Part V

Empirical Applications of Production Theory:

Macroeconomic Data

Chapter V.1

AN AGGREGATE MODEL WITH MULTI-PRODUCT TECHNOLOGIES

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1. Introduction[1]

In macroeconomic models, the production sector has often been specified and estimated without detailed attention to the underlying economic theory. Problems arise due to the inconsistencies of data collection systems which do not permit the straightforward application of the economic theory of production to the building of macro-economic models. Another source of the slackness of the underlying theory is the early prevalence of Keynesian models which were heavily oriented towards demand explanations for aggregate behaviour. Recent efforts have placed more emphasis on supply problems and the increasing interest in economic growth and medium- and long-term planning has shifted the focus of macroeconomic models towards increased emphasis on the structure of the economy.[2]

This paper uses recent developments[3] in the theoretical literature to explore the possibilities of specifying and testing a production sector for

[1]This paper grew out of an unpublished earlier paper (1971). At various stages Dale Jorgenson, Erwin Diewert and Dan McFadden provided assistance. Mel Fuss provided extensive comments that improved the paper. Sole responsibility for the paper remains with the authors.

[2]The models are static and highly aggregate. Hopefully time will change these limitations. The future would seem to contain macroeconomic models that are aggregate microeconomic models.

[3]The papers in this volume by Lau (Chapter I.3) and McFadden (Chapter I.1) contain extensive generalizations of the particular developments used in this paper.

a macro-economic model and provides some estimates of a model for Canada. This is part of a larger effort to construct more satisfactory small scale macro-economic models but only the production sector will be discussed here. In the following sections we will discuss some theoretical problems concerned with multi-product final demand technologies and then proceed to the estimation of a model using Canadian data. The conceptual problems have been discussed by Hall (1973) and Denny (1970) and the empirical work is related to U.S. studies by Burgess (1975, 1976), and Christensen, Jorgenson, and Lau (1973). The theoretical section generalizes the problem of handling multi-product technologies with limited or incomplete information while the empirical part deals with the specific problems in the context of a particular macro-economic model. It will become clear that the theoretical results do not yield any new optimism about the ease of applying theoretical knowledge to empirical work with limited data. However they do clarify the limitations inherent in the methods which are used and will continue to be used.

2. Multi-Product Technologies and Final Demand

The usual statistical sources of aggregate data for a national economy provide information on employment, imports and outputs flowing to final demand. At a lower level of aggregation, information on gross sales or output is often available for at least some sectors, e.g., manufacturing. It is seldom possible to obtain a consistent time series of any length which allocates the gross output of a number of industries to inter-industry and final demand and inventory accumulation. The common procedure in aggregate models has been to assume that total final demand, net of imports, comprises the sole aggregate output of the economy. Although a large number of articles have been published on this simple aggregate model since Solow (1957) popularized it, there has been little work on expanding the model to encompass more outputs or inputs. Recent studies by Denny (1971), Burgess (1976) and Brown, Caves and Christensen (1975) have begun to change this pattern. At lower levels of aggregation final demand or value-added outputs have often been used again without serious consideration of the relationship between the economic theory of production and the imperfect and incomplete data which are available. Although the models which are estimated are those which use final demand output measures, much of the theoretical discussion presented in this section relates to a much wider class of cases which commonly arise.

One of the major problems has been the difficulty of specifying a sensible and estimable functional form for the case of a multi-product technology. The development of flexible functional forms by Diewert (1971) and Christensen et al. (1973) has eliminated many of these problems. The task which concerns us is the possibility of analyzing information about production activities that is incomplete in several particular ways. Three problems are important. First, there is the problem of incomplete data on intermediate inputs and also gross outputs. Second, there is a lack of complete information on input distribution to particular outputs, i.e., production activities. Finally, there is the problem of specifying technologies with multiple outputs given the first two difficulties. The problems are closely related in some cases to the theory of aggregation but these links will not be explored here.

Our approach will be to consider the possibility of specifying a model for production activity under the conditions given above and to indicate the economic meaning that can be given to various cases.

Suppose that we have a vector of outputs $\mathbf{Y} = (Y_1,...,Y_n)$ and of primary inputs $\mathbf{X} = (X_1,...,X_m)$ which are technically related by a production structure which we can summarize as

$$H(\mathbf{Y},\mathbf{X}) = 0.$$

One simplification that has been used in practice is to assume that the technology can be specialized to

$$G(\mathbf{Y}) = F(\mathbf{X}).$$

It is then much easier to specify functional forms for $G(\mathbf{Y})$ and $F(\mathbf{X})$ separately which satisfy the production conditions, usually imposed. The separability imposed by the assumption that we may write $H(\mathbf{Y},\mathbf{X}) = G(\mathbf{Y}) - F(\mathbf{X})$ implies some severe restrictions on the technology.

The following theorem proved by Hall (1973) and Denny (1970) indicates the restrictiveness of attempting to estimate a multi-product technology by assuming that the production frontier is separable:

Theorem I: If each output Y_i is produced with a non-joint[4] *technology* $f_i(\mathbf{X}^i)$ satisfying the conditions that (1) $f_i(\mathbf{x}^i)$ is concave

[4]A multiple output technology is non-joint if the output of any single process depends only on the inputs used in that process and not on the levels of inputs or outputs into any other production process. Hall (1973) has a formal statement of this condition.

and strictly quasi-concave for all $\mathbf{x}^i$; and (2) $f_i(\mathbf{x}^i)$ is linear homogeneous for all i and the production possibility curve $G(\mathbf{Y})$ is non-linear concave then separability of the production frontier, $H(\mathbf{Y},\mathbf{X}) = G(\mathbf{Y}) - F(\mathbf{X})$, implies joint[5] production of the outputs.

Two brief observations can be immediately made. First, the production frontier, $G(\mathbf{Y})$, can be thought of as an aggregation formula and the notion of a single aggregate output such as GNP falls within the confines of our theorem. Secondly, the assumption in the theorem that $G(\mathbf{Y})$ is non-linear concave eliminates the special case of $G(\mathbf{Y})$ being a hyperplane in which case there is only one set of strictly positive prices at which positive outputs Y_i are produced for all i. In that situation $G(\mathbf{Y}) = \sum_i b_i Y_i$ and the individual technologies must be of the form

$$f_i(\mathbf{x}^i) = 1/b_i \cdot H(\mathbf{x}^i).$$

The result can easily be generalized to the case in which there are non-constant returns to scale and non-homotheticity by the following theorem:

> *Theorem* II: Assume that each output Y_i is produced with a non-joint technology, $Y_i = f_i(\mathbf{x}^i)$, satisfying the conditions: (a) $f_i(\mathbf{0}) = 0$; (b) $f_i(\mathbf{x}^i)$ is non-decreasing in $\mathbf{x}^i$; (c) strict monotonicity, i.e., if all components of $\mathbf{x}^i$ are increased then Y_i is increased; (d) $f_i(\mathbf{x}^i)$ is continuous.

Under these assumptions the production frontier is never separable unless the isoquants for each technology, $f_i(\mathbf{x}^i)$ are identical up to a renumbering of the isoquants.

In general, we have the requirement that the assumption of a separable production frontier requires either some type of jointness amongst the underlying production process or else relatively similar isoquants.

The theorems summarized here can be extended to cases in which explicit treatment of intermediate goods is considered. The results are similar.[6] The use of a separable non-joint multiple-output production technology presumes strong restrictions on the production functions for the individual outputs.

[5]See footnote 4 for a definition of non-joint.
[6]The results are contained in an unpublished manuscript, Denny (1971).

3. Jointness and Separability of the Cost Function

Our estimation will be based on the joint cost function. The restrictions on the joint cost function due to the imposition of non-jointness or separability will be specified in this section. Consider the cost function,

$$C = g(\mathbf{w},\mathbf{Q}), \qquad \mathbf{w} = (w_1,\ldots,w_n), \quad \mathbf{Q} = (Q_1,\ldots,Q_m). \tag{1}$$

If the individual outputs Q_i are produced with a separable non-joint production function then for any Q_i,

$$C_i = g_i(\mathbf{w},Q_i). \tag{2}$$

The total cost of producing a vector of outputs Q is then simply the sum of the costs of producing each output separately. Therefore,

$$C = g(\mathbf{w},\mathbf{Q}) = \sum_i g_i(\mathbf{w},Q_i). \tag{3}$$

The multiple output cost function must have the special form given by (3) if the outputs are produced non-jointly.

If the production frontier is separable into a function of outputs and a function of inputs, what does this imply about the cost function? Denny (1970) and Hall (1973) have shown that this implies that the cost function may be written as

$$C = G(\mathbf{w},H(\mathbf{Q})).$$

The separable frontier which is linear homogeneous in the outputs will have the specialized form[7]

$$C = G(\mathbf{w})\cdot H(\mathbf{Q}).$$

This will imply that the aggregate input price index, $G(\mathbf{w})$, has a corresponding aggregate quantity index which is homothetic.

4. The Approximate Translog Cost Function

The translog cost function will be used as an approximation to the multiple output cost function that we wish to estimate,

[7]See Hall (1973) for a proof of this statement.

$$\ln C = \alpha_0 + \sum_i \alpha_i \ln w_i + \sum_k \beta_k \ln Q_k + \tfrac{1}{2}\sum_i \sum_j \gamma_{ij} \ln w_i \ln w_j + \tfrac{1}{2}\sum_k \sum_l \theta_{kl} \ln Q_k \ln Q_l + \sum_i \sum_k \delta_{ik} \ln w_i \ln Q_k, \quad i,j = 1,\dots,m, \quad k,l = 1,\dots,n. \tag{4}$$

The translog function is used in this paper as the quadratic approximation to an arbitrary multiple product cost function around a point of expansion.[8] The cost function is expanded about the point (1,1,1,1,1). At the point of expansion the parameters of the joint cost function are equal to the first- and second-order derivations of the approximation. For this interpretation, the *symmetry constraints* are

$$\gamma_{ij} = \gamma_{ji}, \qquad \theta_{kl} = \theta_{lk}.$$

The cost function is linear homogeneous in the input prices w_i and this requires the parameter restrictions,

$$\sum_i \alpha_i = 1, \qquad \sum_i \gamma_{ij} = 0, \qquad \sum_k \delta_{ik} = 0.$$

For these aggregate production models, constant returns to scale will be imposed and this implies an additional set of restrictions,

$$\sum_k \beta_k = 1, \qquad \sum_k \theta_{kl} = 0, \qquad \sum_i \delta_{ik} = 0.$$

There are other implied constraints on both the θ_{kl}'s and the γ_{ij}'s. From the symmetry and homogeneity constraints, we can obtain the constraints,

$$\sum_l \theta_{kl} = 0, \qquad \sum_j \gamma_{ij} = 0.$$

All of these constraints will be part of the maintained hypothesis.

To estimate the approximate translog cost function, the share equations will be used. The share of factor i in total cost is

$$S_i = \alpha_i + \sum_j \gamma_{ij} \ln w_j + \sum_k \delta_{ik} \ln Q_k. \tag{5}$$

[8]For an extensive and explicit development of the translog as an approximation about a point, see Denny and Fuss (1977).

The use of the share equations implies many of the same parameter restrictions that arise from the imposition of linear homogeneity. Since the shares must add to one,

$$\sum_i \alpha_i = 1, \qquad \sum_j \gamma_{ij} = 0, \qquad \sum_k \delta_{ik} = 0.$$

These are exactly the constraints imposed above because the cost function is linear homogeneous in the input prices.

The share equations above will not provide estimates of the complete cost function. Consequently they will have to be supplemented by either the cost function itself or a behavioural relationship which includes the parameters, θ_{kl} and β_k. One can specify a behavioral relationship between output price and average or marginal cost. Since we do not wish to specify an output–price rule we will use the cost function.

5. Approximate Separability and Non-Jointness of the Translog Cost Function

The weakly separable cost function, $C = G(\mathbf{w},H(\mathbf{Q}))$ is approximated by a Taylor series expansion of

$$\log C = \log G(\log \mathbf{w}, H(\log \mathbf{Q})), \tag{6}$$

about the point, $w_i = 1$, $Q_k = 1$, for all i,k.

Since the parameters of the approximation correspond to first- and second-order derivatives of equation (6), we will consider

$$\frac{\partial \log C}{\partial \log Q_k} = \left[\frac{\partial \log G}{\partial H}\right]\cdot\left[\frac{\partial H}{\partial \log Q_k}\right], \qquad k = 1,\dots,m,$$

$$\frac{\partial^2 \log C}{\partial \log w_i \partial \log Q_k} = \left[\frac{\partial \log G}{\partial H}\cdot \partial \log w_i\right]\cdot\frac{\partial H}{\partial \log Q_k}, \qquad i = 1,\dots,n, \quad k = 1,\dots,m.$$

From these partial derivatives, we can derive the relationship

$$\frac{\partial^2 \log C}{\partial \log w_i \,\partial \log Q_k}\cdot\frac{\partial \log C}{\partial \log Q_l} = \frac{\partial^2 \log C}{\partial \log w_i \,\partial \log Q_l}\cdot\frac{\partial \log C}{\partial \log Q_k}, \qquad i = 1,\dots,n, \quad k,l = 1,\dots,m.$$

In the translog approximation,

$$\frac{\partial \log C}{\partial \log Q_k} = \beta_k \quad \text{and} \quad \frac{\partial^2 \log C}{\partial \log w_i \, \partial \log Q_k} = \delta_{ik}.$$

The parameters of the approximation must satisfy the constraints,

$$\delta_{ik} \cdot \beta_l = \delta_{il} \cdot \beta_k, \qquad \forall i,k,l, \quad k \neq l.$$

In Section 3 above, we noted that if the production technology is linear homogeneous in the outputs, separability of inputs and outputs implies that

$$C = G(\mathbf{w}) \cdot H(\mathbf{Q}).$$

In this case, the function

$$\log C = \log G(\log \mathbf{w}) + \log H(\log \mathbf{Q}) \tag{7}$$

must be expanded. The second-order derivative

$$\frac{\partial^2 \log C}{\partial \log w_i \log Q_k} = 0, \qquad \forall i,k.$$

Since this derivative equals δ_{ik}, the approximation must satisfy the restrictions,

$$\delta_{ik} = 0, \qquad \forall i,k. \tag{8}$$

Tests of the parameter restrictions (8) required for the separability of the production function will be conducted below.

Non-jointness of the cost function requires that

$$C = \sum_k g_k(\mathbf{w}, Q_k), \tag{9}$$

or, with linear homogeneity,

$$C = \sum_k g_k(\mathbf{w}) \cdot Q_K. \tag{10}$$

If we consider equation (9), then the function which must be expanded is

$$\log C = \log \sum_k g_k(\log \mathbf{w}, \log Q_k).$$

Consider the first- and second-order derivatives

$$\frac{\partial \log C}{\partial \log Q_k} = \frac{1}{C}\cdot\frac{\partial g_k}{\partial \log Q_k}, \qquad k = 1,\ldots,m,$$

$$\frac{\partial^2 \log C}{\partial \log Q_k}\cdot\partial \log Q_l = -\frac{1}{C^2}\cdot\frac{\partial g_k}{\partial \log Q_k}\cdot\frac{\partial g_l}{\partial \log Q_l}, \qquad k \neq l, \quad k,l = 1,\ldots,m.$$

These derivatives equal the parameters β_k and θ_{kl}, respectively.
Consequently,

$$\theta_{kl} = -\beta_k \cdot \beta_l, \qquad \forall k,l, \quad k \neq l. \tag{11}$$

In the linear homogeneous case,

$$\log C = \log\left[\sum_k g_k(\log \mathbf{w})\cdot Q_k\right].$$

The same set of restrictions are required.[9]

The impossibility theorems presented above can be illustrated[10] by considering the joint imposition of the approximate constraints for separability and non-jointness of the translog form. Separability of outputs from inputs is implied by the constraints

$$\delta_{ik}\cdot\beta_l = \delta_{il}\cdot\beta_k. \tag{12}$$

In addition to these constraints we wish to impose the constraints for non-jointness. Consider the left-hand side of the separability constraints (12), under the assumption that the technology is non-joint.

$$\delta_{ik} = \frac{\partial^2 \log C}{\partial \log Q_k\, \partial \log w_i}$$

$$= \frac{1}{C}\cdot\frac{\partial^2 C_k}{\partial \log Q_k\, \partial \log w_i} + \frac{\partial C_k}{\partial \log Q_k}\left[-\frac{1}{C^2}\sum_k \frac{\partial C_k}{\partial \log w_i}\right],$$

$$\beta_l\cdot\delta_{ik} = \beta_l\left[\frac{1}{C}\cdot\frac{\partial^2 C_k}{\partial \log Q_k\, \partial \log w_i}\right] - \beta_l\cdot\beta_k\cdot\theta,$$

[9]Direct application of the methods used above to establish the restrictions in other cases will establish this result. The translog function does not permit a distinction between a non-joint technology that is linear homogeneous in all outputs and one that is, *in addition*, linear homogeneous in the micro cost functions.

[10]Professor Fuss suggested this illustration.

where

$$\theta = -\frac{1}{C^2}\left[\sum_k \frac{\partial C_k}{\partial \log w_i}\right].$$

When the technology is non-joint, the separability constraints (12) may be written as

$$\beta_l \left[\frac{1}{C}\cdot\frac{\partial^2 C_k}{\partial \log Q_k \partial \log w_i}\right] = \beta_k \left[\frac{1}{C}\cdot\frac{\partial^2 C_l}{\partial \log Q_l \partial \log w_i}\right]. \tag{13}$$

For ease of exposition, assume that the non-joint production functions are homothetic,

$$C_k(\log \mathbf{w}, \log Q_k) = g_k(\log \mathbf{w})\cdot h_k(\log Q_k).$$

Then,

$$\beta_k = \frac{1}{C}\cdot g_k(\log \mathbf{w})\cdot\frac{\partial h_k}{\partial \log Q_k},$$

$$\frac{\partial C_k}{\partial \log w_i \,\partial \log Q_k} = \frac{\partial g_k}{\partial \log w_i}\cdot\frac{\partial h_k}{\partial \log Q_k},$$

Therefore, we may re-write (13) as

$$\frac{1}{g_k(\log \mathbf{w})}\cdot\frac{\partial g_k}{\partial \log w_i} = \frac{1}{g_l(\log \mathbf{w})}\cdot\frac{\partial g_l}{\partial \log w_i}.$$

Since this holds for all $i,k,l,\ k \neq l$, it implies that

$$g_k(\log \mathbf{w}) = A\cdot g_l(\log \mathbf{w}).$$

Therefore the non-joint cost functions are equivalent up to a scale factor, $A\cdot h_k(\log Q_k)/h_l \log(Q_l)$.

6. An Aggregate Canadian Production Model

The Canadian production sector will be approximated with a model that has two outputs and three inputs. The outputs are consumption (C) and investment (I) goods. The inputs are the services of capital (K) and labour (L) and imported (M) goods. A description of the data used to construct the quantities and prices of the inputs and outputs can be found in the data appendix. The aggregate joint cost function,

$$C = g(p_l, p_k, p_m, Q_C, Q_I),$$

will be approximated using the translog function.

Imports into the Canadian economy are treated as an input into production. The viewpoint taken here is that importing is a decision made by the producing sector. The producer has expectations about the prices available in the domestic and foreign markets for outputs. The domestic prices of factors of production are also known, at least in terms of historical data. The producer's decision is to attempt to maximize the returns from selling products. He can obtain these either by producing them domestically or by importing them. Imports can include either finished products or intermediate goods which will then be further processed. The price of imports includes the tariff duties paid for the goods. As an aggregate model of this process, the technology facing the producer contains three factors of production – imports, capital and labour services.

Table 1 presents some evidence on the distribution of imports for Canada and it is clear from these that the direct purchase of consumer goods is small. The group classified as consumer goods can be treated as a decision to purchase an intermediate input by the retail or wholesale trade sector since for almost all imports of goods flowing to the domestic trade sector there is a significant component of domestic value-added. Roughly 5 to 40 percent of the domestic retail selling price is domestic value-added.

There are many reasons why one would like more detailed information on the producing sector when considering the demand for imports. At the level of aggregation chosen in most macroeconomic

TABLE 1
Canadian imports by economic class (percent).

	1964	1968	1972	1974
Fuels and lubricants	7.31	6.33	5.74	10.47
Industrial materials	27.76	21.65	20.83	21.43
Construction materials	3.66	2.52	2.50	3.10
Materials	*38.73*	*30.40*	*29.07*	*35.00*
Producer's equipment	25.32	21.19	21.39	19.99
Transportation equipment	2.59	4.25	2.39	2.87
Motor vehicles and parts	11.34	25.35	27.53	23.22
Equipment	*39.25*	*50.79*	*51.26*	*46.08*
Consumer goods	19.11	16.48	18.45	17.77
Special items	2.90	2.23	1.26	1.15
Total imports in millions of dollars	7,488	12,358	18,669	31,639

models and studies of production technologies, the behavior of imports can be studied through links with the demand for factors in the importing sector. The usual procedure of linking imports to activity variables in the consumer sector does not provide an adequate description of the import decision. A very small portion of imports are directly purchased due to decisions made by consumers, e.g., travel expenditure. The decision to import is a decision by the business sector and while it may be derived from expectations concerning consumer behavior, e.g., demand expectations, it is properly left in the business sector.

The factor share equations for the translog cost function used in our model may be illustrated with the share of labour,

$$S_l = \alpha_l + \gamma_{ll} \ln p_l + \gamma_{lk} \ln p_k + \gamma_{lm} \ln pm + \delta_{IL} \ln Q_I + \delta_{CL} \ln Q_c.$$

One of the three share equations has to be deleted because the sum of the shares equals one. The cost function will be used as an additional equation.

The constraints for linear homogeneity of the cost function in both factor prices and outputs will be imposed before the parameters are estimated.

7. Estimation and Hypothesis Testing

The model to be estimated consists of the two share equations for capital and labour and the cost function itself. With the constraints for linear homogeneity in factor prices and output imposed, the equations are

$$S_l = \alpha_l + \gamma_{ll} \ln(p_l/p_m) + \gamma_{lk} \ln(p_k/p_m) + \delta_{CL} \ln(Q_C/Q_I),$$

$$S_k = \alpha_k + \gamma_{lk} \ln(p_l/p_m) + \gamma_{kk} \ln(p_k/p_m) + \delta_{CK} \ln(Q_C/Q_I),$$

$$\begin{aligned} \ln C = {} & \alpha_0 + \alpha_l \ln(p_l/p_m) + \alpha_k \ln(p_k/p_m) + \ln p_m + \beta_c \ln(Q_C/Q_I) + \ln Q_I \\ & + \tfrac{1}{2}\gamma_{ll}[\ln(p_l/p_m)]^2 + \gamma_{lk} \ln(p_l/p_m)\cdot\ln(p_k/p_m) + \tfrac{1}{2}\gamma_{kk}[\ln(p_k/p_m)]^2 \\ & + \delta_{LC} \ln(p_l/p_m)\cdot\ln(Q_C/Q_I) + \delta_{KC} \ln(p_k/p_m)\cdot\ln(Q_C/Q_I) \\ & + \tfrac{1}{2}\theta_{CC}[\ln(Q_C/Q_I)]^2 + \rho_t \ln T + \rho_{tt} \ln T^2. \end{aligned}$$

In the cost equation, three variables appear without any parameters. The latter are eliminated by the constraints imposed. The variables themselves are moved to the left-hand side before the parameters are estimated. Two parameters ρ_t and ρ_{tt} have been added to the cost equation. These parameters provide measures of Hicks' neutral technical change during the period. We will test the hypothesis that there is no technical change by imposing the constraints

$$\rho_t = \rho_{tt} = 0.$$

The three equations are estimated as a system with additive error terms on each equation. The procedure used is similar to Zellner's (1962 and 1963) method except that the estimates are iterated until the estimated variance–covariance matrix is diagonal and the parameter estimates converge. Berndt and Christensen (1973a) have called this procedure "iterative Zellner".

Our tests are based on the likelihood ratio method. The likelihood ratio is

$$\lambda = \frac{L_{\max}(\Omega)}{L_{\max}(W)},$$

for the unconstrained (Ω) and constrained (w) maximum value of the likelihood function (L). Wilks (1938) has shown that the test statistic, $-2\log\lambda$, has an asymptotic distribution that is χ^2 with degrees of freedom equal to the number of restrictions.

The estimates of the parameters of the unconstrained case are given in Table 2. The parameters not given in the table can be derived from the following set of equalities:

$$\delta_{LI} = -\delta_{LC}, \qquad \delta_{KI} = -\delta_{KC}, \qquad \delta_{MI} = -\delta_{MC},$$
$$\theta_{CI} = -\theta_{CC}, \qquad \theta_{II} = \theta_{CC}.$$

The test statistics for the hypotheses about the structure of the model are given in Table 3. The hypothesis that there has been no technical

TABLE 2
Parameter estimates for unconstrained case (standard errors in parentheses).

α_0	10.606	(0.004)	γ_{km}	−0.169	(0.043)
α_l	0.467	(0.001)	γ_{mm}	0.134	(0.043)
α_k	0.322	(0.003)	δ_{LC}	0.076	(0.024)
α_m	0.211	(0.003)	δ_{KC}	−0.166	(0.042)
γ_{ll}	−0.021	(0.006)	δ_{MC}	0.089	(0.046)
γ_{lk}	−0.013	(0.009)	β_C	0.551	(0.060)
γ_{lm}	0.034	(0.101)	β_I	0.449	(0.060)
γ_{kk}	0.182	(0.044)	θ_{CC}	0.093	(1.328)
ρ_t	−0.200	(0.007)	ρ_{tt}	−0.042	(0.004)

TABLE 3

	d.f.	$-2 \log \lambda$	$\chi^2(0.05)$
No technical change	2	50.334	5.99
Separability of outputs and inputs	2	7.994	5.99
Non-jointness	1	2.712	3.84

TABLE 4
Parameter estimates for non-joint case.

α_0	10.616	(0.007)	γ_{km}	0.005	(0.039)
α_l	0.466	(0.005)	γ_{mm}	0.005	(0.037)
α_k	0.338	(0.007)	δ_{LC}	−0.007	(0.034)
α_m	0.196	(0.005)	δ_{KC}	0.099	(0.048)
γ_{ll}	−0.030	(0.012)	δ_{MC}	−0.091	(0.032)
γ_{lk}	0.041	(0.017)	β_C	0.471	(0.053)
γ_{lm}	−0.011	(0.011)	β_I	0.529	(0.053)
γ_{kk}	−0.046	(0.046)	θ_{CC}	−0.249	(0.003)
ρ_t	−0.189	(0.011)	ρ_{tt}	−0.452	(0.005)

change is strongly rejected. Total costs of producing the same output level have fallen over time.[11]

The hypothesis that the function representing the technology is separable into a function of inputs and outputs is rejected. Models which attempt to use a single aggregate output or that estimate an aggregation function for outputs independent of inputs are rejected. Burgess (1976) obtained the same results for the U.S. economy. We are unable to reject the hypothesis that the technology is non-joint. Burgess (1976) rejected this hypothesis but the rejection was marginal.

In Table 4, the parameter estimates for the non-joint model are presented. In the next sections we will consider the elasticities of the technology since the direct parameter estimates do not provide an adequate source of information.

7.1. *Elasticities of Demand and Substitution for the Inputs*

The translog cost function has price elasticities of demand e_i and elasticities of substitution σ_{ij} given by the following formulae [Berndt and Christensen (1973a)]:

$$e_i = (\gamma_{ii} + S_i^2 - S_i)/S_i, \qquad i = k,l,m,$$
$$\sigma_{ij} = (\gamma_{ij} + S_iS_j)/S_iS_j, \qquad i \neq j, \quad i,j = k,l,m.$$

[11]The values of the parameters, ρ_t and ρ_{tt}, must be carefully interpreted. The time variable has been normalized to equal 1.0 in 1961.

TABLE 5
Price elasticities of demand; labour, capital and imports, selected years.

	e_l	e_k	e_m
1947	−0.584	−0.831	−0.768
1950	−0.582	−0.838	−0.766
1953	−0.586	−0.822	−0.773
1957	−0.597	−0.826	−0.769
1960	−0.597	−0.809	−0.777
1963	−0.598	−0.806	−0.769
1966	−0.599	−0.822	−0.761
1969	−0.605	−0.822	−0.757
1972	−0.608	−0.819	−0.757

TABLE 6
Elasticities of substitution, selected years, 1947–72.

	σ_{lk}	σ_{lm}	σ_{km}
1947	1.27	0.892	0.713
1950	1.27	0.893	0.701
1953	1.26	0.889	0.730
1957	1.27	0.890	0.743
1960	1.26	0.884	0.780
1963	1.26	0.888	0.763
1966	1.27	0.893	0.741
1969	1.27	0.894	0.753
1972	1.27	0.894	0.762

In Table 5 we have tabulated the price elasticities for selected years. The price elasticities are very stable throughout the post-war period and each is less than one.

The factors can be substituted for each other. Table 6 shows the estimates of the Allen–Uzawa elasticities of substitution.

The possibilities of substitution are largest between labour and capital. However, all three inputs are substitutes and the possibility exists of substituting away from domestic factors towards imported inputs. The values are highly stable because the fitted and actual shares do not vary by a large amount during this period.

7.2. *Sensitivity of the Elasticities for Inputs to the Model Specification*

The maintained hypothesis permitted us to test for non-jointness and separability of the aggregate technology. If the elasticities were the parameters that the investigator wished to study, it might be true that they were insensitive to the specification of the technology. The elasticities in the previous section were calculated given the results of our tests. These elasticities are for the non-joint specification.

For comparison, the elasticities of demand and substitution have been calculated for the maintained hypothesis and the separable case. The maintained hypothesis is the translog approximation to any multiple output cost function and the separable case imposes the approximate constraints for the separability of outputs from inputs.

These results are tabulated in Table 7. The values given in the tables are average values. As Tables 3 and 4 showed, there was very little variation from year to year in the point estimates. This was true for the other two specifications. The average point estimates given in Table 5 are complicated non-linear functions of the point estimates of the estimated parameters. The standard errors of the point estimates for the elasticities are not known. A "casual" reading of the results would suggest the following observations. With the exception of the price elasticity of demand for imports, the price elasticities are not very sensitive to the specification. The elasticity of substitution between imports and capital is the most sensitive of the substitution elasticities. Non-jointness was only marginally accepted and separability was strongly rejected. These results are reflected in the elasticities since both sets of elasticities are quite close for the non-joint and maintained hypothesis. The elasticities in the separable case diverge more sharply from the other two and particularly from the non-joint hypotheses.

With the exception of the import elasticities, the specification of the model does not change the elasticities by large amounts.

TABLE 7
Elasticities of demand and substitution for alternative specifications.

	e_l	e_k	e_m
Non-joint	−0.59	−0.81	−0.77
Maintained	−0.58	−1.07	−1.21
Separable	−0.58	−1.10	−1.45
	σ_{lk}	σ_{lm}	σ_{km}
Non-joint	1.27	0.89	0.75
Maintained	1.21	0.92	1.68
Separable	1.07	1.12	2.02

7.3. *The Marginal Rate of Transformation*

For the aggregate model estimated in this study, we imposed the constraint that the production technology was linear homogeneous in the outputs. This restriction has been enforced throughout the estimation. The linear homogeneity of the technology in the multiple product case implies that the production possibility curve is simply expanded or contracted by a scalar multiple as one expands all inputs by the same scalar multiple. Along any ray in output space the marginal rate of transformation is a constant.

From our approximation to the joint cost function, the marginal rate of transformation can be approximated by the ratio of the marginal costs. With the translog form, the marginal cost of output k is not as easily expressed as the logarithmic marginal cost. This is simply the elasticity of the total cost of all outputs with respect to output k.

In our two output model, with the constraints of the maintained hypothesis imposed, the logarithmic marginal cost for consumption goods is

$$\frac{\partial \ln C}{\partial \ln Q_C} = \beta_C + \theta_{CC} \ln\left(\frac{Q_C}{Q_I}\right) + \delta_{LC} \ln\left(\frac{p_L}{p_M}\right) + \delta_{KC} \ln\left(\frac{p_K}{p_M}\right).$$

The marginal rate of transformation can be expressed as the ratio of the logarithmic marginal costs times the output ratio,

$$\text{MRT} = \frac{\partial \ln C}{\partial \ln Q_C} \Big/ \frac{\partial \ln C}{\partial \ln Q_I} \cdot \frac{Q_I}{Q_C}.$$

In Table 8 the values of the MRT for selected years are tabulated. Column one is for the non-joint model, column two for the maintained hypotheses and column three for the separable cost function.

TABLE 8
Marginal rate of transformation; alternative models, selected years.

	Non-joint	Maintained	Separable
1947	0.97	0.50	0.76
1950	1.06	0.71	0.99
1955	1.01	0.48	0.70
1960	0.90	0.29	0.50
1965	1.29	1.83	2.16
1970	1.43	3.38	3.68
1972	1.63	8.90	8.25

The estimated value of the MRT has increased through time. The non-joint estimates are quite different from the other two cases. While the specification of the model did not appear to sharply effect the elasticities for the factors this is not true for the outputs. This result should be expected since the restrictions imposed relate to the specification of the multiple outputs. The two rejected models tend to show very substantial MRT for the years since 1965. The increase in the MRT in the latter part of the sample is evident in the non-joint case but the values do not increase to the same extreme extent. For this sample, the specification of the model does substantially effect the estimates of the MRT.

7.4. *Conclusions*

For Canada during the post-war period, the technology can be represented by a non-joint aggregate model of two outputs and three inputs. The estimates of the price elasticities of demand and the elasticities of substitution for the inputs are relatively insensitive to alternative specifications that assume jointness or separability in outputs and inputs of the function representing the technology. The estimates of the slope of the production possibility curve are more sensitive to the alternative specifications. Large differences in the estimates of the marginal rate of transformation occur. This is reasonable given that it is the specification of the multiple outputs that differs in the three cases.

8. Appendix: Data

The data used in this study have been developed for the years 1926–72 as a data base for a small-scale general equilibrium model. The description given here is succinct. A similar data base for the United States has been developed by Christensen and Jorgenson (1969, 1970). The major difference between their accounting framework and ours is our exclusion of household durables as an input into production and as an output.

The data for the three inputs will be described first. The value of imports equals the value of imports included in GNE adjusted to exclude payments for imported labour and capital services and to include the value of customs duties. The constant dollar quantity of imports is the constant dollar quantity of imports included in GNE excluding payments to labour and capital. An implicit price index for imports is calculated from the current and constant dollar series.

The price and quantity series for labour input were developed with the assistance of Statistics Canada. Data on the number employed and average man-hours were supplied by the Productivity unit within Statistics Canada. Labour input is adjusted for educational attainment using the method developed by Jorgenson and Griliches (1967). For Canada, information on earnings by education is available only for 1961. The educational attainment of the labour force is available in each census year since 1931. The structure of relative earnings for different levels of educational attainment is held constant throughout the period. The total value of labour payments equals the value of private labour payments including government enterprises plus the imputed value of labour services in the unincorporated sector. The imputation is made separately for agriculture and non-agriculture. The imputed wage is set equal to the average wage in each sector. An implicit price of labour services is derived from the value and quantity series.

The capital input data was derived from information on investment in fourteen sectors. There approximately ninety investment series that were used to calculate capital stocks. The perpetual inventory method was used and the lifetimes were in general eighty percent of the lifetimes used by Statistics Canada (1972, 1974b). The capital stocks were aggregated to provide an aggregate stock and the service flow was assumed to be proportional to the stocks. An implicit price of capital services was calculated from the quantity and value series. The latter was calculated as the sum of all payments that were not payments for imports, labour or indirect taxes on factor inputs.

The output variables are consumption and investment goods. Investment goods include inventories, durable exports, machinery and equipment, and construction produced by the private sector. All non-investment goods are consumer goods. The sub-categories were aggregated to privide an aggregate quantity of the two outputs and an implicit price was calculated. The data is contained in Statistics Canada (1974a).