the fact that this plant has an automated control system.

[19]Unusually low because the company owns its own coal mine.