Chapter IV.2

MEASUREMENT OF THE ELASTICITY OF FACTOR SUBSTITUTION AND BIAS OF TECHNICAL CHANGE*

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1. Introduction

Empirical studies of the relationships between disembodied technical change and aggregate economic growth have frequently measured directly the rate of technical progress. However, measurements of the bias of technical change and the elasticity of substitution between capital and labor have either been based on cross-section data or have employed untested restrictions on the nature of production possibilities. This reflects a non-identifiability of the elasticity and bias in the absence of a priori hypotheses on the structure of technical change.[1] More precisely, given the time series of all observable market phenomena for a single economy with a classical aggregate production function, one finds that the same time series *could* have been generated by an alternative production function having an arbitrary elasticity or bias at the observed points (except that the bias is measurable when the capital/labor ratio is constant). Further, this conclusion is not altered by

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[1]One statement of this result is given in Sato (1970).

the a priori information that technical change is non-retrogressive (except that now the elasticity is measurable and the bias zero when there is no technical progress). This result is shown in Section 5. Notation, equations relating economic growth to the form of the production function, and a representation of the production function in terms of the elasticity are presented in Sections 2–4, respectively.

The identifiability of the elasticity and bias will depend on what is in fact true about the economy and on what the economist assumes a priori to be true (i.e., his maintained hypothesis, or model). One possible outcome of an econometric experiment is an inconsistency of the maintained hypothesis with the observations. Alternately, the maintained hypothesis may be consistent with the observations, whether it is true or not. In the latter case, one may find that economic variables such as the elasticity and bias are exactly identified, identified up to a range of of indeterminacy, or not identified at all (as in the result in Section 5). When every false maintained hypothesis is inconsistent with observations, every case of identification of the economic variables yields the same values. Alternately, in the usual case where a range of false hypotheses are consistent with a given set of observations, a range of values of the economic variables can result from the various "conditionally" identified models. Then in a broader sense it is impossible to identify the true values among these alternatives. The remaining results of the paper deal with identification under commonly used maintained hypotheses.

The assumption that technical progress is factor augmenting is frequently employed in empirical work. Section 6 gives necessary conditions for observations to be consistent with the factor augmentation hypothesis, and gives limits on the values of the elasticity and bias required for consistency. These limits give outer bounds on the range of indeterminacy. Sufficient conditions for the factor augmentation hypothesis to be consistent with observations are given in Section 7. These conditions apply whether or not the hypothesis is in fact true. In Section 8, inner bounds on the range of indeterminacy of the elasticity and bias are established in the case that the factor augmentation hypothesis is true.

The last two sections of the paper explore cases where exact identification is possible. First, if under the maintained hypothesis of factor augmentation a single factor is in fact being augmented, then there is identification at any effective capital/labor ratio which can be shown to be attained more than once. Second, the elasticity and bias are identified if the economist makes a maintained hypothesis, which is in fact true, that technical change is factor augmenting with augmentation coefficients which are known up to a finite number of parameters, provided the data show sufficient variation.[2]

2. Notation

It is assumed that the economy under consideration has production possibilities which can be described by an aggregate production function with constant returns, $y = f(k,t)$ where y is output per worker; k, capital per worker; and t, time. The function is defined on the region $0 \leqq k < +\infty$, $0 \leqq t \leqq t_1 < +\infty$.

The function f is said to be a *classical* production function if it satisfies the following conditions[3] (subscripts denoting partial derivations):

(i) (continuity) $f(k,t)$ is twice continuously differentiable in k and once continuously differentiable in t;
(ii) (positive marginal products) $r = f_k > 0$, $w = f - kf_k > 0$;
(iii) (diminishing marginal rate of substitution) $f_{kk} < 0$.

A classical production function is said to be *neoclassical* on domain R in (k,t) space if, on R, it further satisfies:

(iv) (non-retrogression) $f_t \geqq 0$.

A classical production function is called *strictly neoclassical* on R if it satisfies:

(v) (progression) $f_t > 0$.

Denoting the ratio of the marginal product of labor to that of capital by $p(k,t) = (f - kf_k)/f_k$, we can define the elasticity of substitution $\sigma(k,t)$ and the bias of technical change $D(k,t)$ in terms of its partial derivatives,

$$\frac{1}{\sigma} = \frac{\partial \ln p}{\partial \ln k} = \frac{-kff_{kk}}{f_k(f - kf_k)}, \tag{1}$$

$$D = \frac{-\partial \ln p}{\partial t} = \frac{ff_{kt} - f_kf_t}{f_k(f - kf_k)}. \tag{2}$$

[2]This investigation was motivated by a result of Nerlove (1967).
[3]f is continuous for $0 \leqq k < +\infty$, $0 \leqq t \leqq t_1$, while all other properties are assumed to hold in the region $0 < k < +\infty$, $0 \leqq t \leqq t_1$.

We also employ the following notation:

$$T(k,t) = f_t/f, \tag{3}$$

the rate of technical progress;

$$\pi = kf_k/f, \tag{4}$$

the elasticity of output with respect to capital (equal to the share of capital under competition);

$$s = (f - kf_k)/kf_k = p/k, \tag{5}$$

the ratio of the elasticities of output with respect to labor and with respect to capital (equal to the relative share of labor under competition);

$$v = y/k, \tag{6}$$

the average product of capital.

The time derivative of a function $x(t)$ is denoted by $\dot{x}$, and its growth rate by $\mathring{x} = \dot{x}/x$.

Technical change is said to be factor augmenting if its only impact on the production function is to change the efficiency of inputs. Then, there are factor augmentation coefficients $A(t)$ and $B(t)$ giving the number of efficiency units contained in each measured unit of capital and labor, respectively, and the production function can be written $y = f(k,t) = B(t)g(kA(t)/B(t))$. Factor augmenting technical change is termed capital improving if $\dot{A} > 0$, $\dot{B} = 0$; labor improving if $\dot{A} = 0$, $\dot{B} > 0$, and factor improving if $\dot{A} > 0$, $\dot{B} > 0$.

Letting $x = kA(t)/B(t)$ denote the capital/labor ratio measured in "efficiency" units, formulae (1) and (2) for the elasticity of substitution and the bias of technical change become

$$\frac{1}{\sigma} = \frac{-xgg_{xx}}{g_x(g - xg_x)}, \tag{1a}$$

$$D = \left(1 - \frac{1}{\sigma}\right)(\mathring{A} - \mathring{B}). \tag{2a}$$

The elasticity of output with respect to capital becomes a function solely of the efficiency capital/labor ratio,

$$\pi = xg_x(x)/g(x), \tag{4a}$$

$$s = g(x)/xg_x(x) - 1. \tag{5a}$$

3. Growth Equations

Differentiation of a classical production function yields expressions for the rates of growth of output per worker, the ratio of marginal products, and the marginal products of capital and labor in terms of the indices defined above,[4]

$$\mathring{y} = T + \pi\mathring{k} \quad \text{and} \quad \mathring{v} = T - (1-\pi)\mathring{k}, \tag{7}$$

$$\mathring{p} = \frac{1}{\sigma}\mathring{k} - D, \tag{8}$$

$$\mathring{r} = T + (1-\pi)D - \frac{1-\pi}{\sigma}\mathring{k} = T - (1-\pi)\mathring{p} = \mathring{v} - (1-\pi)\mathring{s}, \tag{9}$$

$$\mathring{w} = T - \pi D + \frac{\pi}{\sigma}\mathring{k} = T + \pi\mathring{p} = \mathring{y} + \pi\mathring{s}. \tag{10}$$

The rate of technical progress can be expressed in terms of inputs and output [from (7)] or in terms of marginal products [from (9) and (10)],

$$\begin{aligned} T &= \mathring{y} - \pi\mathring{k} = \mathring{v} + (1-\pi)\mathring{k} \\ &= \mathring{w} - \pi\mathring{p} = \mathring{r} + (1-\pi)\mathring{p} \\ &= \pi\mathring{v} + (1-\pi)\mathring{y} = \pi\mathring{r} + (1-\pi)\mathring{w}. \end{aligned} \tag{11}$$

Progressive technical change ($T > 0$) implies [from (11)] the inequalities

$$-\frac{1}{1-\pi}\mathring{v} < \mathring{k} < \frac{1}{\pi}\mathring{y}, \tag{12}$$

$$-\frac{1}{1-\pi}\mathring{r} < \mathring{p} < \frac{1}{\pi}\mathring{w}, \tag{13}$$

$$-\left(\frac{1}{1-\pi}\mathring{r} + \frac{1}{\pi}\mathring{y}\right) < \mathring{s} < \left(\frac{1}{\pi}\mathring{w} + \frac{1}{1-\pi}\mathring{v}\right). \tag{14}$$

When technical change is factor augmenting, these formulae reduce to

$$\mathring{y} = \mathring{B} + \pi\mathring{x} \quad \text{and} \quad \mathring{v} = \mathring{A} - (1-\pi)\mathring{x}, \tag{7a}$$

$$\mathring{p} = \frac{1}{\sigma}\mathring{k} - \left(1 - \frac{1}{\sigma}\right)(\mathring{A} - \mathring{B}) = -\mathring{A} + \mathring{B} + \frac{1}{\sigma}\mathring{x}, \tag{8a}$$

$$\mathring{r} = \mathring{A} - \frac{1}{\sigma}(1-\pi)\mathring{x} = \mathring{A} + \frac{s}{\sigma}\mathring{B} - \frac{s}{\sigma}\mathring{y}, \tag{9a}$$

$$\mathring{w} = \mathring{B} + \frac{1}{\sigma}\pi\mathring{x} = \mathring{B} + \frac{1}{s\sigma}\mathring{A} - \frac{1}{s\sigma}\mathring{v}, \tag{10a}$$

$$T = \pi\mathring{A} + (1-\pi)\mathring{B}. \tag{11a}$$

[4]These formulae are derived in Diamond (1965).

From (7a) we obtain the inequality

$$-\frac{1}{1-\pi}\mathring{v} \leqq \mathring{x} \leqq \frac{1}{\pi}\mathring{y}. \tag{12a}$$

Letting $q = pA/B$ denote the ratio of the marginal products of labor and capital measured in efficiency units, (9a) and (10a) imply $\mathring{r} + (1-\pi)\mathring{q} = \mathring{A}$ and $\mathring{w} - \pi\mathring{q} = \mathring{B}$, and hence

$$-\frac{1}{1-\pi}\mathring{r} \leqq \mathring{q} \leqq \frac{1}{\pi}\mathring{w}. \tag{13a}$$

These equations also imply

$$-\frac{1}{1-\pi}\mathring{r} \leqq \frac{1}{\sigma}\mathring{x} \leqq \frac{1}{\pi}\mathring{w}. \tag{14a}$$

Finally, (8a) implies $\mathring{x} = \sigma\mathring{q}$ and

$$\mathring{s} = -\left(1 - \frac{1}{\sigma}\right)\mathring{x} = (1-\sigma)\mathring{q}. \tag{15}$$

TABLE 1
Sign patterns excluded under technical progress.

Pattern	Implied by
(1) $\mathring{k} \leqq 0$ and $\mathring{v} \leqq 0$	Equation (11)
(2) $\mathring{y} \leqq 0$ and $\mathring{k} \geqq 0$	Equation (11)
(3) $\mathring{y} \leqq 0$ and $\mathring{v} \leqq 0$	Equation (11)
(4) $\mathring{r} \leqq 0$ and $\mathring{p} \leqq 0$	Equation (11)
(5) $\mathring{r} \leqq 0$ and $\mathring{w} \leqq 0$	Equation (11)
(6) $\mathring{w} \leqq 0$ and $\mathring{p} \geqq 0$	Equation (11)
(7) $\mathring{s} > 0$, $\mathring{v} \leqq 0$, and $\mathring{r} \geqq 0$	Equation (9)
(8) $\mathring{s} < 0$, $\mathring{v} \geqq 0$, and $\mathring{r} \leqq 0$	Equation (9)
(9) $\mathring{s} > 0$, $\mathring{y} \geqq 0$, and $\mathring{w} \leqq 0$	Equation (10)
(10) $\mathring{s} < 0$, $\mathring{y} \leqq 0$, and $\mathring{w} \geqq 0$	Equation (10)
(11) $\mathring{w} < 0$, $\mathring{r} \geqq 0$, and $\mathring{p} \geqq 0$	Definition of p
(12) $\mathring{y} < 0$, $\mathring{k} \geqq 0$, and $\mathring{v} \geqq 0$	Definition of v
(13) $\mathring{s} < 0$, $\mathring{k} \leqq 0$, and $\mathring{p} \geqq 0$	Definition of s
(14) $\mathring{s} > 0$, $\mathring{k} \geqq 0$, and $\mathring{p} \leqq 0$	Definition of s
(15) $\mathring{s} < 0$, $\mathring{y} \leqq 0$, and $\mathring{r} \leqq 0$	Equation (14)
(16) $\mathring{s} > 0$, $\mathring{v} \leqq 0$, and $\mathring{w} \leqq 0$	Equation (14)

The following two patterns are excluded when technical change is factor augmenting:

(17) $\mathring{v} \leqq 0$ and $\mathring{w} \leqq 0$	Equation (10a)
(18) $\mathring{y} \leqq 0$ and $\mathring{r} \leqq 0$	Equation (9a)

One can obtain from the equations above a number of implications for the possible patterns of signs of observed variables when technical change is progressive. The table above indicates patterns which are excluded (and the equation from which this implication is obtained). One should note from this table that the marginal product of at least one factor must always be increasing; and that the simultaneous observation of a non-increasing marginal product for one factor and non-increasing average product for the other factor contradicts the hypothesis that technical change is factor augmenting (but not necessarily the hypothesis that it is progressive).

4. Production Relations

From the definition of the ratio of marginal products $p(k,t) = f/f_k - k$, or upon rearranging terms

$$f_k/f = [p(k,t) + k]^{-1}, \tag{16}$$

we see that the relationship between p and the derivatives of f can be reversed to yield a relationship between f and the integral of p:

$$\ln f(k,t) = \ln y^*(t) + \int_{k^*(t)}^{k} [p(z,t) + z]^{-1}\, \mathrm{d}z, \tag{17}$$

where $y^*(t) = f(k^*(t),t)$ and $k^*(t)$ is a selected capital/labor path. We shall select the actually observed capital/labor ratio for $k^*(t)$ in constructing production functions. Similarly the relationship in equation (1) between p and σ can be reversed,

$$\ln p(k,t) = \ln p^*(t) + \int_{k^*(t)}^{k} [\sigma(z,t)]^{-1}(\mathrm{d}z/z), \tag{18}$$

where $p^*(t) = p(k^*(t),t)$.

From these relations we see that any classical production function can be represented in the form of equations (17) and (18). Conversely, from any positive, regular[5] functions $k^*(t)$, $p^*(y)$, $y^*(t)$, $\sigma(k,t)$ one can construct a classical production function satisfying these two equations.[6]

[5]We will call the functions k^*, p^*, y^*, σ regular when $k^*(t)$ and $y^*(t)$ are twice-and $p^*(t)$ is once-continuously differentiable and $\mathring{s}^* = \mathring{p}^* - \mathring{k}^*$ reverses sign at most a finite number of times on $0 \leqq t \leqq t_1$, $\sigma(k,t)$ is continuous in (k,t) and $\partial\sigma/\mathrm{d}t$ exists and is continuous in k. From (18), $p(k,t)$ is then continuously differentiable.

[6]This representation of production functions is developed in McFadden (Chapter IV.1).

5. A Non-Identification Theorem

With the assumption of constant returns to scale. the time series of y, k, and p are sufficient statistics for the observable production information on inputs, outputs, and marginal products. Thus, any two production functions which are consistent with equations (7) and (8) at the points given by the time series are empirically indistinguishable. From equation (7) we see that the rate of technical progress is directly measurable. Equation (8) shows the intermingling of the effects of the bias and the elasticity. When technical retrogression is not excluded a priori, the elasticity of substitution can be chosen as an arbitrary (continuously differentiable) positive function and equation (8) then defines the bias for a classical production function from which the observed data could be derived. Equation (8) also shows the measurability of the bias when the capital/labor ratio is constant.

Adding the assumption that technical change is non-retrogressive, one can conclude that the absence of technical progress ($T = 0$) implies the absence of bias ($D = 0$). This result is obtained by differentiating (17) with respect to time. Noting that $\pi = k/(k + p)$, we see that

$$T(k,t) = T^*(t) + \int_{k^*(t)}^{k} \pi(z,t)(1 - \pi(z,t))D(z,t)(\mathrm{d}z/z), \tag{19}$$

where $T^*(t)$ is the rate of technical progress on the observed path. If one had $T^*(t) = 0$ and $D(k^*(t),t) > 0$, then one would obtain $T(k,t)$ negative for k slightly smaller than $k^*(t)$, contradicting non-regressivity. The argument for $D < 0$ is similar.

One can conclude further from equation (8) that the elasticity is known whenever the rate of technical progress is zero and $\mathring{k}$ is non-zero, and that the requirement that the elasticity be positive yields the condition

$$\mathring{k} \gtreqless 0 \quad \text{implies} \quad D \gtreqless -\mathring{p}. \tag{20}$$

Suppose next that observed data are derived from a neoclassical production function, and that the measured rate of technical change in equation (11) is positive. We shall show that this function can be perturbed to yield a neoclassical production function, consistent with the data, which has an arbitrary positive continuously differentiable elasticity at the observed points. From equation (8), we can then obtain the corollary that the data is consistent with an arbitrary bias of the form

$D = (1/\sigma)\mathring{k} - \mathring{p}$, where σ is any positive continuously differentiable function.

Theorem 1. Given positive, regular observed $y^*(t)$, $k^*(t)$, $p^*(t)$, $0 \leqq t \leqq t_1$, for output per worker, capital per worker, and the ratio of marginal products, respectively, generated by a neoclassical production function $y = f(k,t)$ which exhibits positive technical progress on the observed path, and given any positive continuously differentiable function $\sigma^*(t)$ on $0 \leqq t \leqq t_1$, there exists a neoclassical production function $\tilde{f}(k,t)$ which generates the observed series and has an elasticity of substitution $\sigma^*(t)$ along the path $(k^*(t),t)$.

Proof: The method of proof is to perturb the true production function in such a way that it has an arbitrary elasticity on the observed path, but is still consistent with the data and with the condition on non-retrogressive technical change. Let $y = f(k,t)$ denote the true production function, $\sigma(k^*(t),t)$ the true elasticity on the observed path, and $\sigma^*(t)$ the arbitrary elasticity which we wish to impose. Define a function $\theta^*(t) = [\sigma(k^*(t),t)^{-1} - \sigma^*(t)^{-1}]\pi^*(t)(1-\pi^*(t))y^*(t)/k^*(t)^2$, where $\pi^*(t)$ is the observed share of capital. From equation (1), it follows that $f_{kk}(k^*(t),t) + \theta^*(t) = -\pi^*(t)(1-\pi^*(t))y^*(t)/\sigma^*(t)k^*(t)^2 < 0$. Using the continuity properties of the observed path, the production function and its derivatives, and the condition that technical change is progressive on the observed path, one can choose $0 < \delta < 1$ such that $k^*(t) > \delta$, $\delta|\theta_t^*(t)| < 1$, and $T(k,t) > 0$ and $f_{kk}(k,t) + \theta^*(t) < 0$ for $|k - k^*(t)| \leqq \delta$ and $0 \leqq t \leqq t_1$. Then one can choose $0 < \epsilon < 1$ such that $f_t(k,t) > 2\epsilon$, $f_k(k,t) > 2\epsilon$, $f(k,t) > 2\epsilon$, and $f_{kk}(k,t) + \epsilon < 0$ for $|k - k^*(t)| \leqq \delta$ and $0 \leqq t \leqq t_1$.

For the scalars δ, ϵ and a non-negative scalar θ, define a function $\phi(x,\theta)$ for $x \geqq 0$ illustrated in Figure 1, by

$$\phi(x,\theta) = \frac{x^2}{2}\left(\theta - \frac{x}{3A}\right) \quad \text{for} \quad 0 \leqq x < \theta A,$$

$$= \theta^2 A\left(\frac{\theta A}{3} + \frac{1}{2}(x - \theta A)\right) - \frac{D}{3C}(x - \theta A)^3 \quad \text{for} \quad \theta A \leqq x < \frac{\theta(A+B)}{2},$$

$$= \frac{\theta E \delta^2}{32} + \frac{D}{3C}(x - \theta B)^3 \quad \text{for} \quad \frac{\theta(A+B)}{2} \leqq x \leqq \theta B,$$

$$
\begin{aligned}
&= \frac{\theta E \delta^2}{32} && \text{for} \quad \theta B \leqq x < \frac{\delta}{2}, \\
&= \frac{\theta E \delta^2}{32} - \frac{4}{3}\frac{E\theta}{\delta}\left(x - \frac{\delta}{2}\right)^3 && \text{for} \quad \frac{\delta}{2} \leqq x < \frac{5\delta}{8}, \\
&= \frac{\theta E \delta^2}{64} - \frac{\theta E \delta}{8}\left(x - \frac{3\delta}{4}\right) + \frac{4}{3}\frac{E\theta}{\delta}\left(x - \frac{3\delta}{4}\right)^3 && \text{for} \quad \frac{5\delta}{8} \leqq x < \frac{7\delta}{8}, \\
&= \frac{4E\theta}{3\delta}(\delta - x)^3 && \text{for} \quad \frac{7\delta}{8} \leqq x < \delta, \\
&= 0 && \text{for} \quad \delta \leqq x,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \epsilon\delta/8(1 + \theta)(1 + \epsilon + \theta), \\
B &= \delta/8(1 + \theta), \\
C &= B - A = \delta/8(1 + \epsilon + \theta), \\
D &= \epsilon/(1 + \theta), \\
E &= \epsilon\theta^2(3 + 3\theta + 4\epsilon)/24(1 + \theta)^2(1 + \epsilon + \theta)^2.
\end{aligned}
$$

One may verify by direct computation that $\phi(x,\theta)$ is continuous in (x,θ); $\phi_\theta(x,\theta)$, $\phi_x(x,\theta)$, $\phi_{xx}(x,\theta)$, and $\phi_{x\theta}(x,\theta)$ exist and are continuous in (x,θ); and that $\phi(0,\theta) = \phi_x(0,\theta) = 0$ and $\phi_{xx}(0,\theta) = \theta$. One may verify further that for $0 \leqq x \leqq \delta$, the following bounds hold

$$
\begin{aligned}
&-\epsilon \leqq -\epsilon\theta/(1 + \theta) \leqq \phi_{xx} \leqq \theta, \\
&|\phi_x| \leqq \frac{\epsilon}{16}\frac{\theta}{1 + \theta}, \\
&|\phi_\theta| \leqq \frac{\epsilon\delta}{32}\frac{\theta}{1 + \theta},
\end{aligned}
$$

and

$$|\varphi| \leqq \frac{\epsilon}{32}\frac{\theta}{1 + \theta}.$$

Define a perturbed production function

$$\tilde{f}(k,t) = f(k,t) + (\operatorname{sign} \theta^*(t)) \cdot \phi(|k - k^*(t)|, |\theta^*(t)|).$$

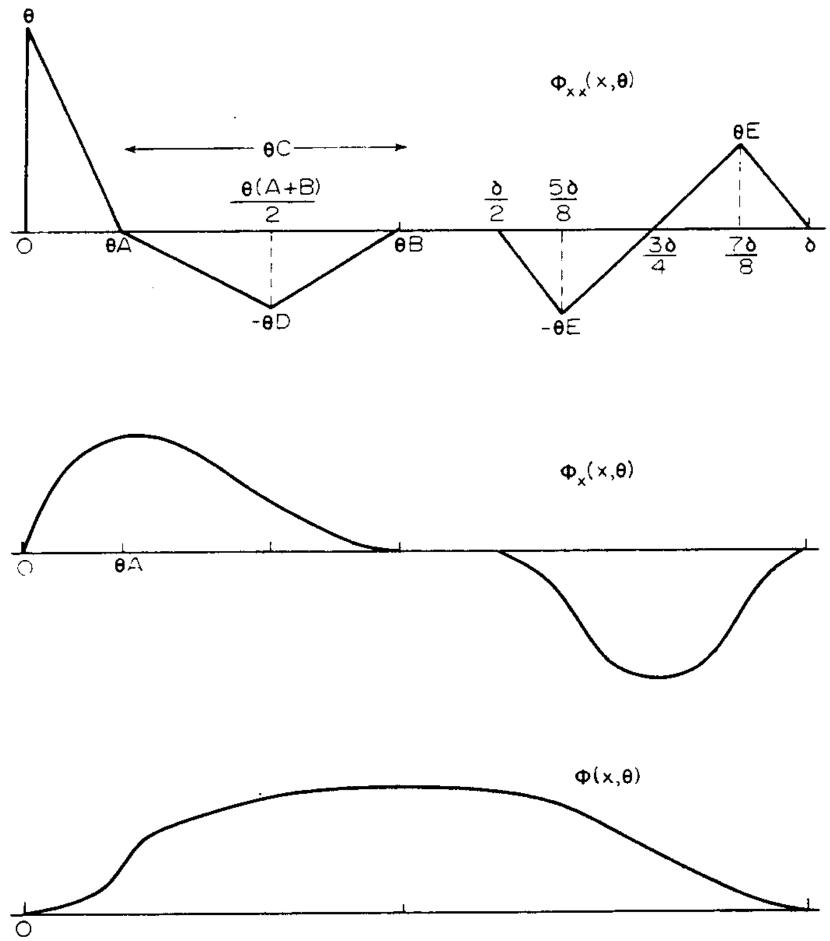

FIGURE 1

With the bounds established above, one can verify that $\tilde{f}(k,t)$ is neoclassical, is consistent with the data, and has the desired elasticity $\sigma^*(t)$ on the observed path. Q.E.D.

6. Necessary Conditions for Factor Augmenting Technical Change

In this section, the true production function is assumed to be strictly neoclassical and to exhibit purely factor augmenting technical progress. We note as a corollary to Theorem 1 that one cannot obtain a definitive acceptance of the factor augmentation hypothesis from aggregate data: these data could have been generated by a neoclassical production function with an arbitrary elasticity of substitution on the observed path, and this elasticity could always be chosen so that the resulting production function is not factor augmenting.[7] However, imposition of the factor augmentation hypothesis implies restrictions on observations which limit the range of non-identification of the elasticity of substitution. Moreover, these restrictions can be viewed as necessary conditions for the consistency of observations with the factor augmentation hypothesis. Hence, if this hypothesis were not true, observations on rates of change of aggregate variables at a single point in time might be sufficient to verify this fact.[8] The following argument establishes this last point and provides the admissible ranges of variables under the factor augmentation hypothesis. First, equations (7a), (9a), and (10a) imply

$$\mathring{x} = -\frac{\sigma \mathring{r}}{1-\pi} + \frac{\sigma \mathring{A}}{1-\pi} = -\frac{\mathring{v}}{1-\pi} + \frac{\mathring{A}}{1-\pi} = \frac{\mathring{y}}{\pi} - \frac{\mathring{B}}{\pi} = \frac{\sigma \mathring{w}}{\pi} - \frac{\sigma \mathring{B}}{\pi}.$$

Then, the average and marginal products of at least one of the factors are increasing at each point in time. Thus, the possible cases of factor augmenting technical progress have either $\mathring{y} > 0$ and $\mathring{w} > 0$, or $\mathring{v} > 0$ and $\mathring{r} > 0$ (or both). In particular, $\mathring{w} \leqq 0$ implies $\mathring{v} > 0$, $\mathring{r} > 0$, while $\mathring{r} \leqq 0$ implies $\mathring{y} > 0$, $\mathring{w} > 0$.

If at a point of time relative shares are unchanging, factor augmentation is consistent with only two hypotheses – either an elasticity equal to one or a constant effective capital/labor ratio [see equation (15a)]. If, further, one of the marginal products is declining, equations (9a) and (10a) imply that we can rule out the possibility of a constant effective capital/labor ratio. (Since the share of a factor equals its marginal product divided by its average product, the two products must move together when relative shares are constant.) This conclusion, and bounds on the elasticity of substitution implied by the other possible configurations of rates of change of shares and marginal products, are summarized in Table 2. The last column of this table gives a concise derivation of each bound using the growth equations of Section 3. [Note from (11) that at least one marginal product must be increasing under technical progress.] From the limits on $\mathring{x}$ in equation (12a) and the

[7]One may, in particular, construct a production function which has $\sigma^*(t) = 1$ on the observed path, so that all changes in relative shares must necessarily be the "consequence" of non-factor augmenting technical change.

[8]A necessary condition for factor augmentation is that the average and marginal products of at least one of the factors be increasing. An example of observations generated by a neoclassical production function which violate this hypothesis is $f(k,t) = e^{0.1t}k^{0.5+0.2t}$, defined for $k \geqq 1$ and $0 \leqq t \leqq 1$ (the definition can be extended to $0 \leqq k \leqq 1$ by a quadratic function of k in such a way that f remains neoclassical) and $k^*(t) = e^{0.3t}$. Then at $t = 0$, $T = 0.1$, $\mathring{v} = -0.15$, and $\mathring{w} = -0.05$.

bounds on σ in Table 2, bounds can be placed on the bias from (2a),

$$D = \left(1 - \frac{1}{\sigma}\right)(\mathring{A} - \mathring{B}) = \left(1 - \frac{1}{\sigma}\right)(\mathring{x} - \mathring{k}). \quad (21)$$

7. Sufficient Conditions for Factor Augmenting Technical Change

The discussion of the last section presented limits on the values of the elasticity of substitution which could possibly be consistent with factor augmenting technical progress. These bounds are necessary, but are not sufficient in the sense of indicating the range of non-identifiability, as observations from different points of time may eliminate some of the possibilities. The theorems in this section will present a range of non-identification, or sufficient bounds for satisfaction of the factor augmentation hypothesis, when the observations satisfy certain conditions. This range is a minimal range of non-identification, as other values may be consistent.

Let $E(t) = A(t)/B(t)$ denote the ratio of factor augmentation coefficients. Provided the average and marginal products of one of the factors increase throughout the period of observation, a range of non-identification is established for $\mathring{E}$ at each point of time. These ranges may be combined in any (continuous) way to give a range of non-identification for $\mathring{E}$, and thus by (8a) for the elasticity. The range will be chosen to guarantee that $\mathring{A}$ and $\mathring{B}$ are non-negative, σ is positive, and the efficiency capital/labor ratio is monotone.

We assume the average and marginal products of labor are increasing throughout the period of observation. Equation (11a) implies $T = \pi\mathring{A} + (1-\pi)\mathring{B} = \mathring{B} + \pi\mathring{E} = \mathring{A} - (1-\pi)\mathring{E}$, and hence

$$-T/(1-\pi) \leqq \mathring{E} \leqq T/\pi. \quad (22)$$

An increasing effective capital/labor ratio requires

$$\mathring{E} > -\mathring{k}, \quad (23)$$

and from equation (8a),

$$\mathring{E} > -\mathring{p}. \quad (24)$$

Theorem 2. Given positive, regular observed series $y^*(t)$, $k^*(t)$, $p^*(t)$, $0 \leqq t \leqq t_1$, of output per worker, capital per worker, and the wage–rentals ratio generated by a neoclassical production function which exhibits technical progress on the observed path, and which has monotonically increasing marginal and average products of labor on the observed path, then for any continuously differentiable function $E^*(t)$ satisfying (22), (23), and (24) there exists a neoclassical production function with factor augmenting change consistent with the data and showing a ratio of factor augmentation coefficients equal to $E^*(t)$.

We note that the possible range of elasticities established by the theorem above may lie strictly inside the necessary bounds established in Table 2; for example, the elasticity

$$\sigma = (\mathring{k} + \mathring{E})/(\mathring{p} + \mathring{E}) \tag{25}$$

is constrained by Theorem 2 to the range $0 < \sigma \leqq \mathring{y}/\mathring{w}$ in the case $\mathring{s} > 0$, $\mathring{w} > 0$, $\mathring{r} > 0$, while the alternative interval $\sigma \geqq \mathring{v}/\mathring{r}$ in Table 2 is now excluded. An analogue of Theorem 2 holds when the marginal and average products of capital are increasing.

Proof: The conditions $-T^*(1-\pi^*) \leqq \mathring{E}^* \leqq T^*/\pi^*$ and $\mathring{E}^* > -\mathrm{Min}(\mathring{k}^*, \mathring{p}^*)$ are non-vacuous under the hypotheses of the theorem since $T^*/\pi^* - (-\mathring{k}^*) = \mathring{y}^*/\pi > 0$ and $T^*/\pi^* - (-\mathring{p}^*) = \mathring{w}^*/\pi > 0$ by equation (11). Further, the range of E^* is non-degenerate, so that this function is not unique.

Given a choice of a continuous function $\mathring{E}^*$ satisfying the conditions above, and taken without loss of generality to satisfy $E^*(0) = 1$, define a function $\sigma(t)$ from equation (25) and define $x^* = E^* k^*$. Now $\mathring{x}^* > 0$ implies $x^*(t)$ can be inverted on $0 \leqq t \leqq t_1$ to obtain a continuous function $t = \tau(x)$, $x^*(0) \leqq x \leqq x^*(t_1)$. Define a new function for $x \geqq 0$ by

$$\begin{aligned} \sigma^*(x) &= \sigma(0) && \text{for} \quad 0 \leqq x < x^*(0), \\ &= \sigma(\tau(x)) && \text{for} \quad x^*(0) \leqq x \leqq x^*(t_1), \\ &= \sigma(t_1) && \text{for} \quad x^*(t_1) < x. \end{aligned} \tag{26}$$

Applying formulae (20) and (21), we can define an efficiency wage–rentals ratio $q(x)$ and output per efficiency worker ratio $g(x)$:

$$\begin{aligned} \ln q(x) &= \ln p^*(0) + \int_{k^*(0)}^{x} \sigma^*(z)^{-1} (\mathrm{d}z/z), \\ \ln g(x) &= \ln y^*(0) + \int_{k^*(0)}^{x} [q(z) + z]^{-1}\, \mathrm{d}z. \end{aligned} \tag{27}$$

Bounds on the elasticity of substitution implied by factor augmenting technical progress.

Case	Sign pattern of variables $\hat{s}$	$\hat{w}$	$\hat{r}$	Bounds on σ	Bounds on other variables	Derivation of bounds and other implications (numbers in parentheses indicate relevant equations)
1.	0	–	+	$\sigma = 1$	$\hat{v} = \hat{r} > 0$, $\hat{y} = \hat{w} < 0$, $\hat{x} < 0$, $\hat{k} < 0$, $\hat{\beta} < 0$	$\hat{v} > 0$ (11); $\hat{x} < 0$ (10a); $\sigma = 1$ (15a); $\hat{y} = \hat{w}$ (15a), (7a); $\hat{r} = \hat{v}$ (9a); $\hat{k} = \hat{\beta} - \hat{s} < 0$.
1a.	0	+	–	$\sigma = 1$	$\hat{v} = \hat{r} < 0$, $\hat{y} = \hat{w} > 0$, $\hat{x} > 0$, $\hat{k} > 0$, $\hat{\beta} > 0$	Symmetric to case 1.
2.	0	0	+	Either $-\hat{r}/(1-\pi) \leqq \hat{x} < 0$ and $\sigma = 1$; or $\hat{x} = 0$ and σ arbitrary	$\hat{v} = \hat{r} > 0$, $\hat{y} = 0$, $\hat{x} \leqq 0$, $\hat{k} < 0$, $\hat{\beta} < 0$	$\hat{v} = \hat{r} > 0$ (9); $\hat{y} = 0$ (10); $\hat{\beta} < 0$ (11); $\hat{k} < 0$ (11); either $\hat{x} < 0$ and $\sigma = 1$ or $\hat{x} = 0$ and σ arbitrary (14a), (15a).
2a.	0	+	0	Either $0 < \hat{x} \leqq \hat{w}/\pi$ and $\sigma = 1$; or $\hat{x} = 0$ and σ arbitrary	$\hat{y} = \hat{w} > 0$, $\hat{v} = 0$, $\hat{x} \geqq 0$, $\hat{k} > 0$, $\hat{\beta} > 0$	Symmetric to case 2.
3.	0	+	+	Either $-\hat{r}/(1-\pi) \leqq \hat{x} \leqq \hat{w}/\pi$, $\hat{x} \neq 0$, and $\sigma = 1$; or $\hat{x} = 0$ and σ arbitrary	$\hat{v} = \hat{r} > 0$, $\hat{y} = \hat{w} > 0$	$\hat{v} = \hat{r} > 0$ (9a); $\hat{y} = \hat{w} > 0$ (10a); either $\sigma = 1$ or $\hat{x} = 0$ (15a).
4.	+	– or 0	+	$1 < \hat{v}/\hat{r} \leqq \sigma$ and if $\hat{w} < 0$, $\sigma \leqq \hat{y}/\hat{w}$	$\hat{y} < \hat{w} \leqq 0$, $\hat{v} > \hat{r} > 0$, $\hat{x} < 0$, $\hat{k} < 0$, $\hat{\beta} < 0$	$\hat{y} < \hat{w} \leqq 0$ (10); $\hat{k} < 0$ (11); $\hat{v} > \hat{r} > 0$ (9); $\hat{\beta} < 0$ (10); $\hat{x} < 0$ (7a); $\sigma > 1$ (15a); $\sigma \geqq \hat{v}/\hat{r}$ (7), (9); if $\hat{w} < 0$, $\sigma \leqq \hat{y}/\hat{w}$ (17), (10) and $\hat{y}/\hat{w} - \hat{v}/\hat{r} = -T\hat{s}/\hat{r}\hat{w} > 0$ (9), (10).
4a.	–	+	– or 0	$1 < \hat{v}/\hat{w} \leqq \sigma$ and if $\hat{r} < 0$, $\sigma \leqq \hat{v}/\hat{r}$	$\hat{y} > \hat{w} > 0$, $\hat{v} < \hat{r} \leqq 0$, $\hat{x} > 0$, $\hat{k} > 0$, $\hat{\beta} > 0$	Symmetric to case 4.
5.	+	+	–	$0 < \sigma \leqq \hat{y}/\hat{w} < 1$ and if $\hat{v} < 0$, $\sigma \geqq \hat{v}/\hat{r}$	$\hat{y} > \hat{w} > 0$, $\hat{v} > \hat{r}$, $\hat{x} > 0$, $\hat{\beta} > 0$	$\hat{\beta} > 0$ (9); $\hat{v} > \hat{r}$ (9); $\hat{y} < \hat{w}$ (10); $\hat{x} > 0$ (9a); $\sigma < 1$ (15a); $\hat{y} > 0$ (12a); $\sigma \leqq \hat{y}/\hat{w}$ (17), (10); $\sigma \geqq \hat{v}/\hat{r}$ (17), (9); $\hat{y}/\hat{w} - \hat{v}/\hat{r} = -T\hat{s}/\hat{r}\hat{w} > 0$ (9), (10).
5a.	–	–	+	$0 < \sigma < \hat{v}/\hat{r} < 1$ and if $\hat{y} < 0$, $\sigma \geqq \hat{y}/\hat{w}$	$\hat{v} > \hat{r} > 0$, $\hat{y} > \hat{w}$, $\hat{x} < 0$, $\hat{\beta} < 0$	Symmetric to case 5.
6.	+	+	0	$0 < \sigma \leqq \hat{y}/\hat{w} < 1$	$\hat{v} > 0$, $0 < \hat{y} < \hat{w}$, $\hat{x} > 0$, $\hat{\beta} > 0$	$\hat{\beta} > 0$ (9); $\hat{v} > 0$ (9); $\hat{y} < \hat{w}$ (10); $\hat{x} > 0$ [if $\hat{x} = 0$, then $\hat{A} = 0$ (9a) implies $\hat{v} = \hat{A} - (1-\pi)\hat{x} = 0$, contradiction]; $\sigma < 1$ (15a); $y > 0$ (12a); $\sigma \leqq y/w$ (17), (10).
6a.	–	0	+	$0 < \sigma \leqq \hat{v}/\hat{r} < 1$	$\hat{y} > 0$, $0 < \hat{v} < \hat{r}$, $\hat{x} < 0$, $\hat{\beta} < 0$	Symmetric to case 6.
7.	+	+	+	Either $\sigma \geqq \hat{v}/\hat{r} > 1$; or $\hat{y} > 0$ and $0 < \sigma \leqq \hat{y}/\hat{w}$	$\hat{v} > \hat{r} > 0$, $\hat{y} < \hat{w}$	$\hat{v} > \hat{r}$ (9), $\hat{y} < \hat{w}$ (10); $\sigma \neq 1$ (8a); if $\sigma > 1$, then $\sigma \leqq \hat{v}/\hat{r}$ (17), (9); if $\sigma < 1$, then $\sigma \leqq \hat{y}/\hat{w}$ and $\hat{y} > 0$.
7a.	–	+	+	Either $\sigma \geqq \hat{y}/\hat{w} > 1$; or $\hat{v} > 0$ and $0 < \sigma \leqq \hat{v}/\hat{r}$	$\hat{y} > \hat{w} > 0$, $\hat{v} < \hat{r}$	Symmetric to 7.

Finally, define $B^*(t) = y^*(t)/g(E^*(t)k^*(t))$. Then the function $y = B^*(t)g(E^*(t)k)$ is a classical production function by the arguments of Section 4, and by construction is neoclassical and consistent with the data. Q.E.D.

The proof of Theorem 2 utilizes the property that the constructed effective capital/labor ratio is monotone, so that it is unnecessary to reconcile elasticities at different points in time corresponding to the same effective capital/labor ratios. The next theorem establishes that when data is generated by a factor augmented neoclassical production function with both augmentation coefficients strictly monotone, a non-degenerate range of non-identification of the elasticity continues to exist even when it can be deduced from the data that the effective capital/labor ratio has reversed direction of change. The proof, as in Theorem 1, involves perturbing the function known to be consistent with the observations and showing that this perturbation is consistent with the observations whenever the same effective capital/labor ratio reappears. This conclusion is subject to two limitations: the bias of technical change is identified when $\mathring{k} = 0$. The elasticity, which is a function solely of the efficiency capital/labor ratio x, is identified for any $x = x^*(t)$ value for which there is a reversal in the trend of relative shares $s^*(t)$. We term $x^*(t)$ a singular value of the true efficiency capital/labor ratio when $\mathring{s}^*(t) = 0$.[9]

> *Theorem* 3. Suppose one is given positive regular observed series $y^*(t)$, $k^*(t)$, $p^*(t)$, $0 \leqq t \leqq t_1$, generated by a neoclassical production function exhibiting purely factor augmenting technical progress, $y = B^*(t)g(kA^*(t)/B^*(t))$, with $\mathring{A}^* > 0$ and $\mathring{B}^* > 0$. Then for any time t^0 such that the value $x^*(t^0)$ of the true efficiency capital/labor ratio is non-singular, the elasticity is not identified [i.e., there exists a neoclassical factor augmenting production function consistent with the data whose elasticity at t^0 does not equal the true value $\sigma^*(x^*(t^0)]$.

Proof: Let $y = B^*(t)g^*(kE^*(t))$ denote the true production function, and let $x_0^* = x^*(t^0)$. By the hypothesis that the observed data are regular,

[9]We take the convention that all efficiency capital/worker ratios equal one at the start of the sample. Then the true efficiency capital/worker ratio, which we denote by $x^*(t)$, is uniquely determined. The meaning of identification at a singular value x_0^* of $x^*(t)$ is that any elasticity of substitution $\sigma(x)$ and path $x(t)$ which are consistent with the data will necessarily have $\sigma(x(t')) = \sigma^*(x(t'))$ at any time t' such that $x^*(t') = x_0^*$.

there are at most a finite number of singular values of $x^*(t)$. Then there are a finite number of distinct times $t^{-m} < \cdots < t^0 < \cdots < t^n$ such that $x^*(t^i) = x_0^*$, since $\mathring{s}^*$ must be zero at some point in each interval (t^i, t^{i+1}). Equation (15a) and non-singularity imply $\mathring{s}^*(t^i) \neq 0$ and the true elasticity of substitution $\sigma^*(x^*(t^i)) \neq 1$.

Consider a closed neighborhood $[x_0^* - \delta,\ x_0^* + \delta]$ of x_0^*, $\delta > 0$. For δ sufficiently small, the inverse image of this neighborhood under the mapping $x = x^*(t)$ is a set of disjoint closed intervals $N_i(\delta) = [t^i - \delta_i',\ t^i + \delta_i'']$, each containing one of the times t^i, with $\mathring{s}^* \neq 0$, $\mathring{x}^* \neq 0$, $\sigma^*(x^*(t)) \neq 1$, and $\mathring{x}^* + \mathring{s}^* \neq 0$ on these intervals.

Now consider a perturbed production function $\hat{g}(x,\theta) = g^*(x) + \theta\psi(|x - x_0^*|)$, where $\psi(x) = x^2(\delta - x)^3$ for $0 \leqq x \leqq \delta$ and $\psi(x) = 0$ otherwise, and θ is an arbitrary constant. Then ψ_x and ψ_{xx} exist and are continuous, with $|\psi| \leqq \delta^5/27$, $|\psi_x| \leqq 3\delta^4/16$, and $|\psi_{xx}| \leqq 16\delta^3/9$. Since g^* is classical, it follows that $\hat{g}(x,\theta)$ is classical for $|\theta|$ sufficiently small. Define the function $u(x,\theta) = -1 + \hat{g}(x,\theta)/x\hat{g}_x(x,\theta)$. Then u is continuously differentiable in x and θ, and satisfies $u(x^*(t),0) = s^*(t)$ for $0 \leqq t \leqq t_1$ and $u(x,\theta) = u(x,0)$ for $x \notin [x_0^* - \delta,\ x_0^* + \delta]$ and $|\theta|$ sufficiently small. It follows from the implicit function theorem that $u(x,\theta) = s^*(t)$ has a solution $x = \hat{x}(t,\theta)$ for θ sufficiently small and $0 \leqq t \leqq t_1$ which is continuously differentiable in (t,θ) and which satisfies $x^*(t) = \hat{x}(t,0)$ for all t and $x^*(t) = \hat{x}(t,\theta)$ for $t \notin N_i(\delta)$, all i.

Define $\hat{E}(t,\theta) = \hat{x}(t,\theta)/k^*(t)$ and $\hat{B}(t,\theta) = y^*(t)/\hat{g}(\hat{x}(t,\theta),\theta)$. Then the production function $y = \hat{B}(t,\theta)\hat{g}(k\hat{E}(t,\theta),\theta)$ generates the observed data and is classical for $|\theta|$ sufficiently small. Since $\mathring{B}(t,0) = \mathring{B}^*(t) > 0$ and $\mathring{E}(t,0) + \mathring{B}(t,0) = \mathring{A}^*(t) > 0$, it follows that $\mathring{A}(t,\theta) > 0$, $\mathring{B}(t,\theta) > 0$, $0 \leqq t \leqq t_1$, for θ sufficiently small. Hence, the perturbed production function is neoclassical. Further,

$$\hat{\sigma}(t,\theta)^{-1} = \sigma^*(t^0)^{-1} - 2\delta^3\theta k^*(t^0)y^*(t^0)A^*(t^0)^2/B^*(t^0)r^*(t^0)w^*(t^0),$$

implying the value $\sigma^*(t^0)$ is not identified. Q.E.D.

8. Identification for Capital Augmentation

If a production function were known a priori to be purely capital augmenting, then by equations (9a) and (11a) we see that we would have identification of the elasticity of substitution and bias of technical change. (This statement and the argument of this entire section holds equally for labor augmentation.) Assuming that it is hypothesized a priori only that technical progress is factor augmenting, but that the true production function is actually purely capital augmenting, one can ask whether this fact can be detected, allowing identification. If the hypotheses of Theorem 2 hold, then that result implies identification is not possible. However, if the true effective capital/labor ratio changes direction in a manner which violates those hypotheses, then the argument below shows that identification becomes possible. (Thus the assumption of strict augmentation in Theorem 3 is a necessary one for that non-identification result.)

> *Theorem* 4. Suppose positive regular observed series $y^*(t)$, $k^*(t)$, $p^*(t)$, $0 \leqq t \leqq t_1$ are generated by a purely capital augmenting production function $y = g(kA^*(t))$, and suppose technical change is hypothesized to be factor augmenting. Suppose there are times $0 \leqq t^1 < t^0 < t^2 \leqq t_1$, such that $\mathring{s}^*(t)$ is of one sign for $t \in [t^1,t^0)$ and of the opposite sign for $t \in (t^0,t^2]$, either $\mathring{r}(t') < 0$ or $\mathring{v}(t') < 0$ for some $t' \in [t^1,t^2]$, and either $\mathring{w}(t'') < 0$ or $\mathring{y}(t'') < 0$ for some $t'' \in [t^1,t^2]$. Then there exists a neighborhood of t^0 on which the production function above is the only neoclassical factor augmenting function consistent with the observations, and the elasticity is identified.

It should be noted that the hypothesis of the theorem on the true production function is consistent with the hypotheses on data. If, for example, g has an elasticity $\sigma(x) < 1$ for all x and $\mathring{k}^*(t)$ is decreasing in t, with $\mathring{k}^*(t^0) = -\mathring{A}^*(t^0)$ and $\mathring{k}^*(t^1) > (\sigma/(1-\pi))\mathring{A}^*(t^1)$, then equations (9a), (10a), and (15a) imply $\mathring{r}(t^1) < 0$, $\mathring{w}(t^2) < 0$, $\mathring{s}^*(t) > 0$ for $t \in [t^1,t^0)$, and $\mathring{s}^*(t) < 0$ and $t \in (t^0,t^2]$.

Proof: We consider the case with $s^*(t) > 0$ for $t \in [t^1,t^0)$, $s^*(t) < 0$ for $t \in (t^0,t^2]$, and $s^*(t^1) \geqq s^*(t^2)$; the remaining symmetric cases are left to the reader. Define a function h from $[t^1,t^0)$ into $(t^0,t^2]$ by $s^*(t) = s^*(h(t))$ for $t \in [t^1,t^0)$. By the implicit function theorem, h is a continuously differentiable function with $h_t(t) = \mathring{s}^*(t)/\mathring{s}^*(h(t)) < 0$, $h(t) \leqq t^2$, and $\lim_{t \to t^0} h(t) = t^0$.

Let x^* denote the true efficiency capital/labor ratio. Suppose there exists a false neoclassical factor augmenting production function which generates the data, and let $\hat{x}$ denote its efficiency capital/labor ratio. By equations (7a) and (12a), $\mathring{y}/\pi = \mathring{x}^* \geqq \mathring{\hat{x}}$.

Equation (5a) implies the relative share s is a function solely of the efficiency capital/labor ratio. Then, since $s^*(t') \neq s^*(t)$ for $t \in [t^1,t^0)$,

$t' \in (t^0, t^2]$ unless $t' = h(t)$, it follows that a value of $\hat{x}(t)$ [or $x^*(t)$] occurring at $t \in [t^1, t^0)$ cannot be repeated at $t' \in (t^0, t^2]$ unless $t' = h(t)$. But equation (14a) implies that the efficiency capital/labor ratio must reverse direction at t^0. Hence, using monotonicity, values of this ratio occurring at $t \in [t^1, t^0)$ must be repeated at a time in $(t^0, t^2]$. This argument establishes $\hat{x}(t) = \hat{x}(h(t))$ [or $x^*(t) = x^*(h(t))$] for $t \in [t^1, t^0)$. Hence, $x^*/\hat{x}$ satisfies

$$\mathring{x}^*(t) - \mathring{\hat{x}}(t) = [\mathring{x}^*(h(t)) - \mathring{\hat{x}}(h(t))]h_t(t). \tag{28}$$

But we have established that the left-hand side of (28) is non-negative, while the right-hand side is the product of a non-negative term and a negative term. Hence, $\mathring{x}^*(t) = \mathring{\hat{x}}(t)$, or $\mathring{E}^*(t) = \mathring{\hat{E}}(t)$, for $t \in [t^1, h(t^1)]$, and the elasticity and bias are identified on this domain, with $\mathring{E}^*(t) = \mathring{y}^*(t)/\pi - \mathring{k}^*(t)$ and $\sigma(t) = \mathring{y}^*(t)/(\mathring{y}^*(t) + \pi\mathring{s}^*(t))$. Q.E.D.

Note that in the proof above, the neighborhood of t^0 on which the conclusion of the theorem holds is $[t^1, t^3]$, where $t^3 = h(t^1)$. This neighborhood need not be "small". Analogous domains hold for the other cases in the proof.

An implication of purely capital augmenting technical change which provides a necessary condition for its presence is that $\mathring{y}^*(t^0) = \mathring{w}^*(t^0) = 0$ at a time t^0 satisfying the hypotheses of Theorem 4.

9. Identification for Finite Parameter Families of Augmentation Functions

The analysis of this paper began with the assumption that technology could be described by a neoclassical production function and that marginal products could be observed. This was seen to be an insufficient set of assumptions to obtain identification. We then added the hypothesis of factor augmentation. While this resulted in a restriction of possible production functions it did not, except in some cases when only one factor was being augmented, give a unique production function consistent with the observations. This is a familiar problem in econometrics, and calls for further assumptions which will permit identification. We consider now one such further assumption which when consistent with the observations implies identification except in unlikely singular cases. Presumably there are many other assumptions which might be employed instead. This one does however correspond to an assumption that has been employed in empirical work.[10]

[10]For example, see David and de Klundert (1965).

Assume that the rates of factor augmentation are hypothesized to be unknown linear combinations of known functions of time,[11]

$$\mathring{A}(t)=\sum_{i=1}^{n} a_i(t)\theta_i \quad \text{and} \quad \mathring{B}(t)=\sum_{i=1}^{n} b_i(t)\theta_i. \tag{29}$$

The set of unknown parameters $\{\theta_1,\dots,\theta_n\}$ is finite and linearly related to observations. From equation (11), the rate of technical progress satisfies

$$T(t)=\sum_{i=1}^{n}(\pi(t)a_i(t)+(1-\pi(t))b_i(t)). \tag{30}$$

Define $\phi_i(t)$ by

$$\phi_i(t)=\pi(t)a_i(t)+(1-\pi(t))b_i(t), \tag{31}$$

and M_{ij} and M_{iT} by

$$M_{ij}=\int_0^{t_1}\phi(t)\phi_j(t)\,\mathrm{d}t \quad \text{and} \quad M_{iT}=\int_0^{t_1}\phi_i(t)T(t)\,\mathrm{d}t, \qquad i,j=1,2,\dots,n. \tag{32}$$

Let M denote the $n\times n$ matrix with coefficients M_{ij}, M_T the column vector of coefficients M_{iT}, and θ the column vector of θ_i. We can then express the identification theorem as Theorem 5.

Theorem 5. If $y^*(t)$, $k^*(t)$, $p^*(t)$, $0\leqq t\leqq t_1$, are positive regular observed series generated by a factor augmenting production function whose factor augmentation coefficients satisfy (29), then a sufficient condition for technical change to be identified is the non-singularity of M. In this case the true values of the coefficients satisfy $\theta^*=M^{-1}M_T$.

Proof: Define the function

$$h(\theta)=\int_0^{t_1}\left(T^*(t)-\sum_{i=1}^{n}\phi_i(t)\theta_i\right)^2\mathrm{d}t.$$

By hypothesis this function has a minimum of zero at the true value of the coefficient vector θ^*. The first-order conditions for minimization are $M_T-M\theta=0$. If M is non-singular these equations have a unique solution which is the true value θ^*. Q.E.D.

[11]Writing these two augmentation functions in terms of the same θ_i does not imply any restriction on forms, for some of the a_i and b_i may be zero.

As an illustration, if the factor augmentation coefficients are exponential ($a_1(t) = 1$, $a_2(t) = 0$, $b_1(t) = 0$, $b_2(t) = 1$), then M will be non-singular if $\pi(t)$ is not constant.

Theorem 5 can be extended to systems which are non-linear in the unknown parameters by use of the global implicit function theorem (global univalence theorem).

10. Extensions and Conclusions

This paper has investigated identification of the elasticity and bias from time series data. However, factor augmenting technical change is formally identical to the introduction of non-observed "quality adjustment" coefficients for measured labor and capital in cross-section analysis. In general, no restriction analogous to "non-retrogression" applies in the cross-sectional case, and the elasticity and bias are clearly non-identified. If one knows a priori, however, that the sample index t has been taken so that one is moving uniformly from low efficiency to high efficiency units in terms of both factors, then Theorems 2–5 have a direct cross-section interpretation.[12]

Our results can also be extended along the lines of Section 9. In this analysis, a "smoothness" condition that technical change is undetermined only up to a finite number of degrees of freedom, along with a non-singularity condition, provided identification. This result can be extended by noting from equations (1), (2), (17), and (18) that if any one of the production function $f(k,t)$, relative share $s(k,t)$, rate of technical change $T(k,t)$, bias $D(k,t)$, or elasticity $\sigma(k,t)$ has a functional form which is known up to a finite number of unknown parameters, then all these functions can be specified up to a finite number of unknown parameters. Then application of an argument to equation (8) paralleling that of Theorem 5 establishes that except for singular cases, the elasticity and bias can be identified.

11. Appendix: Generalization to Non-Constant Returns

The identification theorems in this chapter have made essential use of the neoclassical assumption of constant returns to scale. However, it is

[12]We are indebted to Perry Shapiro for this point.

possible to derive growth accounting conditions without this assumption, and to investigate in this more general context the problems of identification. Cost and profit functions prove useful tools in this generalization. This appendix lists the important growth accounting conditions holding in the absence of constant returns. Let $Y = F(K,L,t)$ denote the production function, $c = C(Y,r,w,t)$ denote its cost function, and (in the case of decreasing returns) $\pi = \Pi(p,r,w,t)$ denote its profit function, with p the price of output. Let $\mu = (KF_k + LF_L)/F$ denote the marginal degree of returns to scale; α_k,α_L the cost shares of capital and labor; and $\beta_\pi,\beta_K,\beta_L$ the shares in total revenue of profit, payments to capital, and payments to labor. Then, $YC_r/C = 1/\mu$ and $\beta_\pi = 1-\mu$, so that with decreasing returns and profit maximization μ is identified. From the production function,

$$\mathring{Y} = \mu\alpha_K\mathring{K} + \mu\alpha_L\mathring{L} + T,$$

with T the rate of technical progress. The cost function yields

$$\mathring{c} = \frac{1}{\mu}\mathring{Y} + \alpha_k\mathring{r} + \alpha_L\mathring{w} + \frac{C_t}{C} = \alpha_K(\mathring{r} + \mathring{K}) + \alpha_L(\mathring{w} + \mathring{L}),$$

while the profit function yields

$$\beta_\pi\mathring{\Pi} = \mathring{p} - \beta_K\mathring{r} - \beta_L\mathring{w} + \beta_\pi\frac{\Pi_t}{\Pi} = \mathring{p} + \mathring{Y} - \beta_K(\mathring{r} + \mathring{K}) - \beta_L(\mathring{w} + \mathring{L}).$$

Combining these equations implies

$$\begin{aligned} T &= -\mu\frac{C_t}{C} = \beta_\pi\frac{\Pi_t}{\Pi} \\ &= \mathring{Y} + \mu\alpha_K\mathring{r} + \mu\alpha_L\mathring{w} - \mu\mathring{c} \\ &= \beta_\pi\mathring{\Pi} + \beta_K\mathring{r} + \beta_L\mathring{w} - \mathring{p} \\ &= \beta_\pi\mathring{\Pi} + \mu\alpha_K\mathring{r} = \mu\alpha_L\mathring{w} - \mathring{p}. \end{aligned}$$

An expression relating bias of technical change and the elasticity of substitution is derived most readily from the cost function. Recall that the elasticity of substitution is defined by

$$\sigma = -\left.\frac{\mathrm{d}\ln(K/L)}{\mathrm{d}\ln(r/w)}\right|_{Y,t\ \mathrm{const.}} = \left.\frac{\mathrm{d}\ln(C_r/C_w)}{\mathrm{d}\ln(r/w)}\right|_{Y,t} = \left.\frac{\mathrm{d}\ln(K/L)}{\mathrm{d}\ln(F_K/F_L}\right|_{Y,t}.$$

The bias of technical change is

$$D_t = \left.\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}t}\right|_{Y,K/L\ \mathrm{const.}}.$$

Then

$$\mathring{w} - \mathring{r} = \frac{1}{\sigma}(\mathring{K} - \mathring{L}) + D_s\mathring{Y} + D_t.$$

Alternately, from $K/L = C_r/C_w$,

$$\mathring{K} - \mathring{L} = \frac{\partial \ln(C_r/C_w)}{\partial \ln(r/w)}(\mathring{r} - \mathring{w}) + \frac{\partial \ln(C_r/C_w)}{\partial \ln Y}\mathring{Y} + \frac{\partial \ln(C_r/C_w)}{\partial t},$$

implying

$$D_s = -\frac{1}{\sigma}\frac{\partial \ln(C_r/C_w)}{\partial \ln Y} = -\frac{1}{\sigma}\left[\frac{YC_{ry}}{C_r} - \frac{YC_{wy}}{C_w}\right],$$

$$D_t = -\frac{1}{\sigma}\frac{\partial \ln(C_r/C_w)}{\partial t} = -\frac{1}{\sigma}\left[\frac{C_{rt}}{C_r} - \frac{C_{wt}}{C_w}\right].$$

We conclude from these growth equations that the rate of technical change T and returns to scale μ are identified under profit maximization. (However note that under the hypothesis solely of cost minimization, there is a non-identification between T and μ.) The elasticity of substitution σ, scale bias D_s, and bias in technical change D_t are in general not identified. Thus, for example, in a growing economy it may be possible to assign an arbitrary elasticity of substitution and bias of technical change, and "explain" observed series solely in terms of scale bias. Or, in the same circumstances, it may be possible to assume homothetic production and "explain" observations solely in terms of bias of technical change. We leave for future research the task of setting out conditions for identification or bounds on non-identification under various true states and maintained hypotheses involving non-constant returns.