Part III

An Analysis of the Concept of Real Value-Added

Chapter III.1

DUALITY, INTERMEDIATE INPUTS AND VALUE-ADDED*

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1. Introduction

Discussions of duality in production theory and in the analysis of cost and profit functions usually center around "completely" dual structures. In other words, the relationships are analyzed between the underlying production structure and its dual, where *all* quantities of commodities (and variable factors) are replaced by their prices or vice versa. However, the same theory can also be extended and applied to "mixed" systems in which a partial set of "primal" variables is replaced by their "dual". An obvious candidate for a "restricted" profit function of this kind is the concept of value-added. The measure is extensively used in empirical work yet the concept has received only very partial treatment in production and duality theory.[1]

In an attempt to fill a small gap in the relevant theoretical literature Section 2 briefly develops the notion of value-added as a "hybrid" function: a production function which is concave in the "primary"

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[1]For the general duality results that are relevant in the present context, see Chapter I.1 by McFadden and Diewert (1974a) (where the term variable rather than restricted profit function is used). Related papers are those by Arrow (1974), Hall (1973), Denny and Pinto (Chapter V.1), Denny and May (Chapter III.3), and Diewert (Chapter III.2). The idea that duality can be affected in a step-by-step approach (and also the mention of value-added in that context) already appeared in Samuelson's classic paper (1953–54).

factors of production and at the same time a restricted profit function which is convex in the prices of the remaining "intermediate" inputs. This in itself is a useful theoretical device which could be of use in other areas.

Here we mainly illustrate some applications of this concept for the analysis of specification bias, in empirical work, due to the leaving out of intermediate inputs. It is shown in Sections 3 and 4 that questions such as the effect of value-added deflation methods and the correct measure of total productivity can be given a more general treatment within the above framework.[2] Section 5 briefly discusses an extension to the case of imperfect competition.

2. Duality and Value-Added

Consider a firm or an industry producing a single (composite)[3] gross output X and exogenously given price π with the aid of a production function,

$$X = X(\mathbf{L},\mathbf{M}), \tag{1}$$

where **L** is an n-vector of given "primary" inputs and **M** is an m-vector of "intermediate" inputs whose prices **p** (also an m-vector) will be assumed to be exogenously given.[4] Under profit maximization with respect to the intermediate inputs we must have

$$\pi X_{\mathbf{M}}(\mathbf{L},\mathbf{M}) = \mathbf{P}, \tag{2}$$

where $X_{\mathbf{M}}$ is the vector $\partial X/\partial\mu$ ($\mu \in \mathbf{M}$).

Nominal value-added is defined as

$$G = \pi X - \mathbf{P'M} \quad (\mathbf{P'} \text{ being the row-vector transpose of } \mathbf{P}).$$

[2]The literature on these questions seems to be remarkably small. The first and only systematic discussions of biases in the use of value-added measures are given by David (1962) and Domar (1961). Two more recent theoretical notes that are directly relevant are McFadden (1967) and Sims (1969). For recent empirical work, see Griliches and Ringstad (1971), and Denny (1974b).

[3]We shall throughout, for simplicity, stick to the single output and production function notions. The analysis could be extended to the case of output vectors and general transformation functions rather than production functions. For this extension, see Diewert's Chapter III.2.

[4]The distinction between "primary" and "intermediate" goods here is of no particular significance except to differentiate between quantities that are included and those which are excluded, and for which prices are substituted.

The Nominal Value-Added function (NVA) is defined as

$$G(\mathbf{L},\pi,\mathbf{P}) = \max_{X,\mathbf{M}} \{\pi X - \mathbf{P}'\mathbf{M} | X \leq X(\mathbf{L},\mathbf{M})\}$$
$$= \pi \hat{X}(\mathbf{L},\mathbf{P},\pi) - \mathbf{P}'\hat{\mathbf{M}}(\mathbf{L},\mathbf{P},\pi), \tag{3}$$

where $\hat{X},\hat{\mathbf{M}}$ are (restricted) profit maximizing functions of the quantities of primary inputs and the prices of output and intermediate inputs. The first-order equilibrium conditions (2) can now be written

$$\pi X_{\mathbf{M}}(\mathbf{L},\hat{\mathbf{M}}(\mathbf{L},\mathbf{P},\pi)) = \mathbf{P}. \tag{2'}$$

For much of our discussion we shall assume the underlying function X to possess continuous first- and second-order partial derivatives and a negative-definite Hessian matrix $\mathbf{H} = (X_{ij})$. It may be further restricted to have all factors gross substitutes so that the off-diagonal elements will be non-negative. The latter will be called the gross substitute (GS) case.

The partial derivatives of $\hat{\mathbf{M}}$ with respect to $\mathbf{L},\mathbf{P},\pi$ in (2') can be written down in the following vector derivative form:

$$\frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{L}} = -X_{\mathbf{MM}}^{-1} X_{\mathbf{ML}} \qquad (m \times n \text{ matrix}), \tag{4}$$

$$\frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{P}} = \frac{1}{\pi} X_{\mathbf{MM}}^{-1} \qquad (m \times n \text{ matrix}), \tag{5}$$

$$\frac{\partial \hat{\mathbf{M}}}{\partial \pi} = -\frac{1}{\pi^2} X_{\mathbf{MM}}^{-1} \mathbf{P}, \tag{6}$$

where $X_{\mathbf{MM}}$ is the $m \times m$ submatrix of the Hessian X_{ij} related to the intermediate inputs $\mathbf{M}$ and $X_{\mathbf{MM}}^{-1}$ is its inverse.[5] $X_{\mathbf{ML}}$ is an $m \times n$ submatrix of the Hessian related to the substitution effects between intermediate and primary inputs.

If we analyze the first and second partial derivatives of the function $G(\mathbf{L},\mathbf{P},\pi)$ we obtain the following properties:

(1) Extending standard duality theory,[6] the dual derivative property here holds simultaneously on the "hybrid" function G; i.e., we have (denot-

[5] We note that in the gross substitute case $\mathbf{X}_{\mathbf{MM}}^{-1}$ will consist of negative elements only. See McKenzie (1960).

[6] See Chapter I.1 by McFadden. For the present mathematical context see Samuelson (1960).

ing the shadow prices of **L** by the n-vector **W***)

$$G_{\mathbf{L}} = \pi X_{\mathbf{L}} = \mathbf{W}^*, \tag{7}$$

$$G_{\pi} = \hat{X}, \qquad G_{\mathbf{P}} = -\hat{\mathbf{M}}. \tag{8}$$

In other words, the derivative of G with respect to the prices gives the optimal gross output and intermediate input bundle; while its derivative with respect to the primary factors gives the optimal primary factor shadow price vector. If G is actually maximized (in competitive markets) with respect to **L** then $\mathbf{W}^* = \mathbf{W}$, the exogenous vector of factor prices of the primary inputs.

(2) The Hessian $G_{\mathbf{LL}}$ of G with respect to the factors **L** is negative definite,[7]

$$\begin{aligned} G_{\mathbf{LL}} &= \pi X_{\mathbf{LL}} + \pi X_{\mathbf{LM}} \cdot \frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{L}} \\ &= \pi (X_{\mathbf{LL}} - X_{\mathbf{LM}} X^{-1}_{\mathbf{MM}} X_{\mathbf{LM}}). \end{aligned} \tag{9}$$

(3) The Hessian of G with respect to the prices **P** is positive-definite,

$$G_{\mathbf{PP}} = -\frac{1}{\pi} X^{-1}_{\mathbf{MM}}. \tag{10}$$

We have thus established the dual concave–convex or *saddle-point property* of the function G with respect to **L**,**P**. We also note that for the mixed derivatives $G_{\mathbf{PL}}$ we get [cf. (4)][8]

$$G_{\mathbf{PL}} = -\frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{L}} = X^{-1}_{\mathbf{MM}} X_{\mathbf{ML}}.^{9} \tag{11}$$

$G_{\mathbf{PL}}$ gives the total response of **M** with respect to a change in **L** (allowing for the adjustment in X).

[7] If **H** is also GS so will be $G_{\mathbf{LL}}$, i.e., we have, in the gross substitute case $G_{lk} < 0$ for $l = k$, $G_{lk} \geq 0$ for $l \neq k$, $l,k \in \mathbf{L}$.

[8] For the output price π, we get

$$G_{\pi\mathbf{P}} = \partial \hat{X} / \partial \mathbf{P} = \partial \hat{\mathbf{M}} / \partial \pi = -(1/\pi^2) X^{-1}_{\mathbf{MM}} \mathbf{P}. \tag{11'}$$

[9] Similarly for the mixed $\mathbf{L} - \pi$ derivatives, we have

$$G_{\pi\mathbf{L}} = X_{\mathbf{L}} - (1/\pi) P' X^{-1}_{\mathbf{MM}} X_{\mathbf{ML}}, \tag{11''}$$

which is the total response of X with respect to **L** allowing for the adjustment of **M**.

(4) If X is linear homogeneous in **L**,**M**, then so is $G, G_{\mathbf{P}}$ and G_π in terms of **L**,

$$\mathbf{L}'G_{\mathbf{L}x} = G'_x, \qquad x = 0, \mathbf{P}, \pi.^{10} \tag{12}$$

(5) Irrespective of whether X is homogeneous or not, G, G_{L} are linear homogenous in terms of prices,

$$G_{y\mathbf{P}}\mathbf{P} + G_{y\pi} = G_y, \qquad y = 0, \mathbf{L}. \tag{13}$$

3. Value-Added in Constant Prices

G was so far defined as the Nominal Value-Added function (NVA). Over time this may vary as prices π,**P** vary.[11] Consider first a modified definition of the value-added function based on *single* output price (π) deflation, call it SVA, and denote it by g. We have $g = G/\pi$ and $\mathbf{P}/\pi = X_{\mathbf{M}} = \mathbf{p}$, say. We derive the following only slightly modified formulae for $g = g(\mathbf{L};\mathbf{p})$:

$$\begin{aligned} &g_{\mathrm{L}} = X_{\mathrm{L}}, \qquad g_{\mathbf{p}} = -\mathbf{M}, \\ &g_{\mathbf{pL}} = X_{\mathrm{MM}}^{-1}X_{\mathrm{ML}}, \qquad g_{\mathbf{pp}} = -X_{\mathrm{MM}}^{-1}, \qquad \text{etc.} \end{aligned} \tag{14}$$

Next consider the *double deflation procedure* for real value-added. Call it DVA for short and denote it by the function F. This is defined as $F = \hat{X} - \hat{\mathbf{M}}'\mathbf{e}$, where **e** denotes a vector of 1's. In other words, we normalize prices in a base period to be all equal to 1 and measure real value-added in base-year output and input prices.

Using our previous definition we can write F as a function of **L** and **p**,

$$F = g + g_{\mathbf{p}}'(\mathbf{e} - \mathbf{p}), \tag{15}$$

from which it follows that

$$F_{\mathbf{p}} = g_{\mathbf{p}} - g_{\mathbf{p}} + g_{\mathbf{pp}}(\mathbf{e} - \mathbf{p}) = -X_{\mathrm{MM}}^{-1}(\mathbf{e} - \mathbf{p}), \tag{16}$$

$$F_{\mathrm{L}} = g_{\mathrm{L}} + g_{\mathbf{pL}}(\mathbf{e} - \mathbf{p}) = X_{\mathrm{L}} + X'_{\mathrm{ML}}X_{\mathrm{MM}}^{-1}(\mathbf{e} - \mathbf{p}) = X_{\mathrm{L}} - X'_{\mathrm{ML}}F_{\mathbf{p}}. \tag{17}$$

Equations (16) and (17) form a convenient frame of reference for the analysis of possible biases that are introduced when DVA production

[10]The subscript 0 means "no subscript".

[11]In cross-section estimation, however, π,**P** may be assumed constant thus justifying use of G as a "pure" production function.

functions are used as a substitute for the "true" gross output relationships.[12] One important criterion is the preservation of relative marginal primary factor productivities.[13] For any pair of primary factors $l, k \in \mathbf{L}$, this condition can be expressed as the vanishing of the following scalar products:

$$\left(\frac{X_{l\mathbf{M}}}{X_l} - \frac{X_{k\mathbf{M}}}{X_k}\right) F_{\mathbf{p}} = 0, \qquad \forall l, k \in \mathbf{L}, \tag{18}$$

since

$$\frac{F_l}{F_k} = \frac{X_l}{X_k}\left(\frac{1-(X'_{\mathbf{M}l}/X_l)F_{\mathbf{p}}}{1-(X'_{\mathbf{M}k}/X_k)F_{\mathbf{p}}}\right) = \frac{X_l}{X_k}\left(\frac{1-(X_{l\mathbf{M}}/X_l)F_{\mathbf{p}}}{1-(X_{k\mathbf{M}}/x_k)F_{\mathbf{p}}}\right).$$

Under what general alternative conditions can (18) be satisfied?

By looking at (18) and (16) one can see that such conditions can conveniently be divided up into cases for which $F_{\mathbf{p}} = 0$, namely $\mathbf{p} = \mathbf{e}$ (constant relative prices), or letting $X^{-1}_{\mathbf{MM}} \to 0$ (constant intermediate factor proportions), and a third case,

$$\frac{X_{l\mathbf{M}}}{X_l} = \frac{X_{k\mathbf{M}}}{X_k},$$

which will turn out to be identical with functional separability of X in $\mathbf{L}$ and $\mathbf{M}$.

Let us first dispose of the case $X^{-1}_{\mathbf{MM}} \to 0$. On inspection this turns out to lead to the limiting case of fixed intermediate input proportions, which is a traditional argument in production function estimation for leaving out intermediate inputs and working only with value-added. This case can be subsumed under the functional separability of X category, since, if fixed proportions obtain, $X = \min\{z(\mathbf{L}), a_1 M_1, \ldots, a_m M_m\}$ which is clearly separable in $\mathbf{L}$ and $\mathbf{M}$. However, we can see heuristically the correspondence of fixed proportions to the case $X^{-1}_{\mathbf{MM}} \to 0$ as follows. From (16), $X^{-1}_{\mathbf{MM}} \to 0 \Rightarrow g_{\mathbf{pp}} \to 0 \Rightarrow \partial\hat{\mathbf{M}}/\partial\mathbf{p} \to 0$. But if the optimal $\mathbf{M}$ is independent of the relative price vector $\mathbf{p}$, $\mathbf{M}$ must be used in fixed proportions to X. The underlying production function is depicted in Figure 1 for a single component of $\mathbf{M}$ and a fixed level of $\mathbf{L}$.

Under our original assumptions we must have $X^{-1}_{\mathbf{MM}} \neq 0$.[14] It follows

[12]Cf. David (1962).

[13]Note that there is nothing in (17) that guarantees $F_{\mathbf{L}} > 0$ (if $X_{\mathbf{L}} > 0$). If we assume gross substitution and $\mathbf{p} \geq \mathbf{e}$, then $F_{\mathbf{L}} \geq X_{\mathbf{L}} \geq 0$. Linear homogeneity, however, would be preserved since $\mathbf{L}'F_{\mathbf{L}} = F$ if $\mathbf{L}'g_{\mathbf{L}} = g$.

[14]However, a row of $X^{-1}_{\mathbf{MM}}$ may be non-zero and yet consist of sufficiently "small" elements to make the above case of "almost" fixed proportions empirically relevant. Strictly speaking $X_{\mathbf{M}}$ would not be defined at the relevant point.

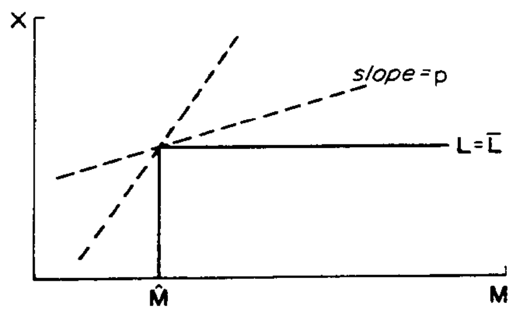

FIGURE 1

that we can have $F_{\mathbf{p}}$ vanish identically if and only if $\mathbf{p}=\mathbf{e}$, i.e., the relative prices of all intermediate inputs stay constant. This second special "composite good" case may be of no less empirical usefulness than the fixed factor proportions case.[15]

Consider now the more interesting third general case in which (18) will be satisfied, namely,

$$\frac{X_{l\mathbf{M}}}{X_l}=\frac{X_{k\mathbf{M}}}{X_k}, \qquad \forall l, k \in \mathbf{L}. \tag{19}$$

Equation (19) can alternatively be written in the form

$$\frac{\partial}{\partial \mathbf{M}}\left(\ln \frac{X_l}{X_k}\right)=0.$$

It follows that X_l/X_k is a function only of factors excluded from **M**, i.e., included in **L**. By Leontief's (1947) functional separability theorem it follows that X can be written in the form $X(A(\mathbf{L}),B(\mathbf{M}))$.

The converse is also true. Suppose X is functionally separable in the above form. It follows that

$$X_{l\mathbf{M}}=\frac{\partial}{\partial \mathbf{M}}(X_A A_l)=A_l X_{A\mathbf{M}}=X_l\frac{X_{A\mathbf{M}}}{X_A}.$$

Thus $X_{l\mathbf{M}}/X_l = X_{A\mathbf{M}}/X_A$ for all $l \in \mathbf{L}$, and (19) follows.

It is important to note that in this analysis we have nowhere had to assume anything about homogeneity or gross substitution.

Let us summarize these results in a theorem:

Theorem 1. Double-deflation of value-added (DVA) leads to a

[15]It is also at least as easy to test empirically. For an extensive analysis of this case, see Diewert (Chapter III.2).

derived value-added production function whose partial derivatives will correctly measure the marginal productivities of the primary factors if each intermediate input satisfies one of three conditions:

(1) these inputs are used in fixed proportions to gross output, or
(2) relative prices of intermediate inputs remain constant, or
(3) the original gross-output production function is functionally separable into the intermediate and all primary inputs.

The theorem is correct for *any* underlying production function whether linearly homogeneous or not.

As Sims (1969) has shown, in the case of functional separability, double deflation can be interpreted as a simple Divisia index. The above theorem complements this finding by showing that other than the two dual composite good cases, functional separability is the only global relative factor-price preserving case.

We note, going back to equation (14), that single deflation value-added (SVA) functions would be free of the above bias even without any special assumptions.[16] It might thus be asked what the practical use of double deflation is. Clearly there are practical applications where the distinction between the g- and the F-functions becomes important. One of these is in the measurement of real total factor productivity which is taken up in the next section. Before we move on, however, let us point to one application in a different field.[17] Consider the case of a small open economy that faces given import and export prices and imposes domestic ad-valorem tariffs and subsidies. If we measure the quantities of all tradeable outputs and intermediate inputs in terms of foreign currency expenditure (or revenue) then unit prices of goods will equal $\mathbf{P}=\mathbf{e}+\mathbf{T}$ where $\mathbf{T}$ is the vector of tariff (subsidy) rates. G will then be value-added in *domestic* prices while F will be value-added in *international* prices. Questions such as the effect of tariff structure and tariff change on resource allocation and real (foreign exchange) output can thus conveniently be analyzed in terms of the various value-added function concepts.[18] The great advantage of working with VA functions here as

[16]This point has been discussed by David (1962) and Diewert (Chapter III.2).
[17]This is discussed in greater detail in another paper by Bruno (1973).
[18]A similar candidate that suggests itself might be the study of domestic value-added tax systems.

well as in other uses is the fact that it is a natural vehicle for aggregation across industries.[19]

4. The Measurement of Total Productivity

The measurement of the "residual" total productivity of an industry or an economy is usually done by assessing the contribution of various primary factors of production to output measured in terms of value-added. The effect of intermediate inputs is for various reasons usually left out.[20] An interesting question in the present context is: Under what alternative set of assumptions could one infer, from the observed estimates in terms of value-added, the "true" measure of total productivity (in terms of the underlying gross output function)? Alternatively, what is the bias introduced by leaving out intermediate inputs?

Suppose the original X function takes the form $X = X(\mathbf{L},\mathbf{M};T)$ where T, a scalar, stands for the "technical progress" shift factor. For the SVA function we have $g = g(\mathbf{L},\mathbf{p};T)$, and

$$\frac{\mathrm{d}g}{\mathrm{d}T} = \frac{\partial g}{\partial T} + g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} + g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T},$$

$$\begin{aligned} g_T = \frac{\partial g}{\partial T} &= \frac{\mathrm{d}g}{\mathrm{d}T} - g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} - g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} \\ &= \frac{\mathrm{d}X}{\mathrm{d}T} - \mathbf{p}'\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}T} - \mathbf{M}'\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} - g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} - g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} \\ &= \frac{\mathrm{d}X}{\mathrm{d}T} - X'_{\mathbf{M}}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}T} + g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} - g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} - g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} \\ &= \frac{\partial X}{\partial T} = X_T. \end{aligned}$$

Therefore the actual estimate of technical progress can be represented by either g_T or X_T. However, the observed estimate of technical progress obtained from the SVA function may ignore variations in $\mathbf{p}$ and be given by

$$\frac{\mathrm{d}g}{\mathrm{d}T} - g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} = X_T + g'_{\mathbf{p}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T},$$

[19]E.g., if we define the economy-wide aggregate value added as $\bar{G} = \Sigma\, G_i$ and denote the complete price vector by $\bar{\mathbf{P}}$ we have $\bar{G}_{\bar{\mathbf{P}}}$ = net sales of *final* goods (domestic purchases and sales of intermediate goods cancel out), $\bar{G}_{\bar{\mathbf{P}}} = \mathbf{W}$, etc.

[20]Exceptions where intermediate goods were explicitly included are empirical studies of Griliches and Ringstad (1971) for Norway, and Denny (1974b) for Canada.

or

$$\frac{dg}{g} - \frac{g'_{\mathbf{L}}}{g}\,d\mathbf{L} = \left(\frac{X}{g}\right)\left(\frac{X_T}{X}\right)dT - \frac{g'_{\mathbf{p}}}{g}\,d\mathbf{p}. \tag{20}$$

The second term on the right-hand side will in general not be zero and thus observations on the left-hand side must give biased estimates of $(X_T/X)\,dT$ (a scaling factor g/X has to be put in anyway). Does double deflation solve this problem?

The derived F-function will be of the form $F = F(\mathbf{L};\mathbf{p},T)$ and we have

$$\begin{aligned} &F(\mathbf{L},\mathbf{p},T) = g(\mathbf{L},\mathbf{p},T) + (\mathbf{e}' - \mathbf{p}')g_{\mathbf{p}}, \\ &F_T = g_T + (\mathbf{e}' - \mathbf{p}')g_{\mathbf{p}T} = g_T + (\mathbf{e}' - \mathbf{p}')X^{-1}_{\mathbf{MM}}X_{\mathbf{M}T}, \end{aligned} \tag{21}$$

or

$$F_T = g_T - F'_{\mathbf{p}}X_{\mathbf{M}T}.$$

For the total differential dF/F we have

$$\frac{dF}{F} = \frac{F'_{\mathbf{L}}}{F}\,d\mathbf{L} + \frac{F'_{\mathbf{p}}}{F}\,d\mathbf{p} + \frac{F_T}{F}\,dT. \tag{22}$$

Suppose we start from observations on changes in double deflated value-added (DVA) dF/F and on factor input changes ($d\mathbf{L}$) and competitive income shares $(g_l/g)\cdot l$ ($l \in \mathbf{L}$). Suppose we want to use these in order to estimate the total productivity change in terms of the original function X (corrected for output scale),

$$q = \left(\frac{X}{F}\right)\frac{X_T}{X}\,dT = \frac{X_T}{F}\,dT. \tag{23}$$

The *observed* rate of change in total productivity is given by

$$\tilde{q} = \frac{dF}{F} - \frac{g'_{\mathbf{L}}}{g}\,d\mathbf{L}. \tag{24}$$

We first note that the bias as developed in (20) can be eliminated if we define $dF/F = dg/g - (g'_{\mathbf{p}}/g)\,d\mathbf{p}$, i.e., measure F as a variable weight divisia index. However, when F is computed by the usual double deflation procedure, fixed base period weights are used and a bias results. This bias can be measured by the difference $\tilde{q} - q$.

We can compare q and $\tilde{q}$ by deriving dF/dT in terms of the SVA function g. Totally differentiating (21) with respect to T we obtain

$$\frac{dF}{dT} = g_T + g'_{\mathbf{L}}\frac{d\mathbf{L}}{dT} + g'_{\mathbf{p}}\frac{d\mathbf{p}}{dT} + (\mathbf{e}' - \mathbf{p}')\left(g_{\mathbf{p}T} + g_{\mathbf{pL}}\frac{d\mathbf{L}}{dT} + g_{\mathbf{pp}}\frac{d\mathbf{p}}{dT}\right) - g'_{\mathbf{p}}\frac{d\mathbf{p}}{dT},$$

$$\frac{\mathrm{d}F}{\mathrm{d}T} = X_T + g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} + (\mathbf{e}' - \mathbf{p}')\left(g_{\mathbf{p}T} + g_{\mathbf{pL}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} + g_{\mathbf{pp}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T}\right),$$

$$\frac{\mathrm{d}F}{\mathrm{d}T} - g'_{\mathbf{L}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} = X_T - F'_{\mathbf{p}}X_{\mathbf{MM}}\left(X^{-1}_{\mathbf{MM}}X_{\mathbf{M}T} + X^{-1}_{\mathbf{MM}}X_{\mathbf{ML}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T} - X^{-1}_{\mathbf{MM}}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T}\right)$$
$$= X_T + F'_{\mathbf{p}}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}T} - X_{\mathbf{M}T} - X_{\mathbf{ML}}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}T}\right). \tag{25}$$

Dividing both sides of (24) by F we obtain

$$\frac{\mathrm{d}F}{F} = \frac{g'_{\mathbf{L}}}{g}\left(\frac{g}{F}\right)\mathrm{d}\mathbf{L} = \frac{X_T}{F}\,\mathrm{d}T + \frac{F'_{\mathbf{p}}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T - X_{\mathbf{ML}}\,\mathrm{d}\mathbf{L}),$$

$$\frac{\mathrm{d}F}{F} - \frac{g'_{\mathbf{L}}}{g}\mathrm{d}\mathbf{L} = \frac{X_T}{F}\,\mathrm{d}T + \frac{F'_{\mathbf{p}}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T - X_{\mathbf{ML}}\,\mathrm{d}\mathbf{L}) + \frac{g'_{\mathbf{L}}}{g}\left(\frac{g-F}{F}\right)\mathrm{d}\mathbf{L}$$

$$= \frac{X_T}{F}\,\mathrm{d}T + \frac{F'_{\mathbf{p}}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T - X_{\mathbf{ML}}\,\mathrm{d}\mathbf{L})$$
$$+ \frac{g'_{\mathbf{L}}}{g}\cdot\frac{1}{F}\cdot F'_{\mathbf{p}}X_{\mathbf{MM}}\,\mathrm{d}\mathbf{L} \tag{26}$$

$$= \frac{X_T}{F}\,\mathrm{d}T + \frac{F'_{\mathbf{p}}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T + Z_{\mathbf{ML}}\,\mathrm{d}\mathbf{L}),$$

where

$$Z_{\mathbf{ML}} \doteq \frac{g'_{\mathbf{L}}X_{\mathbf{MM}}}{g} - X_{\mathbf{ML}}.$$

Finally, using (26), (23), and (24), we obtain

$$\tilde{q} = q + \frac{F'_{\mathbf{p}}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T + Z_{\mathbf{ML}}\,\mathrm{d}\mathbf{L}). \tag{27}$$

For the two dual special cases mentioned in Section 3, $F_{\mathbf{p}} = 0$ and $q = \tilde{q}$. This would, for example, be the case for small changes in F when using a divisia index so that always $\mathbf{p} = \mathbf{e}$, i.e., shifting base-year weights in measuring $\mathrm{d}F/F$ will lead to unbiased estimates of q.[21]

What if $F_{\mathbf{p}} \neq 0$? Consider the expression (27). The bias can be simplified if $Z_{\mathbf{ML}} = 0$. If $Z_{\mathbf{ML}} = 0$, then $X_{\mathbf{L}}X_{\mathbf{MM}}/X_{\mathbf{M}} - X_{\mathbf{ML}} = 0$, or

$$\frac{X_l}{X_{\mathbf{M}l}} = \frac{X_k}{X_{\mathbf{M}k}}, \qquad l \in \mathbf{L}, \quad k \in \mathbf{L}. \tag{28}$$

[21]We should keep in mind that to get at X_T/X a scaling factor must always be brought in [as in (23) where $X_T/X = (1 - \boldsymbol{\gamma}'\mathbf{e})q$ for $\boldsymbol{\gamma} = \mathbf{M}/X$, the base-year share of intermediate goods]. Domar (1961) makes this point in the context of the Cobb–Douglas case.

It can be shown that in order for (28) to hold the production function $X = X(\mathbf{L},\mathbf{M})$ must be additively separable of the form

$$X(\mathbf{L},\mathbf{M}) = H\left(J(\mathbf{L}) + \sum_{\mathbf{M}_k \in \mathbf{M}} \alpha_k \mathbf{M}_k\right).$$

Suppose we assume additive separability (as is the case in most empirical work, e.g., Cobb–Douglas, C.E.S.), then (27) takes the form

$$\tilde{q} = q - \frac{(\mathbf{e}' - \mathbf{p}')X_{\mathbf{MM}}^{-1}}{F}(\mathrm{d}\mathbf{p} - X_{\mathbf{M}T}\,\mathrm{d}T).$$

In general, it is hard to make any a priori predictions about the direction of bias. One can expect $X_{\mathbf{M}T} > 0$. If all $\mu \in \mathbf{M}$ are gross substitutes then $X_{\mathbf{MM}}^{-1}$ will consist of negative elements. But even in this case, only if intermediate goods prices are non-increasing ($\mathbf{p} \leq \mathbf{e}, \mathrm{d}\mathbf{p} \leq 0$) can we unambiguously say that $\tilde{q} < q$.

If one had full information on $\mathbf{M}$ and $\mathbf{p}$ over time then, of course, the above discussion of the bias is irrelevant since one could always set up a divisia index or else estimate the separate effect of the intermediate inputs directly. More often than not, however, there might exist only a point estimate of $\mathbf{M}/X$ and a rough indication of the direction of movement of $\mathbf{p}$, so that such considerations may be of use. Let us summarize:

Theorem 2. The measurement of total productivity change from constant price value-added figures (F) will be unbiased (except for a scaling factor F/X) if each intermediate input ($\mu \in \mathbf{M}$) left out is either used in fixed proportions or else a divisia (shifting weight) index is used to obtain F from the underlying intermediate input ($\mathbf{M}$) and gross output (X) figures. The bias in the case of single output price deflation is given by (20) and for double deflation with constant base-year weights is given by (27).

The technique of step-by-step elimination of factors for productivity measurement need not, of course, stop at the level of "primary" factors as conventionally defined. For example, suppose we measure F correctly and we have $F = F(l,k;T)$ where l = labor and k = capital. Now suppose we are in an economy in which producers plan their investments on the basis of a long-run rate of interest (r) that is assumed to be fixed. We can now further net out $F - rk$ and are left with a "net" valued-added (i.e., wage) function that will only be a function of the remaining primary factor labor. If the production function for F is assumed Cobb–Douglas and linearly homogenous and technical progress is exponential there is no way to choose between making projections on the basis of a Cobb–Douglas function with l and k as arguments or on the basis of a naive fixed exponential output per unit of labor (F/l) growth model. In the absence of data on the capital stock the latter is often used in practical long-term projection work. With the above rationalization in mind this may not be as naive and mis-specified as it might look at first sight.[22]

5. Introducing Imperfect Competition

So far we have analyzed the value-added function under the assumption that prices π,**P** were given exogeneously. Is there any sense in which one could work with a concept like the G-function when there is imperfect competition in the output and/or intermediate input markets and prices become endogenous? For simplicity we shall only deal with the case in which the demand elasticity for X (ϵ, say) and the supply elasticity for each μ (θ_μ, say) are given and fixed. Instead of (2) we now have

$$X_\mu = \frac{\mathbf{P}_\mu(1 + 1/\theta_\mu)}{\pi(1 - 1/\epsilon)}, \qquad \forall \mu \in \mathbf{M}. \tag{29}$$

Substituting for **L** from (29) into $G = \pi(X)X - P(\mathbf{M})'\mathbf{M}$ and now looking at G as a function of **L** only, gives

$$G_\mathbf{L} = \left(1 - \frac{1}{\epsilon}\right) X_L \qquad (= \text{marginal value product of } \mathbf{L}). \tag{30}$$

If X is linearly homogenous we get

$$\begin{aligned} G'_\mathbf{L}\mathbf{L} &= \left(1 - \frac{1}{\epsilon}\right) X'_\mathbf{L}\mathbf{L} = \pi \left(1 - \frac{1}{\epsilon}\right)(X - X'_\mathbf{M}\mathbf{M}) \\ &= G - \frac{\pi}{\epsilon} X - \sum_\mu \frac{P_\mu \mu}{\theta_\mu} < G. \end{aligned} \tag{31}$$

The G function thus preserves the relative marginal value products of

[22]Looked at from a different point of view this would be an application of the non-substitution theorem.

L. Its degree of homogeneity will be lower than that of the underlying X function.[23]

The extension of the VA function concept to the case of imperfect competition can be of use in both production function estimation as well as for the analysis of an economy trading in an imperfect world market.

6. Conclusions

In this paper we have utilized the duality between profit and production functions to analyze double deflated value-added (DVA) functions. DVA functions will correctly measure marginal productivities of primary factors when at least one of the following restrictive assumptions are satisfied: (1) intermediate goods are used in fixed proportion to gross output, (2) prices of intermediate goods relative to that of gross output remain constant, or (3) the gross output production function is weakly separable in the primary and intermediate factor groups. When base weight DVA functions are used to measure growth in total productivity, functional separability is no longer sufficient to eliminate the resultant bias. Also, in this case the direction of the bias is unknown unless the price of intermediate goods relative to the price of output is non-increasing.

[23]Thus we might allow for increasing returns in X and yet have a "well-behaved" value-added function.