Appendix A.3

CONVEX ANALYSIS

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1. Introduction

This appendix lists mathematical properties of convex sets and convex conjugate functions used in developing the theory of cost and profit functions. Familiarity with the basic concepts of analysis (e.g., open and closed sets in Euclidean space, interior and closure of sets, compactness, continuity of real-valued functions) is assumed at the level of Rosenlicht (1968) or Bartle (1964). Brief introductions to the theory of convex sets can be found in Karlin (1959) or Mangasarian (1969). More advanced surveys are in Busemann (1958), Fan (1959), Fenchel (1953), Grunbaum (1967), Klee (1963, 1969), and Valentine (1964). The definitive reference work on the topic is Rockafellar (1970).

2. Notation

The results in this appendix will be stated for sets and functions in an N-dimensional Euclidean space $\mathbf{E}^N$. Subsets of $\mathbf{E}^N$ are denoted by boldface Roman caps (e.g., $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$), and points in $\mathbf{E}^N$ by lower case, boldface Roman letters (e.g., $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$). Real numbers are denoted by lower case Greek letters (e.g., α, β, σ, θ). Real-valued functions on $\mathbf{E}^N$ are denoted by Roman caps (e.g., F, G, H). The interior and closure of a set $\mathbf{Y}$ are denoted by $\operatorname{int} \mathbf{Y}$ and $\bar{\mathbf{Y}}$, respectively. The algebraic sum of non-empty sets $\mathbf{Y}$, $\mathbf{Z}$ is defined by $\mathbf{Y} + \mathbf{Z} = \{\mathbf{y} + \mathbf{z} | \mathbf{y} \in \mathbf{Y}, \mathbf{z} \in \mathbf{Z}\}$. The set of points in $\mathbf{Y}$, but not in $\mathbf{Z}$, is denoted by $\mathbf{Y} \setminus \mathbf{Z}$.

3. Hyperplanes

A real-valued linear function P on $\mathbf{E}^N$ can be represented by a vector $\mathbf{p} \in \mathbf{E}^N$, with the value of P at $\mathbf{u} \in \mathbf{E}^N$ given by the inner product $P(\mathbf{u}) = \mathbf{p}\cdot\mathbf{u}$. A *hyperplane* is a set $\mathbf{H}(\mathbf{p},\alpha) = \{\mathbf{v} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{v} = \alpha\}$, where $\mathbf{p} \in \mathbf{E}^N$ is non-zero. Note that a hyperplane is a level set of a non-identically zero real-valued linear functional. The sets $\mathbf{H}^-(\mathbf{p},\alpha) = \{\mathbf{v} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{v} \leqq \alpha\}$ and $\mathbf{H}^+(\mathbf{p},\alpha) = \{\mathbf{v} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{v} \geqq \alpha\}$ are termed the *closed half-spaces* determined by the hyperplane $\mathbf{H}(\mathbf{p},\alpha)$, and $\mathbf{p}$ is termed the *normal* to $\mathbf{H}(\mathbf{p},\alpha)$. A hyperplane $\mathbf{H}(\mathbf{p},\alpha)$ is a *barrier* to a non-empty set $\mathbf{Y}$ if $\mathbf{Y}$ is contained in a closed half-space determined by $\mathbf{H}(\mathbf{p},\alpha)$, and $\mathbf{H}(\mathbf{p},\alpha)$ *supports* $\mathbf{Y}$ if it is a barrier to $\mathbf{Y}$ and intersects $\bar{\mathbf{Y}}$.

4. Convex Sets

A set $\mathbf{Y}$ is *convex* if $\mathbf{u}, \mathbf{v} \in \mathbf{Y}, 0 < \theta < 1$ implies $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in \mathbf{Y}$. The *(closed) convex hull* of a set $\mathbf{X}$ is the intersection of all the (closed) convex sets containing $\mathbf{X}$. The convex hull of $\mathbf{X}$ is denoted by $[\mathbf{X}]$. The closed convex hull of $\mathbf{X}$ equals the closure of the convex hull of $\mathbf{X}$, and is denoted by $\overline{[\mathbf{X}]}$.

5. Affine Subspaces

A set $\mathbf{F}$ is a *flat* (or affine subspace) if $\mathbf{u},\mathbf{v} \in \mathbf{F}$ implies $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in \mathbf{F}$ for all real θ. Note that $\mathbf{E}^N$ itself is a flat; that points, lines, and hyperplanes in $\mathbf{E}^N$ are flats; that an arbitrary intersection of flats is a flat; that all flats are closed and convex; and that a non-empty flat is a translation of a linear subspace of $\mathbf{E}^N$. The *affine hull* of a set $\mathbf{X}$ is the intersection of all flats containing $\mathbf{X}$, and is denoted by aff $\mathbf{X}$. The *relative interior* of $\mathbf{X}$, denoted by intr $\mathbf{X}$, is the interior of $\mathbf{X}$ in the relative topology of $\mathbf{X}$ as a subset of aff $\mathbf{X}$; i.e., the set of points in $\mathbf{X}$ which are not in the closure of (aff $\mathbf{X}$)$\setminus\mathbf{X}$.

6. Separation of Sets

Non-empty sets $\mathbf{Y},\mathbf{Z}$ have the *separation property* if there exists a hyperplane $\mathbf{H}(\mathbf{p},\alpha)$ such that $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\alpha)$ and $\mathbf{Z} \subseteq \mathbf{H}^+(\mathbf{p},\alpha)$. They have the *strong separation property* if there exist parallel hyperplanes $\mathbf{H}(\mathbf{p},\alpha)$ and $\mathbf{H}(\mathbf{p},\beta)$ such that $\alpha < \beta$, $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\alpha), \mathbf{Z} \subseteq \mathbf{H}^+(\mathbf{p},\beta)$.

7. Cones

A set $\mathbf{K}$ is a *cone* with vertex at the origin if $\mathbf{v} \in \mathbf{K}$ and $\theta > 0$ imply $\theta\mathbf{v} \in \mathbf{K}$. $\mathbf{K}$ is a cone with vertex at $\mathbf{v}$ if $\mathbf{K} - \{\mathbf{v}\}$ is a cone with vertex at the origin. Hereafter, all cones are defined with vertex at the origin unless specified otherwise. The *(closed) cone with vertex* **v** *spanned by* a set **Y** is defined as the intersection of all (closed) cones with vertex **v** which contain **Y**, and denoted by $\mathbf{K}_{\mathbf{v}}\mathbf{Y}$ (for the closed cone, $\bar{\mathbf{K}}_{\mathbf{v}}\mathbf{Y}$). If **v** is the origin, the subscript is omitted. The *asymptotic cone* (recession cone) of a set **Y**, denoted by **AY**, is defined as follows: $\mathbf{v} \in \mathbf{AY}$ if and only if there exist a sequence $\mathbf{v}^k \in \mathbf{Y}$ and a sequence of positive real numbers θ_k such that $\theta_k \to 0$ and $\theta_k \mathbf{v}^k \to \mathbf{v}$. A cone **K** is *pointed* if $\mathbf{v}^i \in \bar{\mathbf{K}}$, $i = 1,\dots,m$, for finite m and $\sum_{i=1}^m \mathbf{v}^i = \mathbf{0}$ implies $\mathbf{v}^i = \mathbf{0}$, $i = 1,\dots,m$. A set **Y** is *semi-bounded* if **AY** is pointed.

8. Polar and Normal Cones

The *polar cone* of a set **Y** is the set $\mathbf{PY} = \{\mathbf{p} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{v} \leqq 0$ for all $\mathbf{v} \in \mathbf{Y}\}$. The *normal cone at* **v** of a set **Y** is the set of all $\mathbf{p} \in \mathbf{E}^N, \mathbf{p} \neq \mathbf{0}$, such that $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\mathbf{p}\cdot\mathbf{v})$, and is denoted by $\mathbf{N}(\mathbf{Y},\mathbf{v})$. The *normal cone* of **Y** is the union of $\mathbf{N}(\mathbf{y},\mathbf{v})$ for all $\mathbf{v} \in \mathbf{E}^N$, and is denoted by **NY**. Clearly, $\mathbf{p} \in \mathbf{NY}$ if and only if $\sup\{\mathbf{p}\cdot\mathbf{y} | \mathbf{y} \in \mathbf{Y}\} < +\infty$.

9. Convex Functions

A non-empty set **X** and real-valued function F with domain **X** are denoted by $\langle F,\mathbf{X}\rangle$. We say $\langle F,\mathbf{X}\rangle$ is *convex* if **X** is convex, and $\mathbf{u}, \mathbf{v} \in \mathbf{X}$, $0 < \theta < 1$ implies $F(\theta\mathbf{u} + (1-\theta)\mathbf{v}) \leqq \theta F(\mathbf{u}) + (1-\theta)F(\mathbf{v})$. We say $\langle F,\mathbf{X}\rangle$ is *positively linear homogeneous* if **X** is a cone and $\mathbf{v} \in \mathbf{X}$, $\theta > 0$ implies $F(\theta\mathbf{v}) = \theta F(\mathbf{v})$. We say $\langle F,\mathbf{X}\rangle$ is *closed* if the following conditions hold: (1) $\mathbf{v} \in \bar{\mathbf{X}} \setminus \mathbf{X}$ implies $\lim_{\mathbf{u}\in\mathbf{X},\mathbf{u}\to\mathbf{v}} \inf F(\mathbf{u}) = +\infty$; and (2) $\mathbf{v} \in \mathbf{X}$ implies $\lim_{\mathbf{u}\in\mathbf{X},\mathbf{u}\to\mathbf{v}} \inf F(\mathbf{u}) = F(\mathbf{v})$.[1] The *support function* $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle$ of a non-empty set **Y** is a real-valued function defined by $G^{\mathbf{Y}}(\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{y} | \mathbf{y} \in \mathbf{Y}\}$ for $\mathbf{p} \in \mathbf{NY}$.

10. Properties of Convex Sets

A list of well-known properties of non-empty sets **Y**,**Z** follows. For completeness, references to proofs, or abbreviated or shortened proofs, are given for the non-trivial results.

10.1 $\mathbf{Y} \subseteq [\mathbf{Y}] \subseteq [\bar{\mathbf{Y}}] \subseteq \overline{[\mathbf{Y}]}$.

[1] The notation $\lim_{\mathbf{u}\in\mathbf{X},\mathbf{u}\to\mathbf{v}} \inf F(\mathbf{u}) = F(\mathbf{v})$ is defined as follows: Given $\epsilon > 0$, there exists an open ball **N** about **v** with radius at most ϵ such that

$$\epsilon > |F(\mathbf{v}) - \inf_{\substack{\mathbf{u}\in\mathbf{N}\cap\mathbf{X} \\ \mathbf{u}\neq\mathbf{v}}} F(\mathbf{u})|.$$

10.2 $\mathbf{Y}$ convex implies intr $\mathbf{Y} \neq \emptyset$ and the following sets convex: int $\mathbf{Y}$, intr $\mathbf{Y}$, $\bar{\mathbf{Y}}$, $\mathbf{AY}$, $\mathbf{K_vY}$ for any $\mathbf{v} \in \mathbf{E}^N$.
Proof: Fenchel (1953, Ch. II, Results 11, 15, 17, 22), Rockafellar (1970, Thm 6.2).

10.3 $\mathbf{Y}$ bounded implies $\overline{[\mathbf{Y}]}$ bounded and $\mathbf{AY} = \{\mathbf{0}\}$.

10.4 $\mathbf{Y}$ compact implies $[\mathbf{Y}]$ compact.
Proof: Grunbaum (1967, 2.3.5).

10.5 $\mathbf{Y},\mathbf{Z}$ convex implies $\mathbf{Y} \cap \mathbf{Z}$ and $\mathbf{Y} + \mathbf{Z}$ convex.

10.6 $\mathbf{Y}$ a convex cone implies $\mathbf{Y} = \mathbf{Y} + \mathbf{Y}$.

10.7 $\mathbf{0} \in \mathbf{Y},\mathbf{Z}$ implies $\mathbf{KY} \subseteq \mathbf{K}(\mathbf{Y}+\mathbf{Z}) \subseteq \mathbf{KY} + \mathbf{KZ}$.

10.8 $\mathbf{0} \in \mathbf{Y},\mathbf{Z}$ and $\mathbf{Y},\mathbf{Z}$ convex implies $\mathbf{K}(\mathbf{Y}+\mathbf{Z}) = \mathbf{KY} + \mathbf{KZ}$.
Proof: Use 10.6 and 10.7.

10.9 $\mathbf{Y}$ closed and convex implies $\mathbf{Y} = \mathbf{Y} + \mathbf{AY}$.
Proof: Winter (forthcoming), Rockafellar (1970, Thm. 8.3). If $\mathbf{v} \in \mathbf{Y} + \mathbf{AY}$, then there exist $\mathbf{u} \in \mathbf{Y}$, $\mathbf{w} \in \mathbf{AY}$ with $\mathbf{v} = \mathbf{u} + \mathbf{w}$. Then there exist $\mathbf{w}^k \in \mathbf{Y}$, $\theta_k \geqq 0$ with $\theta_k \mathbf{w}^k \to \mathbf{w}$ and $\theta_k \to 0$, implying $(1-\theta_k)\mathbf{u} + \theta_k \mathbf{w}^k \in \mathbf{Y}$ for k large, and $(1-\theta_k)\mathbf{u} + \theta_k \mathbf{w}^k \to \mathbf{v} \in \mathbf{Y}$.

10.10 Every $\mathbf{v} \in [\mathbf{Y}]$ is expressible in the form $\mathbf{v} = \sum_{i=0}^{N} \theta_i \mathbf{v}^i$ with $\theta_i \geqq 0$, $\sum_{i=0}^{N} \theta_i = 1$, and $\mathbf{v}^i \in \mathbf{Y}$.

10.11 Convex sets $\mathbf{Y},\mathbf{Z}$ have the separation property if and only if one or both of the following are true: (1) (intr $\mathbf{Y}$) $\cap$ (intr $\mathbf{Z}$) $= \phi$, or (2) $\mathbf{Y} \cup \mathbf{Z}$ lies in a hyperplane.
Proof: Klee (1969, Thm. 2.1), Grunbaum (1967, 2.2.2), Rockafellar (1970, Thm. 11.3).

10.12 For $\mathbf{Y}$ convex, $\mathbf{Y}$ and $\{\mathbf{z}\}$ have the separation property if and only if $\mathbf{z} \notin$ int $\mathbf{Y}$. There exists a hyperplane $H(\mathbf{p},\alpha)$ with $\mathbf{z} \in H(\mathbf{p},\alpha)$, $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\alpha)$, and (intr $\mathbf{Y}$) $\cap \mathbf{H}(\mathbf{p},\alpha) = \emptyset$ if and only if $\mathbf{z} \notin$ intr $\mathbf{Y}$.
Proof: Klee (1969, Thm. 1.1), Rockafellar (1970, Thm. 11.3).

10.13 $\mathbf{Y},\mathbf{Z}$ convex and disjoint, $\mathbf{Y}$ closed, $\mathbf{Z}$ compact implies $\mathbf{Y},\mathbf{Z}$ have the strong separation property.
Proof: Grunbaum (1967, 2.2.1), Rockafellar (1970, Corol. 11.4.2).

10.14 $\mathbf{Y}$ convex, $\mathbf{Z}$ a non-empty flat, $\mathbf{Z} \cap$ intr $\mathbf{Y} = \emptyset$ implies the existence of a hyperplane $\mathbf{H}(\mathbf{p},\alpha)$ such that $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\alpha)$, $\mathbf{Z} \subseteq \mathbf{H}(\mathbf{p},\alpha)$, and $\mathbf{H}(\mathbf{p},\alpha) \cap$ intr $\mathbf{Y} = \emptyset$.
Proof: Rockafellar (1970, Thm. 11.2).

10.15 $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{p},\alpha)$ implies $\mathbf{p} \in \mathbf{NY}$.

10.16 int $\mathbf{P}(\mathbf{AY}) \subseteq \mathbf{NY} \subseteq \mathbf{P}(\mathbf{AY})$.
Proof: Rockafellar (1970, Corol. 14.2.1). If $\mathbf{p} \notin \mathbf{P}(\mathbf{AY})$, then $\mathbf{p}\cdot\mathbf{v} > 0$ for some $\mathbf{v} \in \mathbf{AY}$. There exist $\mathbf{v}^j \in \mathbf{Y}$, $\theta_j \geqq 0$ such that

$|\mathbf{v}^j| \to +\infty$ and $\theta_j \mathbf{v}^j \to \mathbf{v}$, implying $\mathbf{p}\cdot\mathbf{v}^j \to +\infty$, and hence $\mathbf{p} \notin \mathbf{NY}$. If $\mathbf{p} \notin \mathbf{NY}$, then there exist $\mathbf{v}^j \in \mathbf{Y}$ with $\mathbf{p}\cdot\mathbf{v}^j \to +\infty$. Then, $\mathbf{v}^j/|\mathbf{v}^j|$ has a subsequence converging to $\mathbf{v} \in \mathbf{AY}$ with $\mathbf{p}\cdot\mathbf{v} \geqq 0$. Then, $(\mathbf{p}+\epsilon\mathbf{v})\cdot\mathbf{v} > 0$ for all $\epsilon > 0$, implying $\mathbf{p}+\epsilon\mathbf{v} \notin \mathbf{P}(\mathbf{AY})$ and hence $\mathbf{p} \notin \operatorname{int} \mathbf{P}(\mathbf{AY})$.

10.17 $\mathbf{Y} \subseteq \mathbf{Z}$ implies $\mathbf{PZ} \subseteq \mathbf{PY}$.

10.18 $\mathbf{PY}$ is closed and convex, and $\mathbf{NY}$ is convex.

10.19 $\mathbf{Y} \subseteq \mathbf{P}(\mathbf{PY}) = \overline{[\mathbf{KY}]}$.

10.20 $\mathbf{0} \in \mathbf{Y},\mathbf{Z}$ implies $\mathbf{P}(\mathbf{Y}+\mathbf{Z}) = (\mathbf{PY}) \cap (\mathbf{PZ})$.

10.21 $\mathbf{Y} \cap \overline{\mathbf{PY}} \subseteq \{\mathbf{0}\}$.

10.22 $\mathbf{AY} = \overline{\mathbf{AY}} = \mathbf{A}\bar{\mathbf{Y}}$ and $NY = N\bar{Y}$.

11. Semi-Bounded Sets

The next series of results give properties of semibounded sets. Most of these properties can be obtained as consequences of theorems of Fenchel (1953), Grunbaum (1967), or Rockafellar (1970); however, we shall give direct proofs which are somewhat simpler.

11.1. *Lemma.* If $\mathbf{Y}$ is a closed pointed cone, then (1) there exists a positive scalar μ such that for any finite set of points $\mathbf{v}^i \in \mathbf{Y}$, $i = 1,\dots,m$, it follows that $|\mathbf{v}^i| \leqq \mu|\sum_{i=1}^m \mathbf{v}^i|$; and (2) $[\mathbf{Y}]$ is a closed pointed cone.

Proof: Rockafellar (1970, Corol. 9.1.2). We first establish the existence of a μ depending in general on m for which (1) is valid. Suppose, for fixed m, no μ with the required property exists. Then, there exist $\mathbf{v}^{ij} \in \mathbf{Y}$ such that $\sum_{i=1}^m \mathbf{v}^{ij} = \mathbf{u}^j$ with $|\mathbf{u}^j| \leqq 1$ and $\lambda_j = 1/\max_i |\mathbf{v}^{ij}| \to 0$. Hence, there is a subsequence of j (retain notation) such that $\lambda_j \mathbf{v}^{ij} \to \mathbf{w}^i \in \mathbf{Y}$ for $i = 1,\dots,m$ and $\sum_{i=1}^m \mathbf{w}^i = \mathbf{0}$. Since $1 \leqq \sum_{i=1}^m \lambda_j |\mathbf{v}^{ij}| \leqq m$, at least one $\mathbf{w}^i \neq \mathbf{0}$. This contradicts the hypothesis that $\mathbf{Y}$ is pointed.

We next prove that $[\mathbf{Y}]$ is closed. If $\mathbf{v}^i \in [\mathbf{Y}]$, $\mathbf{v}^i \to \mathbf{v}$, then there exist $\mathbf{v}^{ij} \in \mathbf{Y}$, $j = 0,\dots,N$, such that $\mathbf{v}^i = \sum_{j=0}^N \mathbf{v}^{ij}$. By the result just proved, the $\mathbf{v}^{ij}$ are bounded, and hence there is a subsequence of i (retain notation) such that $\mathbf{v}^{ij} \to \mathbf{u}^j \in \mathbf{Y}$. Then, $\mathbf{v} = \sum_{j=0}^N \mathbf{u}^j \in [\mathbf{Y}]$, and $[\mathbf{Y}]$ is closed.

If $[\mathbf{Y}]$ were not pointed, then there would exist non-zero $\mathbf{v}^i \in [\mathbf{Y}]$ with $\sum_{i=0}^m \mathbf{v}^i = 0$ (since $[\mathbf{Y}]$ is closed). Since each $\mathbf{v}^i = \sum_{j=0}^N \mathbf{v}^{ij}$ for some $\mathbf{v}^{ij} \in \mathbf{Y}$, the implication $\sum_{i=0}^m \sum_{j=0}^N \mathbf{v}^{ij} = \mathbf{0}$ would be obtained, contradicting the hypothesis that $\mathbf{Y}$ is pointed. Hence, (2) is verified.

By (2), the condition (1) proved for fixed m can be applied to $[\mathbf{Y}]$, establishing μ such that $|\mathbf{v}^i| \leqq \mu|\sum_{i=1}^2 \mathbf{v}^i|$ for $\mathbf{v}^i \in [\mathbf{Y}]$. But the sum of any finite sequence $\mathbf{u}^j \in [\mathbf{Y}]$ can be written as $\sum_{j=1}^m \mathbf{u}^j = \mathbf{u}^k + \mathbf{w}$ with $\mathbf{w} =$

$\sum_{j=1,j\neq k}^{m} \mathbf{u}^j \in [\mathbf{Y}]$, implying $|\mathbf{u}^k| \leqq \mu|\sum_{j=1}^{m}\mathbf{u}^j|$ for $k = 1,...,m$. This verifies (1). Q.E.D.

11.2. *Lemma.* If $\mathbf{Y}$ is non-empty and semi-bounded, then (1) given a positive scalar λ, there exists a positive scalar μ such that for any finite set of points $\mathbf{v}^i \in \mathbf{Y}$, and scalars $\theta_i \geqq 0, i = 1,...,m$, with $|\sum_{i=1}^{m} \theta_i \mathbf{v}^i| \leqq \lambda$, it follows that $|\theta_i \mathbf{v}^i| \leqq \mu$; (2) $\mathbf{A}[\mathbf{Y}] = \overline{[\mathbf{AY}]}$; (3) $[\mathbf{Y}]$ is semi-bounded; and (4) if $\mathbf{Y}$ is closed, then $[\mathbf{Y}] + \mathbf{A}[\mathbf{Y}] = \overline{[\mathbf{Y}]}$.

Proof: The argument parallels that of 11.1. If (1) is violated for fixed m, there exist $\mathbf{v}^{ij} \in \mathbf{Y}$ and $\theta_{ij} \geqq 0$ such that $\sum_{i=1}^{m} \theta_{ij}\mathbf{v}^{ij} = \mathbf{u}^j$ with $|\mathbf{u}^j| \leqq \lambda$ and $\lambda_j = 1/\max_i|\theta_{ij}\mathbf{v}^{ij}| \to 0$, implying the existence of limit points $\mathbf{w}^i \in \mathbf{AY}$ of the $\lambda_j\theta_{ij}\mathbf{v}^{ij}$ such that $\sum_{i=1}^{m} \mathbf{w}^i = \mathbf{0}$ and $\sum_{i=1}^{m} |\mathbf{w}^i| \geqq 1$, contradicting the hypothesis that $\mathbf{Y}$ is semi-bounded.

We next show $\mathbf{A}[\mathbf{Y}] \subseteq \overline{[\mathbf{AY}]}$. If $\mathbf{v} \in \mathbf{A}[\mathbf{Y}]$, there exist $\mathbf{u}^j \in [\mathbf{Y}]$, $\theta_j \geqq 0$, $\mathbf{v}^{ij} \in \mathbf{Y}$, $\lambda_{ij} \geqq 0$ such that $\theta_j \to 0$, $\theta_j\mathbf{u}^j \to \mathbf{v}$, $\sum_{i=0}^{N} \lambda_{ij} = 1$, and $\mathbf{u}^j = \sum_{i=0}^{N} \lambda_{ij}\mathbf{v}^{ij}$. By 11.2(1) for fixed m, $\theta_j\lambda_{ij}\mathbf{v}^{ij}$ is bounded in j, and hence there is a subsequence of j (retain notation) such that $\theta_j\lambda_{ij}\mathbf{v}^{ij} \to \mathbf{v}^i \in \mathbf{AY}$. Then, $\mathbf{v} = \sum_{i=0}^{N} \mathbf{v}^i \in [\mathbf{AY}]$. By 10.2, $\mathbf{A}[\mathbf{Y}]$ is convex. Then, $\mathbf{AY} \subseteq \mathbf{A}[\mathbf{Y}]$ implies $[\mathbf{AY}] \subseteq \mathbf{A}[\mathbf{Y}]$. Hence, (2) is verified.

$\mathbf{Y}$ semi-bounded implies $\mathbf{AY}$ pointed, which implies $[\mathbf{AY}]$ pointed by 11.1(2), which implies in turn $\mathbf{A}[\mathbf{Y}]$ pointed by 11.2(1). Hence, $[\mathbf{Y}]$ is semi-bounded, proving (3). Applying 11.2(1) for fixed m to the set $[\mathbf{Y}]$, we obtain for given λ a scalar μ such that $\mathbf{v}^i \in [\mathbf{Y}]$, $\theta_i \geqq 0$, and $|\theta_1\mathbf{v}^1 + \theta_2\mathbf{v}^2| \leqq \lambda$ implies $|\theta_i\mathbf{v}^i| \leqq \mu$. For any finite set of $\mathbf{v}^i \in [\mathbf{Y}]$ and $\theta_i \geqq 0$, $i = 1,...,m$ with $|\sum_{i=1}^{m} \theta_i\mathbf{v}^i| \leqq \lambda$, define $\alpha = \sum_{i=1,i\neq k}^{m} \theta_i$. Without loss, assume $\alpha > 0$ and define $\mathbf{u} = \alpha^{-1}\sum_{i=1,i\neq k}^{m} \theta_i\mathbf{v}^i \in [\mathbf{Y}]$. Then, $\sum_{i=1}^{m} \theta_i\mathbf{v}^i = \theta_k\mathbf{v}^k + \alpha\mathbf{u}$, implying $|\theta_k\mathbf{v}^k| \leqq \mu$. This verifies 11.2(1) for all m.

We next prove (4). By 10.2 and 10.5, $[\mathbf{Y}] + \mathbf{A}[\mathbf{Y}]$ is convex and contains $[\mathbf{Y}]$, and by 10.9 and 10.22, $[\mathbf{Y}] + \mathbf{A}[\mathbf{Y}] \subseteq \overline{[\mathbf{Y}]} + \mathbf{A}\overline{[\mathbf{Y}]} = \overline{[\mathbf{Y}]}$. Hence, it is sufficient to show $[\mathbf{Y}] + \mathbf{A}[\mathbf{Y}]$ closed. Since $\overline{[\mathbf{Y}]}$ is semi-bounded by (3), $[\mathbf{Y}] + \mathbf{A}[\mathbf{Y}]$ is semi-bounded. Suppose $\mathbf{v}^j \in [\mathbf{Y}] + \mathbf{A}[\mathbf{Y}]$, $\mathbf{v}^j \to \mathbf{v}$. Then, there exist $\mathbf{u}^j \in [\mathbf{Y}]$, $\mathbf{w}^j \in \mathbf{A}[\mathbf{Y}]$, $\mathbf{u}^{ij} \in \mathbf{Y}$, $\theta_{ij} \geqq 0$ such that $\mathbf{v}^j = \mathbf{u}^j + \mathbf{w}^j$, $\sum_{i=0}^{N} \theta_{ij} = 1$, and $\mathbf{u}^j = \sum_{i=0}^{N} \theta_{ij}\,\mathbf{u}^{ij}$. By (1), $\mathbf{w}^j$ and $\theta_{ij}\mathbf{u}^{ij}$ are bounded. Hence there exists a subsequence of j (retain notation) such that $\mathbf{w}^j \to \mathbf{w} \in \mathbf{A}[\mathbf{Y}]$, $\theta_{ij}\mathbf{u}^{ij} \to \mathbf{x}^i$, and $\theta_{ij} \to \theta_i$ with $\sum_{i=0}^{N} \theta_i = 1$. Let $\mathbf{I}$ denote the set of i indices with $\theta_i > 0$, and $\mathbf{J}$ denote the set of remaining indices, and let $y^i = \mathbf{x}^i/\theta_i$ for $i \in \mathbf{I}$. Then, $\mathbf{y}^i \in \mathbf{Y}$, $i \in \mathbf{I}$, and $\mathbf{y} = \sum_I \theta_i\mathbf{y}^i \in [\mathbf{Y}]$. For $i \in \mathbf{J}$, $\mathbf{x}^i \in \mathbf{AY}$. Hence, $\mathbf{u} = \mathbf{w} + \sum_J \mathbf{x}^i \in \mathbf{A}[\mathbf{Y}]$, implying $\mathbf{v} = \mathbf{y} + \mathbf{u} \in [\mathbf{Y}] + \mathbf{A}[\mathbf{Y}]$. This verifies (4). Q.E.D.

11.3. *Lemma.* If $\mathbf{Y}$ is a closed cone, then the following conditions are equivalent: (1) $\mathbf{Y}$ is pointed; (2) $[\mathbf{Y}]\cap[-\mathbf{Y}]=\{\mathbf{0}\}$; (3) int $\mathbf{PY}\neq\emptyset$; and $\mathbf{p}\in\text{int}\,\mathbf{PY}$, $\mathbf{v}\in\mathbf{Y}$, $\mathbf{v}\neq\mathbf{0}$ implies $\mathbf{p}\cdot\mathbf{v}<0$; (4) there exists $\mathbf{p}\in\mathbf{E}^N$ such that $\mathbf{v}\in\mathbf{Y}$, $\mathbf{v}\neq\mathbf{0}$, implies $\mathbf{p}\cdot\mathbf{v}<0$.

Proof: (4)→(3). If $\mathbf{v}\in\mathbf{Y}$, $|\mathbf{v}|=1$, then $\mathbf{p}\cdot\mathbf{v}\leqq-\alpha<0$. For $\mathbf{q}\in\mathbf{E}^N$, $|\mathbf{q}|\leqq\alpha/2$, $(\mathbf{p}+\mathbf{q})\cdot\mathbf{v}\leqq-\alpha+|\mathbf{q}\cdot\mathbf{v}|\leqq-\alpha/2$, implying $\mathbf{p}+\mathbf{q}\in\mathbf{PY}$, and $\mathbf{p}\in\text{int}\,\mathbf{PY}$. The second part of (3) is a trivial consequence of the first part.

(3)→(2). $\mathbf{v}\in[\mathbf{Y}]\cap[-\mathbf{Y}]$ implies $\mathbf{q}\cdot\mathbf{v}=0$ for all $\mathbf{q}\in\mathbf{PY}$. If $\mathbf{p}\in\text{int}\,\mathbf{PY}$, then $\mathbf{p}+\alpha\mathbf{v}\in\mathbf{PY}$ for α small positive, implying $\mathbf{v}\cdot\mathbf{v}=0$, or $\mathbf{v}=\mathbf{0}$.

(2)→(1). If $\mathbf{Y}$ is not pointed there exist $\mathbf{v}^i\in\mathbf{Y}$ such that $\mathbf{v}^0\neq\mathbf{0}$ and $\sum_{i=0}^m\mathbf{v}^i=\mathbf{0}$, implying $-\mathbf{v}^0=\sum_{i=1}^m\mathbf{v}^i\in[\mathbf{Y}]$ and contradicting (2).

(1)→(4). Define the set $\mathbf{Z}=\{\mathbf{v}\in\mathbf{Y}\,|\,|\mathbf{v}|\geqq1\}$. Then, $\mathbf{Z}$ is closed and $\mathbf{AZ}=\mathbf{Y}$, implying $\mathbf{Z}$ semi-bounded. By 11.2 (4) and 10.9, $[\mathbf{Z}]+\mathbf{A}[\mathbf{Z}]=[\mathbf{Z}]$ is closed. If $\mathbf{0}\in[\mathbf{Z}]$, then there exist $\mathbf{v}^i\in\mathbf{Z}$, $\theta_i\geqq0$ such that $\sum_{i=0}^N\theta_i=1$ and $\sum_{i=0}^N\theta_i\mathbf{v}^i=\mathbf{0}$, contradicting (1). Hence, $\mathbf{0}\notin[\mathbf{Z}]$ and by 10.13 there exists $\mathbf{p}$ and $\alpha>0$ such that $\mathbf{p}\cdot\mathbf{v}\leqq-\alpha$ for all $\mathbf{v}\in[\mathbf{Z}]$. Then, $\mathbf{p}$ satisfies (4). Q.E.D.

11.4. *Lemma.* For $\mathbf{Y},\mathbf{Z}$ non-empty, the following conditions hold: (1) $\mathbf{AY}\subseteq\mathbf{A}(\mathbf{Y}\cup\mathbf{Z})=(\mathbf{AY})\cup(\mathbf{AZ})\subseteq\mathbf{A}(\mathbf{Y}+\mathbf{Z})$; (2) $\mathbf{Y},\mathbf{Z}$ convex implies $\mathbf{AY}+\mathbf{AZ}\subseteq\mathbf{A}(\mathbf{Y}+\mathbf{Z})$; (3) $\mathbf{Y}\cup\mathbf{Z}$ semi-bounded implies $\mathbf{A}(\mathbf{Y}+\mathbf{Z})\subseteq\mathbf{AY}+\mathbf{AZ}\subseteq\mathbf{A}[\mathbf{Y}\cup\mathbf{Z}]$.

Proof: (1) $\mathbf{v}\in\mathbf{AY}$ implies there exist $\mathbf{v}^i\in\mathbf{Y}$, $\theta_i\geqq0$ such that $\theta_i\to0$, $\theta_i\mathbf{v}^i\to\mathbf{v}$. Take any $\mathbf{w}\in\mathbf{Z}$. Then $\mathbf{v}^i+\mathbf{w}\in\mathbf{Y}+\mathbf{Z}$, and $\theta_i(\mathbf{v}^i+\mathbf{w})\to\mathbf{v}\in\mathbf{A}(\mathbf{Y}+\mathbf{Z})$. The remaining conditions are immediate.

(2) Using (1) and 10.6, $\mathbf{AY}+\mathbf{AZ}\subseteq\mathbf{A}(\mathbf{Y}+\mathbf{Z})+\mathbf{A}(\mathbf{Y}+\mathbf{Z})=\mathbf{A}(\mathbf{Y}+\mathbf{Z})$.

(3) If $\mathbf{v}\in\mathbf{A}(\mathbf{Y}+\mathbf{Z})$, then there exist $\mathbf{u}^j\in\mathbf{Y}$, $\mathbf{w}^j\in\mathbf{Z}$, $\theta_j\geqq0$ such that $\theta_j\to0$ and $\theta_j(\mathbf{u}^j+\mathbf{w}^j)\to\mathbf{v}$. By 11.2(1), $\theta_j\mathbf{u}^j$ and $\theta_j\mathbf{w}^j$ are bounded, and there exists a subsequence (retain notation) with $\theta_j\mathbf{u}^j\to\mathbf{u}\in\mathbf{AY}$ and $\theta_j\mathbf{w}^j\to\mathbf{w}\in\mathbf{AZ}$, implying $\mathbf{v}=\mathbf{u}+\mathbf{w}\in\mathbf{AY}+\mathbf{AZ}$. Finally, $\mathbf{v}=\mathbf{u}+\mathbf{w}$, $\mathbf{u}\in\mathbf{AY}$, $\mathbf{w}\in\mathbf{AZ}$ imply $\mathbf{v}\in\mathbf{A}[\mathbf{Y}\cup\mathbf{Z}]$ immediately if $\mathbf{u}=\mathbf{0}$ or $\mathbf{w}=\mathbf{0}$, and if $\mathbf{u}$, $\mathbf{w}\neq\mathbf{0}$, imply the existence of $\mathbf{u}^j\in\mathbf{Y}$, $\mathbf{w}^j\in\mathbf{Z}$, $\theta_j,\lambda_j\geqq0$ such that $\theta_j\mathbf{u}^j\to\mathbf{u}$, $\lambda_j\mathbf{w}^j\to\mathbf{w}$, $\theta_j\to0$, $\lambda_j\to0$. Then, $\mathbf{v}^j=(\theta_j+\lambda_j)^{-1}(\theta_j\mathbf{u}^j+\lambda_j\mathbf{w}^j)\in[\mathbf{Y}\cup\mathbf{Z}]$ for j large, and $(\theta_j+\lambda_j)\mathbf{v}^j\to\mathbf{v}\in\mathbf{A}[\mathbf{Y}\cup\mathbf{Z}]$. Q.E.D.

11.5. *Lemma.* $\mathbf{Y},\mathbf{Z}$ non-empty and closed, and $\mathbf{Y}\cup\mathbf{Z}$ semi-bounded implies $\mathbf{Y}+\mathbf{Z}$ closed and semi-bounded.

Proof: $\mathbf{Y}+\mathbf{Z}$ is semi-bounded by 11.4(3). If $\mathbf{v}^i\in\mathbf{Y}+\mathbf{Z}$, $\mathbf{v}^i\to\mathbf{v}$, then there exist $\mathbf{u}^i\in\mathbf{Y}$, $\mathbf{w}^i\in\mathbf{Z}$ such that $\mathbf{v}^i=\mathbf{u}^i+\mathbf{w}^i$. By 11.2(1), $\mathbf{u}^i,\mathbf{w}^i$ are bounded, and there exists a subsequence (retain notation) with $\mathbf{u}^i\to\mathbf{u}\in\mathbf{Y}$, $\mathbf{w}^i\to\mathbf{w}\in\mathbf{Z}$, implying $\mathbf{v}=\mathbf{u}+\mathbf{w}\in\mathbf{Y}+\mathbf{Z}$. Q.E.D.

11.6. *Lemma.* If **Y** is non-empty, convex, and semi-bounded, then $\emptyset \neq \text{int}\,\mathbf{AY}$ if and only if **NY** is pointed.

Proof: Since $\mathbf{AY} = \mathbf{P}(\mathbf{N\bar{Y}})$, 11.3 implies the result. Q.E.D.

12. Properties of Convex Functions

The next series of results give properties of convex functions, particularly support functions. General treatments of this topic can be found in Fenchel (1953, Ch. III), Karlin (1959, 7.5), and Rockafellar (1970, Sects. 10, 13, 25).

12.1. *Lemma.* If $\langle F,\mathbf{X}\rangle$ is convex, then (1) F is continuous on intr **X**, and is uniformly Lipschitzian on any compact subset **Y** of intr **X** (i.e., given **Y**, there exists μ such that $|F(\mathbf{x}) - F(\mathbf{y})| \leqq \mu|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x},\mathbf{y} \in \mathbf{Y}$.

(2) If $\text{int}\,\mathbf{X} \neq \emptyset$, then F possesses a first and second differential in a set $\mathbf{Y} \subseteq \text{int}\,\mathbf{X}$, with $(\text{int}\,\mathbf{X})\setminus\mathbf{Y}$ a set of Lebesgue measure zero. The vector of first-order partial derivatives of F, denoted by $\mathbf{F}'$ and termed the *gradient*, is continuous in **Y**. At each point in **Y**, the matrix of second-order partial derivatives of F, denoted by $\mathbf{F}''$ and termed the *Hessian*, is symmetric (i.e., derivatives are independent of the order of differentiation) and the quadratic form $Q(\mathbf{v},\mathbf{F}'')$ of any vector **v** and the matrix $\mathbf{F}''$ is non-negative (i.e., $\mathbf{F}''$ is a non-negative definite matrix). Further, $F(\mathbf{z}) = F(\mathbf{x}) + \mathbf{F}'(\mathbf{x})(\mathbf{z} - \mathbf{x}) + \frac{1}{2}Q(\mathbf{z} - \mathbf{x},\mathbf{F}''(\mathbf{x})) + i(|\mathbf{z} - \mathbf{x}|^2)$ for $\mathbf{x} \in \mathbf{Y}$, $\mathbf{z} \in \text{int}\,\mathbf{X}$, where $i(\alpha)$ is a term satisfying $\lim_{\alpha\to 0^+} i(\alpha)/\alpha = 0$.

Proof: For (1) see Fenchel (1953, Ch. III, Results 21, 23, 34), Popoviciu (1945), or Rockafellar (1970, Thms. 10.1 and 10.4). For (2) see Reidemeister (1921), Alexandrov (1939), Rockafellar (1970, Thm. 25.5), and Busemann and Feller (1935–36). Q.E.D.

12.2. *Definition.* If $\langle F,\mathbf{X}\rangle$ is convex and closed, define $\langle H,\mathbf{Y}\rangle$ by $\mathbf{y} \in \mathbf{Y}$ if and only if $\sup_{\mathbf{x}\in\mathbf{X}}\{\mathbf{y}\cdot\mathbf{x} - F(\mathbf{x})\} < +\infty$, and $H(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbf{X}}\{\mathbf{y}\cdot\mathbf{x} - F(\mathbf{x})\}$ for $\mathbf{y} \in \mathbf{Y}$. $\langle H,\mathbf{Y}\rangle$ is termed the *conjugate dual* of $\langle F,\mathbf{X}\rangle$, and will be denoted by $\langle H,\mathbf{Y}\rangle = D\langle F,\mathbf{X}\rangle$. The following theorem is due to Fenchel, and is proved in Karlin (1959, 7.5.2 and 7.5.3), or Rockafellar (1970, Thm. 12.2).

12.3. *Theorem.* If $\langle F,\mathbf{X}\rangle$ is convex and closed, then the conjugate dual $\langle H,\mathbf{Y}\rangle = D\langle F,\mathbf{X}\rangle$ has $\mathbf{Y} \neq \emptyset$ and is convex and closed. Further, $D\langle H,\mathbf{Y}\rangle = \langle F,\mathbf{X}\rangle$.

12.4. *Lemma.* If $\mathbf{Y}$ is non-empty, closed, and semi-bounded, then the support function $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle$ is convex, closed, and positively linear homogeneous, and $\overline{[\mathbf{Y}]} = \{\mathbf{y} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{y} \leqq G^{\mathbf{Y}}(\mathbf{p}) \text{ for all } \mathbf{p} \in \mathbf{NY}\}$.

Proof: Rockafellar (1970, Thm. 13.2). Define $\langle F,\mathbf{X}\rangle$ with $\mathbf{X} = \overline{[\mathbf{Y}]}$ and $F(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \mathbf{X}$. Then, $\langle G,\mathbf{V}\rangle = D\langle F,\mathbf{X}\rangle$ is defined for $\mathbf{p} \in \mathbf{V}$ if and only if $\sup_{\mathbf{x}\in\mathbf{X}} \mathbf{p}\cdot\mathbf{x} = \sup_{\mathbf{y}\in\mathbf{Y}} \mathbf{p}\cdot\mathbf{y} < +\infty$, and $G(\mathbf{p}) = \sup_{\mathbf{y}\in\mathbf{Y}} \mathbf{p}\cdot\mathbf{y}$ for $\mathbf{p} \in \mathbf{V}$. Hence, $\mathbf{V} = \mathbf{NY}$ and $G = G^{\mathbf{Y}}$. By 12.3, $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle$ is convex and closed. Since $\mathbf{NY}$ is a cone, the positive linear homogeneity of $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle$ is a consequence of the definition of $G^{\mathbf{Y}}$. Finally, by 12.3, $D\langle G^{\mathbf{Y}},\mathbf{NY}\rangle = \langle F,\mathbf{X}\rangle$ and $\mathbf{x} \in \mathbf{X}$ if and only if $\sup_{\mathbf{p}\in\mathbf{NY}}\{\mathbf{p}\cdot\mathbf{x} - G^{\mathbf{Y}}(\mathbf{p})\} < +\infty$. Using the positive linear homogeneity, $\mathbf{x} \in \mathbf{X}$ if and only if $\mathbf{p}\cdot\mathbf{x} \leqq G^{\mathbf{Y}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{NY}$. Q.E.D.

12.5. *Lemma.* If $\langle F,\mathbf{X}\rangle$ is convex, closed, and positively linear homogeneous, and $\operatorname{int}\mathbf{X} \neq \emptyset$, then $\mathbf{Y} = \{\mathbf{y} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{y} \leqq F(\mathbf{p}) \text{ for all } \mathbf{p} \in \mathbf{X}\}$ is non-empty, convex, closed, and semi-bounded, and $\langle F,\mathbf{X}\rangle$ is the support function of $\mathbf{Y}$.

Proof: By 12.3 and the homogeneity argument used in the proof of 12.4, the conjugate dual of $\langle F,\mathbf{X}\rangle$ is $\langle G,\mathbf{Y}\rangle$, with $\mathbf{Y}$ the set given in the statement of this lemma, and $G(\mathbf{y}) = 0$ for $\mathbf{y} \in \mathbf{Y}$. Hence, by 12.3, $\mathbf{Y}$ is non-empty and convex. The closedness of $\mathbf{Y}$ is immediate from its definition. Since $\mathbf{p} \in \mathbf{X}$ implies $\mathbf{p}\cdot\mathbf{y} \leqq F(\mathbf{p})$ for all $\mathbf{y} \in \mathbf{Y}$, $\mathbf{X} \subseteq \mathbf{NY} \subseteq \mathbf{P}(\mathbf{AY})$, implying $\mathbf{AY} \subseteq \mathbf{PX}$. Since $\operatorname{int}\mathbf{X} \neq \emptyset$, $\mathbf{PX}$ is pointed by 11.3 (3), and hence $\mathbf{Y}$ is semi-bounded. By the argument of 12.4, $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle = D\langle G,\mathbf{Y}\rangle$, implying $\langle G^{\mathbf{Y}},\mathbf{NY}\rangle = \langle F,\mathbf{X}\rangle$ by 12.3. Q.E.D.

12.6. *Definition.* A set $\mathbf{X}$ is a *polytope* if it is the convex hull of a finite set of points. $\mathbf{X}$ is *boundedly polyhedral* if its intersection with any polytope is a polytope.

12.7. *Lemma.* If $\langle F,\mathbf{X}\rangle$ is convex and closed, and $\mathbf{X}$ is boundedly polyhedral, then $\langle F,\mathbf{X}\rangle$ is continuous; i.e., $\mathbf{v} \in \mathbf{X}$ implies $\lim_{\mathbf{u}\in\mathbf{X},\mathbf{u}\to\mathbf{v}} F(\mathbf{u}) = F(\mathbf{v})$.

Proof: Gale, Klee, and Rockafellar (1968, Thm. 2) establish $\lim_{\mathbf{u}\in\mathbf{X},\mathbf{u}\to\mathbf{v}} \sup F(\mathbf{u}) = F(\mathbf{v})$. Since $\langle F,\mathbf{X}\rangle$ is closed, the result follows. Q.E.D.

13. Properties of Maximand Correspondences

We now establish properties of maximands of $\mathbf{p}\cdot\mathbf{y}$ for $\mathbf{y}$ in a closed, semi-bounded set $\mathbf{Y}$. In this and succeeding sections, we shall deal with pairs of Euclidean spaces $\mathbf{E}^N$ and $\mathbf{E}^M$. The spaces in which sets lie will be clear from the context.

13.1. *Definition.* A mapping $\mathbf{\Phi}$ from a non-empty set $\mathbf{Z}\subseteq\mathbf{E}^M$ into *subsets* of $\mathbf{E}^N$ (i.e., $\mathbf{\Phi}(\mathbf{z})\subseteq\mathbf{E}^N$ for each $\mathbf{z}\in\mathbf{Z}$) is termed a *set-valued function* and denoted by $\langle\mathbf{\Phi},\mathbf{Z}\rangle$. If $\mathbf{\Phi}(\mathbf{z})$ is non-empty for all $\mathbf{z}\in\mathbf{Z}$, then $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ is termed a *correspondence.* We say a set-valued function or correspondence $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ is *convex-valued* (or closed-, compact-, or semi-bounded-valued) if $\mathbf{\Phi}(\mathbf{z})$ is convex (or closed, compact, or semi-bounded) for each $\mathbf{z}\in\mathbf{Z}$. If $\bigcup_{\mathbf{z}\in\mathbf{Z}}\mathbf{\Phi}(\mathbf{z})$ is bounded (or semi-bounded), we say $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ has *bounded range* (or semi-bounded range).

13.2. *Definition.* A correspondence $\langle\mathbf{\Phi},\mathbf{Z}\rangle$, $\mathbf{Z}\subseteq\mathbf{E}^M$, $\mathbf{\Phi}(\mathbf{z})\subseteq\mathbf{E}^N$, is *upper hemicontinuous* if $\mathbf{z}^j\in\mathbf{Z}$, $\mathbf{z}^j\to\mathbf{z}\in\mathbf{Z}$, $\mathbf{y}^j\in\mathbf{\Phi}(\mathbf{z}^j)$, $\mathbf{y}^j\to\mathbf{y}$ implies $\mathbf{y}\in\mathbf{\Phi}(\mathbf{z})$. $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ is *lower hemicontinuous* if $\mathbf{z}^j\in\mathbf{Z}$, $\mathbf{z}^j\to\mathbf{z}\in\mathbf{Z}$, $\mathbf{y}\in\mathbf{\Phi}(\mathbf{z})$ implies there exist $\mathbf{y}^j\in\mathbf{\Phi}(\mathbf{z}^j)$ with $\mathbf{y}^j\to\mathbf{y}$. $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ is *strongly upper hemicontinuous* if it is upper hemicontinuous and $\mathbf{z}^j\in\mathbf{Z}$, $\mathbf{z}^j\to\mathbf{z}\in\mathbf{Z}$, $\mathbf{y}^j\in\mathbf{\Phi}(\mathbf{z}^j)$, $\theta_j\geqq 0$, $\theta_j\to 0$, $\theta_j\mathbf{y}^j\to\mathbf{y}$ implies $\mathbf{y}\in\mathbf{A\Phi}(\mathbf{z})$. $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ is (*strongly*) *continuous* if it is lower hemicontinuous and (strongly) upper hemicontinuous.

13.3 *Lemma.* A correspondence $\langle\mathbf{\Phi},\mathbf{Z}\rangle$ has the following properties: (1) if it is upper hemicontinuous, then it is closed-valued; (2) if it is continuous and convex-valued, then it is strongly continuous; (3) suppose it is lower hemicontinuous and convex-valued, with $\operatorname{int}\mathbf{\Phi}(\mathbf{z})\neq\emptyset$ for $\mathbf{z}\in\mathbf{Z}$. Then for any sequence $\mathbf{z}^k\in\mathbf{Z}$, $\mathbf{z}^k\to\mathbf{z}^0\in\mathbf{Z}$ and any compact set $\mathbf{R}\subseteq\operatorname{int}\mathbf{\Phi}(\mathbf{z}^0)$, there exists k_0 such that $\mathbf{R}\subseteq\operatorname{int}\mathbf{\Phi}(\mathbf{z}^k)$ for $k\geqq k_0$.

Proof: The first proposition is an immediate consequence of the definition of upper hemicontinuity. To show the second, suppose $\mathbf{z}^j\in\mathbf{Z}$, $\mathbf{z}^j\to\mathbf{z}\in\mathbf{Z}$, $\mathbf{y}^j\in\mathbf{\Phi}(\mathbf{z}^j)$, $\theta_j\geqq 0$ such that $\theta_j\to 0$ and $\theta_j\mathbf{y}^j\to\mathbf{y}$. Take any $\mathbf{v}\in\mathbf{\Phi}(\mathbf{z})$. By lower hemicontinuity, there exist $\mathbf{v}^j\in\mathbf{\Phi}(\mathbf{z}^j)$, $\mathbf{v}^j\to\mathbf{v}$. For any $\lambda>0$, $0\leqq\theta_j\lambda<1$ for j large, and $\theta_j\lambda\mathbf{y}^j+(1-\theta_j\lambda)\mathbf{v}^j\in\mathbf{\Phi}(\mathbf{z}^j)$ with $\theta_j\lambda\mathbf{y}^j+(1-\theta_j\lambda)\mathbf{v}^j\to\lambda\mathbf{y}+\mathbf{v}\in\mathbf{\Phi}(\mathbf{z})$, by upper hemicontinuity. Hence, $\mathbf{y}\in\mathbf{A\Phi}(\mathbf{z})$.

To prove (3), first consider a sequence $\mathbf{z}^k\in\mathbf{Z}$, $\mathbf{z}^k\to\mathbf{z}^0\in\mathbf{Z}$, and a vector $\mathbf{y}^*\in\operatorname{int}\mathbf{\Phi}(\mathbf{z}^0)$. Let $\mathbf{e}^j$ denote the jth unit vector in $\mathbf{E}^N$, and $\mathbf{e}_N^0$ denote a vector of ones in $\mathbf{E}^N$. For a positive scalar δ, define $\mathbf{y}^0=\mathbf{y}^*-\delta\mathbf{e}_N^0$ and

$\mathbf{y}^j = \mathbf{y}^* + \delta \mathbf{e}^j$, $j = 1,\ldots,N$. For δ sufficiently small, $\mathbf{y}^j \in \operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^0)$, $j = 0,\ldots,N$, and $\mathbf{y}^* = (\sum_{j=0}^N \mathbf{y}^j)/(N+1)$. By the lower hemicontinuity of $\boldsymbol{\Phi}$, there exist $\mathbf{y}^{jk} \in \operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^k)$ with $\mathbf{y}^{jk} \to \mathbf{y}^j$, $j = 0,\ldots,N$. Treating the $\mathbf{y}$ vectors as column vectors, form $(N+1)$-dimensional matrices $\mathbf{A}_k$ and $\mathbf{A}_0$ and $(N+1)$-dimensional column vectors $\mathbf{a}$ and $\boldsymbol{\theta}$ satisfying

$$\mathbf{A}_k = \left[\begin{array}{cccc} \mathbf{y}^{0k} & \mathbf{y}^{1k} & \cdots & \mathbf{y}^{Nk} \\ \hline 1 & 1 & \cdots & 1 \end{array}\right], \qquad \mathbf{A}_0 = \left[\begin{array}{cccc} \mathbf{y}^{0} & \mathbf{y}^{1} & \cdots & \mathbf{y}^{N} \\ \hline 1 & 1 & \cdots & 1 \end{array}\right],$$

$$\mathbf{a} = \left[\begin{array}{c} \mathbf{y}^* \\ \hline 1 \end{array}\right], \qquad \boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_N \end{bmatrix}.$$

Then, $\mathbf{A}_0$ is non-singular, and $\mathbf{A}_0\boldsymbol{\theta} = \mathbf{a}$ has a unique solution $\boldsymbol{\theta}_0 = (N+1)^{-1}\, \mathbf{e}^0_{N+1}$. Since $\mathbf{A}_k \to \mathbf{A}_0$, it follows that $\mathbf{A}_k$ is non-singular for k large, and $\mathbf{A}_k^{-1} \to \mathbf{A}_0^{-1}$. Define $\boldsymbol{\theta}_k = \mathbf{A}_k^{-1}\mathbf{a}$. Then, $\boldsymbol{\theta}_k \to \boldsymbol{\theta}_0$, implying $\boldsymbol{\theta}_k$ non-negative for all sufficiently large k. Then, for all sufficiently large k, the polytope with vertices $\mathbf{y}^* - (\delta/2)\mathbf{e}^0_N$ and $\mathbf{y}^* + (\delta/2)\mathbf{e}^j$ for $j = 1,\ldots,N$, contains $\mathbf{y}^*$ in its interior and is contained in $\boldsymbol{\Phi}(\mathbf{z}^k)$. Therefore, for each $\mathbf{y}^* \in \operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^0)$, there exists an open neighborhood $\mathbf{N}_{\mathbf{y}^*}$ of $\mathbf{y}^*$ and an index $k_{\mathbf{y}^*}$ such that $\mathbf{N}_{\mathbf{y}^*} \subseteq \operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^k)$ for $k = 0$ and $k \geqq k_{\mathbf{y}^*}$. If $\mathbf{R}$ is a compact subset of $\operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^0)$, then the neighborhoods $\mathbf{N}_{\mathbf{y}^*}$ for $\mathbf{y}^* \in \mathbf{R}$ cover $\mathbf{R}$, implying the existence of a finite subcovering of $\mathbf{R}$. Then, for $k \geqq k_{\mathbf{y}^*}$ for each $\mathbf{y}^*$ in the finite subcovering, $\mathbf{R} \subseteq \operatorname{int} \boldsymbol{\Phi}(\mathbf{z}^k)$. Q.E.D.

13.4. *Definition.* For a non-empty, closed, semi-bounded set $\mathbf{Y} \subseteq \mathbf{E}^N$, define $\mathbf{p}^{\mathbf{Y}} = \{\mathbf{p} \in \mathbf{NY} \mid G^{\mathbf{Y}}(\mathbf{p}) = \mathbf{p}\cdot\mathbf{y} \text{ for some } \mathbf{y} \in \mathbf{Y}\}$ and $\boldsymbol{\Phi}^{\mathbf{Y}}(\mathbf{p}) = \{\mathbf{y} \in \mathbf{Y} \mid G^{\mathbf{Y}}(\mathbf{p}) = \mathbf{p}\cdot\mathbf{y}\}$ for $\mathbf{p} \in \mathbf{P}^{\mathbf{Y}}$. We term $\langle \boldsymbol{\Phi}^{\mathbf{Y}},\mathbf{P}^{\mathbf{Y}}\rangle$ the *maximand correspondence* of $\mathbf{Y}$.

13.5. *Lemma.* If $\mathbf{Y} \subseteq \mathbf{E}^N$ is non-empty, closed, and semi-bounded, then (1) $\mathbf{Q} \equiv \operatorname{int} \mathbf{NY} \subseteq \mathbf{P}^{\mathbf{Y}}$; (2) $\langle \boldsymbol{\Phi}^{\mathbf{Y}},\mathbf{P}^{\mathbf{Y}}\rangle$ is closed-valued and upper hemicontinuous; (3) $\langle \boldsymbol{\Phi}^{\mathbf{Y}},\mathbf{Q}\rangle$ is compact-valued; (4) for any compact non-empty set $\mathbf{R} \subseteq \mathbf{Q}$, $\langle \boldsymbol{\Phi}^{\mathbf{Y}},\mathbf{R}\rangle$ has bounded range; and (5) if $\mathbf{Y}$ is convex, then $\langle \boldsymbol{\Phi}^{\mathbf{Y}},\mathbf{P}^{\mathbf{Y}}\rangle$ is convex-valued, and if $\mathbf{p} \in \mathbf{Q}$, $\mathbf{y} \in \boldsymbol{\Phi}^{\mathbf{Y}}(\mathbf{p})$, $\mathbf{y} \notin \operatorname{intr} \boldsymbol{\Phi}^{\mathbf{Y}}(\mathbf{p})$, then there exists $(\mathbf{y}_j,\mathbf{p}^j) \to (\mathbf{y},\mathbf{p})$ with $\mathbf{y}^j \in \boldsymbol{\Phi}^{\mathbf{Y}}(\mathbf{p}^j)$ and $\mathbf{p}^j \in \mathbf{Q}$ not proportional to $\mathbf{p}$.

Proof: We first show that for any non-empty compact set $\mathbf{R} \subseteq \mathbf{Q}$, $\bigcup_{\mathbf{z}\in\mathbf{R}} \boldsymbol{\Phi}(\mathbf{z})$ is non-empty and bounded. Taking $\mathbf{R} = \{\mathbf{p}\}$, this will verify (1). Suppose there exist $\mathbf{R} \subseteq \mathbf{Q}$, $\mathbf{R} \neq \boldsymbol{\emptyset}$, $\mathbf{R}$ compact, $\mathbf{p}^j \in \mathbf{R}$, $\mathbf{y}^j \in \mathbf{Y}$ such that $|\mathbf{y}^j| \to +\infty$ and $\mathbf{p}^j\cdot\mathbf{y}^j - G^{\mathbf{Y}}(\mathbf{p}^j) \to 0$. By the compactness of $\mathbf{R}$ and the continuity of $G^{\mathbf{Y}}$, we can assume $\mathbf{p}^j \to \mathbf{p} \in \mathbf{R}$ and $\mathbf{p}^j\cdot\mathbf{y}^j \to G^{\mathbf{Y}}(\mathbf{p})$. Then, there exists a subsequence of j (retain notation) such that $\mathbf{y}^j/|\mathbf{y}^j| \to \mathbf{y} \in \mathbf{AY}$, $\mathbf{y} \neq \mathbf{0}$, and $\mathbf{p}\cdot\mathbf{y} = 0$, contradicting $\mathbf{p} \in \mathbf{Q}$ by 11.3(3). Hence, for $\mathbf{p} \in \mathbf{R}$, an optimizing sequence $\mathbf{y}^j \in \mathbf{Y}$ with $\mathbf{p}\cdot\mathbf{y}^j \to G^{\mathbf{Y}}(\mathbf{p})$ is bounded, and has a subsequence converging to $\mathbf{y} \in \mathbf{Y}$ with $\mathbf{p}\cdot\mathbf{y} = G^{\mathbf{Y}}(\mathbf{p})$. This proves $\bigcup_{\mathbf{z}\in\mathbf{R}} \Phi(\mathbf{z})$ non-empty and bounded, verifying (1) and (4). If $\mathbf{p}^j \in \mathbf{P}^{\mathbf{Y}}$, $\mathbf{p}^j \to \mathbf{p} \in \mathbf{P}^{\mathbf{Y}}$, $\mathbf{y}^j \in \Phi^{\mathbf{Y}}(\mathbf{p}^j)$, $\mathbf{y}^j \to \mathbf{y}$, then $\mathbf{p}^j\cdot\mathbf{y}^j \geqq \mathbf{p}^j\cdot\mathbf{w}$ for $\mathbf{w} \in \mathbf{Y}$, implying in the limit $\mathbf{p}\cdot\mathbf{y} \geqq \mathbf{p}\cdot\mathbf{w}$ for $\mathbf{w} \in \mathbf{Y}$, and hence $\mathbf{y} \in \Phi^{\mathbf{Y}}(\mathbf{p})$. Hence, $\langle \Phi^{\mathbf{Y}}, \mathbf{P}^{\mathbf{Y}} \rangle$ is upper hemicontinuous, and therefore closed-valued. This verifies (2). By (2) and (4), $\langle \Phi^{\mathbf{Y}}, \mathbf{Q} \rangle$ is compact-valued, verifying (3).

Suppose in (5) that $\mathbf{Y}$ is convex. Then, $\mathbf{y},\mathbf{z} \in \Phi^{\mathbf{Y}}(\mathbf{p})$ implies $\mathbf{p}\cdot(\theta\mathbf{y} + (1-\theta)\mathbf{z}) \geqq \mathbf{p}\cdot\mathbf{w}$ for $\mathbf{w} \in \mathbf{Y}$, $0 < \theta < 1$, and hence $\Phi^{\mathbf{Y}}(\mathbf{p})$ convex. Suppose $\mathbf{p} \in \mathbf{Q}$, $\mathbf{y} \in \Phi^{\mathbf{Y}}(\mathbf{p})$, $\mathbf{y} \notin \operatorname{intr} \Phi^{\mathbf{Y}}(\mathbf{p})$. We first show $\mathbf{y} \notin \operatorname{intr} \mathbf{Y}$. If $\Phi^{\mathbf{Y}}(\mathbf{p}) = \mathbf{Y}$, this result is immediate. Alternately, there exists $\mathbf{w} \in \mathbf{Y}$, $\mathbf{w} \notin \Phi^{\mathbf{Y}}(\mathbf{p})$, and hence $\mathbf{p}\cdot\mathbf{y} > \mathbf{p}\cdot\mathbf{w}$. If $\mathbf{y} \in \operatorname{intr} \mathbf{Y}$, then for small $\theta > 0$, $(1+\theta)\mathbf{y} - \theta\mathbf{w} \in \mathbf{Y}$, implying the contradictory inequality $\mathbf{p}\cdot\mathbf{y} \leqq \mathbf{p}\cdot\mathbf{w}$. Two cases will be distinguished:

Case 1. $\Phi^{\mathbf{Y}}(\mathbf{p}) \cap (\operatorname{intr} \mathbf{Y}) \neq \emptyset$. By 10.12, there exists $\mathbf{q} \neq \mathbf{0}$ such that $\mathbf{Y} \subseteq \mathbf{H}^-(\mathbf{q},\mathbf{q}\cdot\mathbf{y})$ and $(\operatorname{intr} \mathbf{Y}) \cap \mathbf{H}(\mathbf{q},\mathbf{q}\cdot\mathbf{y}) = \phi$. This implies $\mathbf{q}$ not proportional to $\mathbf{p}$. Defining $\mathbf{p}^j = (1 - j^{-1})\mathbf{p} + j^{-1}\mathbf{q}$ and $\mathbf{y}^j = \mathbf{y}$, the sequence $(\mathbf{y}^j,\mathbf{p}^j) \to (\mathbf{y},\mathbf{p})$ satisfies (5).

Case 2. $\Phi^{\mathbf{Y}}(\mathbf{p}) \cap \operatorname{intr} \mathbf{Y} = \emptyset$. Since $\mathbf{y} \notin \operatorname{intr} \Phi^{\mathbf{Y}}(\mathbf{p})$, there exists a sequence $\mathbf{z}^j$ in the flat spanned by $\Phi^{\mathbf{Y}}(\mathbf{p})$ such that $\mathbf{z}^j \notin \Phi^{\mathbf{Y}}(\mathbf{p})$ and $\mathbf{z}^j \to \mathbf{y}$. Then, in particular, $\mathbf{z}^j$ is in the flat spanned by $\mathbf{Y}$ and is in the hyperplane $\mathbf{H}(\mathbf{p},\mathbf{p}\cdot\mathbf{y})$. Choose $\mathbf{w} \in \operatorname{intr} \mathbf{Y}$. Then, there exists θ_j, $0 < \theta_j < 1$, such that $\mathbf{y}^j = \theta_j\mathbf{z}^j + (1-\theta_j)\mathbf{w} \in \mathbf{Y}$ and $\mathbf{y}^j \notin \operatorname{intr} \mathbf{Y}$. If a subsequence had $\theta_j \to \theta < 1$, then $\mathbf{y}^j \to \theta\mathbf{y} + (1-\theta)\mathbf{w} \in \operatorname{intr} \mathbf{Y}$, contradicting the closedness of $\mathbf{Y} \setminus (\operatorname{intr} \mathbf{Y})$. Hence, $\theta_j \to 1$ and $\mathbf{y}^j \to \mathbf{y}$. By 10.12, for each $\mathbf{y}^j$ there exists $\mathbf{p}^j$ such that $|\mathbf{p}^j| = 1$ and $\mathbf{p}^j\cdot\mathbf{y}^j \geqq \mathbf{p}^j\cdot\mathbf{w}$ for all $\mathbf{w} \in \mathbf{Y}$. Since $\mathbf{y}^j \notin \Phi^{\mathbf{Y}}(\mathbf{p})$, $\mathbf{p}^j$ is not proportional to $\mathbf{p}$. Take a subsequence of j with $\mathbf{p}^j$ converging to a vector $\mathbf{q}$. If $\mathbf{q}$ is a positive multiple of $\mathbf{p}$, then after the $\mathbf{p}^j$ are rescaled by dividing by this quantity, $(\mathbf{y}^j,\mathbf{p}^j) \to (\mathbf{y},\mathbf{p})$ is the required sequence. If $\mathbf{q}$ is a negative multiple of $\mathbf{p}$, then $\mathbf{Y} \subseteq \mathbf{H}(\mathbf{p},\mathbf{p}\cdot\mathbf{y})$, contradicting this case. If $\mathbf{q}$ is not proportional to $\mathbf{p}$, then $\mathbf{y}^j = \mathbf{y}$ and $\mathbf{p}^j = (1 - j^{-1})\mathbf{p} + j^{-1}\mathbf{q}$ gives the required sequence. This verifies (5). Q.E.D.

13.6. *Remark.* The *interior* of the set $\mathbf{P}^{\mathbf{Y}}$ on which the support function $G^{\mathbf{Y}}$ achieves a maximum is a non-empty convex set when $\mathbf{Y}$ is closed and semi-bounded, and its closure is also convex; i.e., we have the string of inclusions

$$\emptyset \neq \operatorname{int} \mathbf{P}(\mathbf{AY}) \subseteq \mathbf{P}^{\mathbf{Y}} \subseteq \mathbf{NY} \subseteq \mathbf{P}(\mathbf{AY}),$$

with $\mathbf{P(AY)}$ closed and convex. Nevertheless, $\mathbf{P^Y}$ itself, need not be convex. One example has been given by Winter (forthcoming); a second follows: define the set

$$\mathbf{Y} = \{\mathbf{y} \in \mathbf{E}^3 | y_2, y_3 \leqq 0, y_1 \leqq (y_2 + y_3)^{-1} < 0\}.$$

This set is closed and convex. For $\mathbf{p}^1 = (0,0,1)$, a maximum of $\mathbf{p}^1 \cdot \mathbf{y}$ for $\mathbf{y} \in \mathbf{Y}$ is attained at $\mathbf{y} = (-1,-1,0)$. For $\mathbf{p}^2 = (0,1,0)$, a maximum is attained at $\mathbf{y} = (-1,0,-1)$. But for $\mathbf{p} = \mathbf{p}^1 + \mathbf{p}^2 = (0,1,1)$, the supremum of $\mathbf{p} \cdot \mathbf{y}$ for $\mathbf{y} \in \mathbf{Y}$, equal to zero, is approached by $y_2 + y_3 = y_1^{-1}$ and $y_1 \to \infty$, but is not achieved.

13.7. *Definition.* For a convex, closed, positively linear homogeneous real-valued function $\langle F, \mathbf{X} \rangle$, define a set-valued function $\langle \mathbf{\Gamma}, \operatorname{intr} \mathbf{X} \rangle$, $\mathbf{\Gamma}(\mathbf{x}) \subseteq \mathbf{E}^N$, by $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{x})$ if and only if for all $\mathbf{z} \in \mathbf{X}$,

$$(\mathbf{z} - \mathbf{x}) \cdot \mathbf{y} \leqq \lim_{\theta \to 0^+} \inf (F((1-\theta)\mathbf{x} + \theta \mathbf{z}) - F(\mathbf{x}))/\theta.$$

$\langle \mathbf{\Gamma}, \operatorname{intr} \mathbf{X} \rangle$ is termed the ***sub-differential*** of $\langle F, \mathbf{X} \rangle$. Note that if F is differentiable at $\mathbf{x} \in \operatorname{int} \mathbf{X}$, then $\lim_{\theta \to 0}(F((1-\theta)\mathbf{x} + \theta \mathbf{z}) - F(\mathbf{x}))/\theta$ exists for all $\mathbf{z} \in \mathbf{E}^N$, and $\mathbf{y}$ is unique and equals the gradient (i.e., the vector of partial derivatives of F).

13.8. *Lemma.* If $\langle F, \mathbf{X} \rangle$ is convex, closed, and positively linear homogeneous, then (1) the sub-differential $\langle \mathbf{\Gamma}, \operatorname{intr} \mathbf{X} \rangle$ is a convex-valued upper hemicontinuous correspondence, with $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{x})$, $\mathbf{x} \in \operatorname{intr} \mathbf{X}$, if and only if $F(\mathbf{x}) = \mathbf{y} \cdot \mathbf{x}$ and $F(\mathbf{z}) \geqq \mathbf{y} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbf{X}$; (2) if $\operatorname{int} \mathbf{X} \neq \emptyset$, $\langle \mathbf{\Gamma}, \operatorname{int} \mathbf{X} \rangle$ is compact-valued; (3) F is differentiable at $\mathbf{x} \in \operatorname{int} \mathbf{X}$ with a vector of partial derivatives $\mathbf{y}$ if and only if $\mathbf{\Gamma}(\mathbf{x}) = \{\mathbf{y}\}$; and (4) if $\mathbf{x} \in \operatorname{int} \mathbf{X} \neq \emptyset$ and $\mathbf{Y} = \{\mathbf{w} \in \mathbf{E}^N | \mathbf{x} \cdot \mathbf{w} \leqq F(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}\}$, then $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{x})$ if and only if $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{x} \cdot \mathbf{y} \geqq \mathbf{x} \cdot \mathbf{w}$ for all $\mathbf{w} \in \mathbf{Y}$.

Proof: See also Rockafellar (1970, Sect. 23). (a) Consider the set $\mathbf{A} = \{(\mathbf{x}, \xi) \in \mathbf{E}^{N+1} | \mathbf{x} \in \mathbf{X},\ \xi \geqq F(\mathbf{x})\}$. One can easily show that $\mathbf{A}$ is convex and closed, and that for any $\mathbf{x} \in \operatorname{intr} \mathbf{X}$, $(\mathbf{x}, F(\mathbf{x})) \notin \operatorname{intr} \mathbf{A}$. By 10.12, there exists $(\mathbf{y}, -\eta) \in \mathbf{E}^{N+1}$, $(\mathbf{y}, -\eta) \neq 0$, such that $\mathbf{y} \cdot \mathbf{x} - \eta F(\mathbf{x}) \geqq \mathbf{y} \cdot \mathbf{z} - \eta \xi$ for all $(\mathbf{z}, \xi) \in \mathbf{A}$, with the inequality strict for $(\mathbf{z}, \xi) \in \operatorname{intr} \mathbf{A}$. Taking $\mathbf{z} = \mathbf{x}$ and $\xi > F(\mathbf{x})$ yields $(\mathbf{z}, \xi) \in \operatorname{intr} \mathbf{A}$ and implies $\eta > 0$. Normalize $\eta = 1$. For $\mathbf{z} \in \mathbf{X}$, $0 < \theta < 1$, the inequality yields $\mathbf{y} \cdot (\theta \mathbf{z} + (1-\theta)\mathbf{x}) - \mathbf{y} \cdot \mathbf{x} = \theta \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \leqq F(\theta \mathbf{z} + (1-\theta)\mathbf{x}) - F(\mathbf{x})$. Letting $\theta \to 0^+$, this implies $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{x})$. Hence, $\langle \mathbf{\Gamma}, \operatorname{intr} \mathbf{X} \rangle$ is a correspondence.

(b) We next show that $(F(\theta \mathbf{z} + (1-\theta)\mathbf{x}) - F(\mathbf{x}))/\theta$ is a non-decreasing function of positive θ for $\mathbf{z} \in \mathbf{X}$, $\mathbf{x} \in \operatorname{intr} \mathbf{X}$. Suppose $0 < \theta_1 < \theta_2$ and let

$\alpha = \theta_1/\theta_2$. By the convexity of F, $F(\alpha(\theta_2\mathbf{z} + (1-\theta_2)\mathbf{x}) + (1-\alpha)\mathbf{x}) = F(\theta_1\mathbf{z} + (1-\theta_1)\mathbf{x}) \leqq \alpha F(\theta_2\mathbf{z} + (1-\theta_2)\mathbf{x}) + (1-\alpha)F(\mathbf{x})$, or $(F(\theta_1\mathbf{z} + (1-\theta_1)\mathbf{x}) - F(\mathbf{x}))/\theta_1 \leqq (F(\theta_2\mathbf{z} + (1-\theta_2)\mathbf{x}) - F(\mathbf{x}))/\theta_2$.

(c) Suppose $\mathbf{x} \in \operatorname{intr} \mathbf{X}$, $\mathbf{y} \in \Gamma(\mathbf{x})$. For any $\mathbf{v} \in \mathbf{E}^N$ with $\mathbf{z} = \mathbf{x} + \mathbf{v} \in \mathbf{X}$, paragraph (b) and the definition of Γ imply $\mathbf{v}\cdot\mathbf{y} \leqq F(\mathbf{v}+\mathbf{x}) - F(\mathbf{x})$. Taking $\mathbf{v} = -\mathbf{x}/2$, homogeneity implies $\mathbf{x}\cdot\mathbf{y} \geqq F(\mathbf{x})$. Taking $\mathbf{v} \in \mathbf{X}$, convexity implies $\mathbf{v}\cdot\mathbf{y} \leqq F(\mathbf{v})$. Hence, we have established $\mathbf{y} \in \Gamma(\mathbf{x})$ if and only if $\mathbf{x}\cdot\mathbf{y} = F(\mathbf{x})$ and $\mathbf{z}\cdot\mathbf{y} \leqq F(\mathbf{z})$ for all $\mathbf{z} \in \mathbf{X}$. Since F is continuous on intr $\mathbf{X}$ by 12.1, the results that $\langle \Gamma, \operatorname{intr} \mathbf{X}\rangle$ is convex-valued and upper hemicontinuous follow immediately from this characterization, verifying (1).

(d) In (2), (3), (4), int $\mathbf{X} \neq \phi$ is assumed. Then, by 12.5, $\mathbf{Y}$ defined in (4) is non-empty, convex, closed, and semi-bounded, and $\langle F, \mathbf{X}\rangle$ is the support function of $\mathbf{Y}$. From the characterization of $\langle \Gamma, \operatorname{intr} \mathbf{X}\rangle$ established in (c), (4) holds and $\Gamma(\mathbf{x}) = \Phi^{\mathbf{Y}}(\mathbf{x})$ for $\mathbf{x} \in \operatorname{int} \mathbf{X}$. By 13.5, $\langle \Gamma, \operatorname{int} \mathbf{X}\rangle$ is compact-valued, verifying (2). We noted previously that if F is differentiable at $\mathbf{x} \in \operatorname{int} \mathbf{X}$, then the definition of Γ implies the "only if" implication in (3). The converse implication in (3) is a consequence of the definition of differentiability – a detailed argument is given by Fenchel (1953, Ch. 3, result 32) or Rockafellar (1970, Thm. 25.1). Q.E.D.

13.9. *Corollary.* If $\mathbf{Y}$ is non-empty, closed, and semi-bounded, $\langle G^{\mathbf{Y}}, \mathbf{NY}\rangle$ is the support function of $\mathbf{Y}$, $\langle \Gamma, \operatorname{int} \mathbf{NY}\rangle$ is the subdifferential of $G^{\mathbf{Y}}$, and $\langle \Phi^{\mathbf{Y}}, \operatorname{int} \mathbf{NY}\rangle$ is the maximand correspondence of $\mathbf{Y}$, then $\Gamma(\mathbf{p}) = [\Phi^{\mathbf{Y}}(\mathbf{p})]$ for $\mathbf{p} \in \operatorname{int} \mathbf{NY}$.

Proof: $\mathbf{p} \in \operatorname{int} \mathbf{NY}$ and $\mathbf{y} \in \Phi^{\mathbf{Y}}(\mathbf{p}) \Rightarrow \mathbf{p}\cdot\mathbf{y} \geqq \mathbf{p}\cdot\mathbf{w}$ for all $\mathbf{w} \in \mathbf{Y} \Rightarrow \mathbf{p}\cdot\mathbf{y} \geqq \mathbf{p}\cdot\mathbf{w}$ for all $\mathbf{w} \in [\mathbf{Y}] \Rightarrow \mathbf{y} \in \Phi^{[\mathbf{Y}]}(\mathbf{p})$. Alternately, $\mathbf{y} \in \Phi^{[\mathbf{Y}]}(\mathbf{p}) \Rightarrow$ [by 11.2(4)] $\mathbf{y} = \mathbf{u} + \mathbf{v}$ with $\mathbf{u} \in [\mathbf{Y}]$, $\mathbf{v} \in \mathbf{A}[\mathbf{Y}]$. But $\mathbf{v} \neq \mathbf{0} \Rightarrow \mathbf{p}\cdot\mathbf{v} < 0$ for $\mathbf{p} \in \operatorname{int} \mathbf{NY}$. Hence, $\mathbf{v} = \mathbf{0}$ and $\mathbf{y} \in [\mathbf{Y}] \Rightarrow \mathbf{y} = \sum_{i=0}^{m} \theta_i \mathbf{y}^i$ with $\mathbf{y}^i \in \mathbf{Y}$, $\theta_i \geqq 0$, $\sum_{i=0}^{m} \theta_i = 1$, and $\mathbf{p}\cdot\mathbf{y} \geqq \mathbf{p}\cdot\mathbf{w}$ for all $\mathbf{w} \in [\mathbf{Y}] \Rightarrow \mathbf{y}^i \in \Phi^{\mathbf{Y}}(\mathbf{p}) \Rightarrow \mathbf{y} \in [\Phi^{\mathbf{Y}}(\mathbf{p})]$. Hence, $[\Phi^{\mathbf{Y}}(\mathbf{p})] = \Phi^{[\mathbf{Y}]}(\mathbf{p}) = \Gamma(\mathbf{p})$ by 12.4 and 13.8. Q.E.D.

14. Exposed Sets

The next series of results establish relationships between a closed semi-bounded set and the set of all its maximands.

14.1. *Lemma.* If $\mathbf{Y}$ is non-empty, convex, closed, and semi-bounded, $\mathbf{Z}$ is non-empty, convex, and compact, and $\mathbf{Y} \cap \mathbf{Z} = \emptyset$, then there exists an open set $\mathbf{W} \subseteq \operatorname{int} \mathbf{NY}$ and scalars α, β such that $\mathbf{w}\cdot\mathbf{y} \leqq \alpha < \beta \leqq \mathbf{w}\cdot\mathbf{z}$ for all $\mathbf{w} \in \mathbf{W}$, $\mathbf{y} \in \mathbf{Y}$, $\mathbf{z} \in \mathbf{Z}$. In particular, $\mathbf{w} \in \mathbf{W}$ may be chosen so that $\mathbf{w}\cdot\mathbf{y}$ is maximized over $\mathbf{y} \in \mathbf{Y}$ at a unique point.

Proof: By 10.13, there exist $\mathbf{p}$, α_1, β_1 such that $\mathbf{p}\cdot\mathbf{y} \leqq \alpha_1 - \beta_1 < \alpha_1 + \beta_1 \leqq \mathbf{p}\cdot\mathbf{z}$ for $\mathbf{y} \in \mathbf{Y}$, $\mathbf{z} \in \mathbf{Z}$. Since $\mathbf{Z}$ is compact, $\text{int}\,\mathbf{NY} \neq \emptyset$, and $\langle G^{\mathbf{Y}}, \mathbf{NY}\rangle$ is closed and convex, we can choose $\mathbf{q} \in \text{int}\,\mathbf{NY}$ such that $|\mathbf{q}-\mathbf{p}| < \beta_1/2(1+\max_{\mathbf{Z}}|\mathbf{z}|)$ and $G^{\mathbf{Y}}(\mathbf{q}) - G^{\mathbf{Y}}(\mathbf{p}) < \beta_1/2$, implying $\mathbf{q}\cdot\mathbf{y} \leqq \alpha_1 - \beta_1 < \alpha_1 + \beta_1 \leqq \mathbf{q}\cdot\mathbf{z}$ for $\mathbf{y} \in \mathbf{Y}$, $\mathbf{z} \in \mathbf{Z}$, $\beta_2 = \beta_1/2$. Then, a small open neighborhood $\mathbf{W}$ of $\mathbf{q}$ is contained in $\text{int}\,\mathbf{NY}$, and by the continuity of $G^{\mathbf{Y}}$ and the compactness of $\mathbf{Z}$ can be taken so that the strict separation is preserved. Since $G^{\mathbf{Y}}$ is differentiable almost everywhere in $\text{int}\,\mathbf{NY}$ (Lemma 12.1) and $\mathbf{w}\cdot\mathbf{y}$ achieves a unique maximum on $\mathbf{Y}$ if $G^{\mathbf{Y}}$ is differentiable at $\mathbf{w}$, the last conclusion is immediate. Q.E.D.

14.2. *Definition.* $\mathbf{X}$ is an *exposed set* of a non-empty, closed, semi-bounded set $\mathbf{Y}$ if $\mathbf{X}$ is the intersection of $\mathbf{Y}$ and a supporting hyperplane, i.e., $\mathbf{\Phi}^{\mathbf{Y}}(\mathbf{p}) = \mathbf{X}$ for some $\mathbf{p} \in \mathbf{NY}$. If $\mathbf{X} = \{\mathbf{y}\}$, $\mathbf{y}$ is termed an *exposed point*. Let $\mathbf{Y}^{\#}$ denote the set of exposed points of $\mathbf{Y}$.

14.3. *Lemma.* If $\mathbf{Y}$ is non-empty, convex, closed, and semi-bounded, then $\mathbf{Y} = \overline{[\mathbf{Y}^{\#}]} + \mathbf{AY}$.

Proof: By 10.9, $\mathbf{Z} \equiv \overline{[\mathbf{Y}^{\#}]} + \mathbf{AY} \subseteq \mathbf{Y}$, implying $\mathbf{AZ} = \mathbf{AY}$, and hence $\mathbf{P}(\mathbf{AZ}) = \mathbf{P}(\mathbf{AY})$. If $\mathbf{y} \in \mathbf{Y}$, $\mathbf{y} \notin \mathbf{Z}$, then by 14.1 there exists an open set $\mathbf{W} \subseteq \text{int}\,\mathbf{NZ} = \text{int}\,\mathbf{NY}$ such that $\mathbf{p}\cdot\mathbf{z} \leqq \alpha < \beta \leqq \mathbf{p}\cdot\mathbf{y}$ for all $\mathbf{z} \in \mathbf{Z}$, $\mathbf{p} \in \mathbf{W}$. Choose $\mathbf{p} \in \mathbf{W}$ such that $G^{\mathbf{Y}}(\mathbf{p})$ is differentiable. Then, there exists a unique $\mathbf{v} \in \mathbf{Y}$ such that $G^{\mathbf{Y}}(\mathbf{p}) = \mathbf{p}\cdot\mathbf{v} \leqq \mathbf{p}\cdot\mathbf{y}$. But then $\mathbf{v} \in \mathbf{Y}^{\#} \subseteq \mathbf{Z}$, contradicting the inequality $\mathbf{p}\cdot\mathbf{z} \leqq \alpha$. Hence, $\mathbf{Z} = \mathbf{Y}$. Q.E.D.

15. Conjugate Correspondences

Thus far, we have investigated properties of the support function of a single set $\mathbf{Y}$. We now list properties of a family of support functions corresponding to a parametric family of sets $\mathbf{Y}$. This topic does not seem to have been investigated in the mathematical literature, although the work of Rockafellar (1970) on perturbations is closely related. Throughout this section, we shall consider a non-empty set of parameters $\mathbf{V} \subseteq \mathbf{E}^M$ and a mapping from $\mathbf{v} \in \mathbf{V}$ into non-empty subsets $\mathbf{Y}(\mathbf{v}) \subseteq \mathbf{E}^N$. Then, $\langle \mathbf{Y}, \mathbf{V}\rangle$ is a correspondence.

15.1. *Definition.* For a correspondence $\langle \mathbf{Y}, \mathbf{V}\rangle$ mapping $\mathbf{v} \in \mathbf{V} \subseteq \mathbf{E}^M$ into $\mathbf{Y}(\mathbf{v}) \subseteq \mathbf{E}^N$ which has $\mathbf{Y}(\mathbf{v})$ closed and semi-bounded for each $\mathbf{v} \in \mathbf{V}$, let $\mathbf{AY}(\mathbf{v})$ and $\mathbf{NY}(\mathbf{v})$ denote the asymptotic cone and normal cone of $\mathbf{Y}(\mathbf{v})$, respectively. Then, $\langle \mathbf{AY}, \mathbf{V}\rangle$ and $\langle \mathbf{NY}, \mathbf{V}\rangle$ are correspondences. Define the sets

$$\mathbf{D} = \{(\mathbf{v},\mathbf{p}) \in \mathbf{E}^{M+N} | \mathbf{v} \in \mathbf{V}, \mathbf{p} \in \mathbf{NY}(\mathbf{v})\},$$

$$\mathbf{D}^0 = \{(\mathbf{v},\mathbf{p}) \in \mathbf{E}^{M+N} | \mathbf{v} \in \mathbf{V}, \mathbf{p} \in \operatorname{int} \mathbf{NY}(\mathbf{v})\}.$$

On the domain $\mathbf{D}$, define the support function G by

$$G(\mathbf{v},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{y} | \mathbf{y} \in \mathbf{Y}(\mathbf{v})\}.$$

On the domain $\mathbf{D}^0$, let $\mathbf{\Gamma}(\mathbf{v},\mathbf{p})$ denote the sub-differential of G, and let

$$\mathbf{\Phi}(\mathbf{v},\mathbf{p}) = \{\mathbf{y} \in \mathbf{Y}(\mathbf{v}) | \mathbf{p}\cdot\mathbf{y} \geqq \mathbf{p}\cdot\mathbf{w} \text{ for all } \mathbf{w} \in \mathbf{Y}(\mathbf{v})\}$$

denote the maximand correspondence. The abbreviated notation $\langle G,\mathbf{D}\rangle$,$\langle \mathbf{\Gamma},\mathbf{C}^0\rangle$, $\langle \mathbf{\Phi},\mathbf{D}^0\rangle$ will also be used for these mappings.

15.2. *Lemma.* If $\langle \mathbf{Y},\mathbf{V}\rangle$ is a strongly continuous semi-bounded-valued correspondence, then (1) $\langle \mathbf{AY},\mathbf{V}\rangle$ is a upper hemicontinuous correspondence; (2) $\langle \mathbf{NY},\mathbf{V}\rangle$ is a lower hemicontinuous correspondence; and (3) if $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{R} \subseteq \operatorname{int} \mathbf{NY}(\mathbf{v})$ is non-empty and compact, then there exists j_0 such that

$$\mathbf{R} \subseteq \operatorname{int} \mathbf{NY}(\mathbf{v}^j) \quad \text{for} \quad j \geqq j_0,$$

and

$$\mathbf{Z} = \bigcup_{j \geqq j_0} \bigcup_{\mathbf{p} \in \mathbf{R}} \mathbf{\Phi}(\mathbf{v}^j,\mathbf{p})$$

is bounded.

Proof: If $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{y}^j \in \mathbf{AY}(\mathbf{v}^j)$, $\mathbf{y}^j \to \mathbf{y}$, then there exist $\mathbf{w}^j \in \mathbf{Y}(\mathbf{v}^j)$ and $\theta_j \geqq 0$ such that $\theta_j < j^{-1}$ and $|\theta_j \mathbf{w}^j - \mathbf{y}^j| < j^{-1}$, implying $\theta_j \mathbf{w}^j \to \mathbf{y}$. Then, $\mathbf{y} \in \mathbf{AY}(\mathbf{v})$ by strong continuity. This verifies (1).

Consider $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{R} \subseteq \operatorname{int} \mathbf{NY}(\mathbf{v})$, $\mathbf{R}$ non-empty and compact. Suppose for an infinite subsequence of j, there exists $\mathbf{p}^j \in \mathbf{R}$ with $\mathbf{p}^j \notin \operatorname{int} \mathbf{NY}(\mathbf{v}^j)$, implying $\mathbf{p}^j\cdot\mathbf{y}^j \geqq 0$ for some $\mathbf{y}^j \in \mathbf{AY}(\mathbf{v}^j)$ with $|\mathbf{y}^j| = 1$. Then, there exists a subsequence of $(\mathbf{p}^j,\mathbf{y}^j)$ converging to $(\mathbf{p},\mathbf{y})$ with $\mathbf{p} \in \mathbf{R}$, $\mathbf{y} \in \mathbf{AY}(\mathbf{v})$ by (1), and $\mathbf{p}\cdot\mathbf{y} \geqq 0$, contradicting the definition of $\mathbf{R}$. Hence, there exists j_0 such that $\mathbf{R} \subseteq \operatorname{int} \mathbf{NY}(\mathbf{v}^j)$ for $j \geqq j_0$. Next suppose there exists $\mathbf{p}^j \in \mathbf{R}$, $\mathbf{y}^j \in \mathbf{\Phi}(\mathbf{v}^j,\mathbf{p}^j)$ for $j \geqq j_0$ with $\mathbf{y}^j$ unbounded. Then there exists a subsequence of j with $\mathbf{y}^j/|\mathbf{y}^j|$ converging to $\mathbf{u} \in \mathbf{AY}(\mathbf{v})$ by strong continuity and $\mathbf{p}^j$ converging to $\mathbf{p} \in \mathbf{R}$. But for any $\mathbf{w} \in \mathbf{Y}(\mathbf{v})$, by lower hemicontinuity there exists $\mathbf{w}^j \in \mathbf{Y}(\mathbf{v}^j)$ with $\mathbf{w}^j \to \mathbf{w}$, and $\mathbf{p}^j\cdot\mathbf{y}^j = G(\mathbf{v}^j,\mathbf{p}^j) \geqq$

$\mathbf{p}^j\cdot\mathbf{w}^j \to \mathbf{p}\cdot\mathbf{w}$, implying $\mathbf{p}^j\cdot\mathbf{y}^j/|\mathbf{y}^j| \to \mathbf{p}\cdot\mathbf{u} \geqq 0$ and contradicting the definition of $\mathbf{R}$. This verifies (3).

Finally, suppose $(\mathbf{v},\mathbf{p}) \in \mathbf{D}$, $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v}$. There exist $\mathbf{p}^i \in \operatorname{int}\mathbf{NY}(\mathbf{v})$, $\mathbf{p}^i \to \mathbf{p}$. $\{\mathbf{p},\mathbf{p}^i\}$ is a compact subset of $\operatorname{int}\mathbf{NY}(\mathbf{v})$, and thus by (3) there exists j_0 such that $\mathbf{p}^i \in \mathbf{NY}(\mathbf{v}^{j_0+i})$, verifying (2). Q.E.D.

15.3. *Lemma.* If $\langle\mathbf{Y},\mathbf{V}\rangle$ is a strongly continuous semi-bounded-valued correspondence, then (1) for fixed $\mathbf{v} \in \mathbf{V}$, the support function $\langle G,\mathbf{NY}(\mathbf{v})\rangle$ is a convex, closed, positively linear homogeneous function of $\mathbf{p} \in \mathbf{NY}(\mathbf{v})$; (2) $\langle G,\mathbf{D}^0\rangle$ is continuous [i.e., $G(\mathbf{v},\mathbf{p})$ is continuous jointly in $\mathbf{v}$ and $\mathbf{p}$ at each $(\mathbf{v},\mathbf{p}) \in \mathbf{D}^0$]; (3) $\langle G,\mathbf{D}\rangle$ is lower hemicontinuous; i.e., if $(\mathbf{v},\mathbf{p}) \in \mathbf{D}$, $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v}$, then

$$G(\mathbf{v},\mathbf{p}) = \liminf_{\substack{\mathbf{p}^j \in \mathbf{NY}(\mathbf{v}^j)\\ \mathbf{p}^j \to \mathbf{p}}} G(\mathbf{v}^j,\mathbf{p}^j);$$

(4) If $(\mathbf{v}^j,\mathbf{p}^j) \in \mathbf{D}$, $(\mathbf{v}^j,\mathbf{p}^j) \to (\mathbf{v},\mathbf{p})$ with $\mathbf{v} \in \mathbf{V}$, $\mathbf{p} \notin \mathbf{NY}(\mathbf{v})$, then $\lim_j G(\mathbf{v}^j,\mathbf{p}^j) = +\infty$.

Proof: Result 12.4 implies (1). We next establish an inequality used to prove (3) and (4). Suppose $(\mathbf{v}^j,\mathbf{p}^j) \in \mathbf{D}$ is a sequence converging to $(\mathbf{v},\mathbf{p})$, $\mathbf{v} \in \mathbf{V}$, for which the limit of $G(\mathbf{v}^j,\mathbf{p}^j)$, possibly infinite, exists. Take $\mathbf{y}^i \in \mathbf{Y}(\mathbf{v})$ with $\mathbf{p}\cdot\mathbf{y}^i \to G(\mathbf{v},\mathbf{p})$. By the lower hemicontinuity of $\mathbf{Y}$, there exists a subsequence j_i and points $\mathbf{w}^i \in \mathbf{Y}(\mathbf{v}^{j_i})$ such that $|\mathbf{w}^i - \mathbf{y}^i| < i^{-1}$ and $|\mathbf{p}^{j_i}\cdot\mathbf{w}^i - \mathbf{p}\cdot\mathbf{y}^i| < i^{-1}$. Then,

$$G(\mathbf{v}^{j_i},\mathbf{p}^{j_i}) \geqq \mathbf{p}^{j_i}\cdot\mathbf{w}^i,$$

and

$$G(\mathbf{v},\mathbf{p}) = \lim_i \mathbf{p}\cdot\mathbf{y}^i \leqq \lim_j G(\mathbf{v}^j,\mathbf{p}^j).$$

If $\mathbf{p} \notin \mathbf{NY}(\mathbf{v})$, then $G(\mathbf{v},\mathbf{p}) = +\infty$, and (4) holds. If $\mathbf{p} \in \mathbf{NY}(\mathbf{v})$, then $G(\mathbf{v},\mathbf{p}) \leqq \liminf_{\substack{(\mathbf{v}^j,\mathbf{p}^j) \in \mathbf{D}\\ (\mathbf{v}^j,\mathbf{p}^j) \to (\mathbf{v},\mathbf{p})}} G(\mathbf{v}^j,\mathbf{p}^j)$.

Next suppose $(\mathbf{v}^j,\mathbf{p}^j) \in \mathbf{D}$, $(\mathbf{v}^j,\mathbf{p}^j) \to (\mathbf{v},\mathbf{p}) \in \mathbf{D}^0$. For some j_0, the set $\mathbf{Z} = \{\mathbf{p},\mathbf{p}^{j_0},\mathbf{p}^{j_0+1},\ldots\}$ is a compact subset of $\operatorname{int}\mathbf{NY}(\mathbf{v})$. By 15.2(3) and 13.5, $\mathbf{Z} \subseteq \operatorname{int}\mathbf{NY}(\mathbf{v}^j)$ and there exists a bounded sequence $\mathbf{y}^j \in \mathbf{\Phi}(\mathbf{v}^j,\mathbf{p}^j)$ for j large. Then, there exists a subsequence of j with $\mathbf{y}^j$ converging to $\mathbf{y} \in \mathbf{Y}(\mathbf{v})$ by upper hemicontinuity. Then, retaining the same notation for this subsequence, $\lim_j G(\mathbf{v}^j,\mathbf{p}^j) = \lim_j \mathbf{p}^j\cdot\mathbf{y}^j = \mathbf{p}\cdot\mathbf{y} \leqq G(\mathbf{v},\mathbf{p})$. Since the opposite inequality was shown to hold above, this verifies (2).

Finally, suppose $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{p} \in \mathbf{NY}(\mathbf{v})$. By the closedness and convexity of G in $\mathbf{p}$ for fixed $\mathbf{v}$, there exists a sequence $\mathbf{p}^i \in \operatorname{int} \mathbf{NY}(\mathbf{v})$ such that $\mathbf{p}^i \to \mathbf{p}$ and $G(\mathbf{v},\mathbf{p}) = \lim_i G(\mathbf{v},\mathbf{p}^i)$. By (2) proved above, for each i there exists j_i such that $\mathbf{p}^i \in \mathbf{NY}(\mathbf{v}^j)$ and $|G(\mathbf{v},\mathbf{p}^i) - G(\mathbf{v}^j,\mathbf{p}^i)| < i^{-1}$ for $j \geqq j_i$. Take $q^j = \mathbf{p}^i$ for $j_i \leqq j < j_{i+1}$. Then, $\lim_j G(\mathbf{v}^j,q^j) = G(\mathbf{v},\mathbf{p})$, and (3) holds. Q.E.D.

15.4. *Lemma.* If $\langle \mathbf{Y},\mathbf{V}\rangle$ is a strongly continuous, semi-bounded-valued correspondence, then the maximand correspondence $\langle \mathbf{\Phi},\mathbf{D}^0\rangle$ is upper hemicontinuous.

Proof: Suppose $(\mathbf{v}^j,\mathbf{p}^j) \in \mathbf{D}^0$, $(\mathbf{v}^j,\mathbf{p}^j) \to (\mathbf{v},\mathbf{p}) \in \mathbf{D}^0$, $\mathbf{y}^j \in \mathbf{\Phi}(\mathbf{v}^j,\mathbf{p}^j)$, $\mathbf{y}^j \to \mathbf{y}$. For any $\mathbf{w} \in \mathbf{Y}(\mathbf{v})$, by lower hemicontinuity there exist $\mathbf{w}^j \in \mathbf{Y}(\mathbf{v}^j)$, $\mathbf{w}^j \to \mathbf{w}$. Then, $\mathbf{p}^j \cdot \mathbf{w}^j \leqq \mathbf{p}^j \cdot \mathbf{y}^j$, implying in the limit that $\mathbf{p} \cdot \mathbf{w} \leqq \mathbf{p} \cdot \mathbf{y}$. Since $\mathbf{y} \in \mathbf{Y}(\mathbf{v})$ by upper hemicontinuity, this implies $\mathbf{y} \in \mathbf{\Phi}(\mathbf{v},\mathbf{p})$. Q.E.D.

15.5. *Lemma.* Suppose $\mathbf{V} \subseteq \mathbf{E}^M$ is non-empty, $\langle \mathbf{K},\mathbf{V}\rangle$ is a lower hemicontinuous correspondence mapping $\mathbf{v} \in \mathbf{V}$ into $\mathbf{K}(\mathbf{v}) \subseteq \mathbf{E}^N$, with $\mathbf{K}(\mathbf{v})$ a pointed convex cone and $\operatorname{int} \mathbf{K}(\mathbf{v}) \neq \emptyset$ for each $\mathbf{v} \in \mathbf{V}$. Define

$$\mathbf{D} = \{(\mathbf{v},\mathbf{p}) \in \mathbf{E}^{M+N} | \mathbf{v} \in \mathbf{V}, \mathbf{p} \in \mathbf{K}(\mathbf{v})\},$$

$$\mathbf{D}^0 = \{(\mathbf{v},\mathbf{p}) \in \mathbf{D} | \mathbf{p} \in \operatorname{int} \mathbf{K}(\mathbf{v})\},$$

and suppose $\langle \mathbf{F},\mathbf{D}\rangle$ is a closed real-valued function with $\langle F,\mathbf{D}^0\rangle$ continuous (for definitions, see 15.3) such that for each fixed $\mathbf{v} \in \mathbf{V}$, $\langle F,\mathbf{K}(\mathbf{v})\rangle$ is convex, closed, and positively linear homogeneous (as a function of $\mathbf{p}$). Then, the correspondence $\langle \mathbf{Y},\mathbf{V}\rangle$ defined by $\mathbf{Y}(\mathbf{v}) = \{\mathbf{Y} \in \mathbf{E}^N | \mathbf{p} \cdot \mathbf{y} \leqq F(\mathbf{v},\mathbf{p})$ for all $\mathbf{p} \in \mathbf{K}(\mathbf{v})\}$ is strongly continuous, convex-valued, semi-bounded-valued, with $\operatorname{int} \mathbf{AY}(\mathbf{v}) \neq \emptyset$.

Proof: By 12.5, $\mathbf{Y}(\mathbf{v})$ is non-empty, closed, convex, and semi-bounded. By 11.6 and 10.9, $\operatorname{int} \mathbf{Y}(\mathbf{v}) \neq \emptyset$. Suppose $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{y}^j \in \mathbf{Y}(\mathbf{v}^j)$, $\mathbf{y}^j \to \mathbf{y}$. If $\mathbf{p} \in \operatorname{int} \mathbf{K}(\mathbf{v})$, then by lower hemicontinuity there exist $\mathbf{p}^j \in \mathbf{K}(\mathbf{v}^j)$ with $\mathbf{p}^j \to \mathbf{p}$, implying $\mathbf{p}^j \cdot \mathbf{y}^j \leqq F(\mathbf{v}^j,\mathbf{p}^j)$. Taking the limit, $\mathbf{p} \cdot \mathbf{y} \leqq F(\mathbf{v},\mathbf{p})$ by the continuity of $\langle F,\mathbf{D}^0\rangle$. For any $\mathbf{p} \in \mathbf{K}(\mathbf{v})$, there exist $\mathbf{p}^i \in \operatorname{int} \mathbf{K}(\mathbf{v})$, $\mathbf{p}^i \to \mathbf{p}$, and $F(\mathbf{v},\mathbf{p}^i) \to F(\mathbf{v},\mathbf{p})$ by the closedness of $\langle F,\mathbf{D}\rangle$. Hence, $\mathbf{p}^i \cdot \mathbf{y} \leqq F(\mathbf{v},\mathbf{p}^i)$ implies in the limit $\mathbf{p} \cdot \mathbf{y} \leqq F(\mathbf{v},\mathbf{p})$. Therefore, $\mathbf{y} \in \mathbf{Y}(\mathbf{v})$, implying $\langle \mathbf{Y},\mathbf{V}\rangle$ upper hemicontinuous.

Next suppose $\mathbf{v}^j \in \mathbf{V}$, $\mathbf{v}^j \to \mathbf{v} \in \mathbf{V}$, $\mathbf{y} \in \operatorname{int} \mathbf{Y}(\mathbf{v})$. Then, there exists $\alpha > 0$ such that $\mathbf{y} + \alpha\mathbf{p} \in \mathbf{Y}(\mathbf{v})$ for all $\mathbf{p} \in \mathbf{E}^N$ with $\mathbf{p} \cdot \mathbf{p} = 1$. Hence, $\mathbf{p} \cdot (\mathbf{y} + \alpha\mathbf{p}) \leqq F(\mathbf{v},\mathbf{p})$, or $\mathbf{p} \cdot \mathbf{y} \leqq F(\mathbf{v},\mathbf{p}) - \alpha$, for all $\mathbf{p} \in \mathbf{K}(\mathbf{v})$ with $\mathbf{p} \cdot \mathbf{p} = 1$. Suppose that for an infinite subsequence of j, $\mathbf{y} \notin \mathbf{Y}(\mathbf{v}^j)$. Then, there exists $\mathbf{p}^j \in \operatorname{int} \mathbf{K}(\mathbf{v}^j)$

such that $\mathbf{p}^j \cdot \mathbf{p}^j = 1$ and $\mathbf{p}^j \cdot \mathbf{y} > F(\mathbf{v}^j, \mathbf{p}^j)$. Extract a subsequence (retain notation) with $\mathbf{p}^j \to \mathbf{p}$. Then, the closedness of $\langle F, \mathbf{D} \rangle$ implies in the limit the inequality $\mathbf{p} \cdot \mathbf{y} \geqq F(\mathbf{v}, \mathbf{p})$. This condition also implies $\mathbf{p} \in \mathbf{K}(\mathbf{v})$. But then the inequality $\mathbf{p} \cdot \mathbf{y} \leqq F(\mathbf{v}, \mathbf{p}) - \alpha$ is contradicted. Therefore, $\mathbf{y} \in \operatorname{int} \mathbf{Y}(\mathbf{v})$ implies $\mathbf{y} \in \mathbf{Y}(\mathbf{v}^j)$ for j large. Finally, for any $\mathbf{y} \in \mathbf{Y}(\mathbf{v})$, there exist $\mathbf{y}^i \in \operatorname{int} \mathbf{Y}(\mathbf{v})$, $\mathbf{y}^i \to \mathbf{y}$. Then, there exists a subsequence j_i of j such that $\mathbf{y}^i \in \mathbf{Y}(\mathbf{v}^{j_i})$. Hence, $\langle \mathbf{Y}, \mathbf{V} \rangle$ is lower hemicontinuous, and therefore continuous. By 13.3, it is strongly continuous. Q.E.D.

15.6. *Lemma.* If $\langle \mathbf{Y}, \mathbf{V} \rangle$ is a lower hemicontinuous convex-valued correspondence, then there exists a continuous function $\mathbf{y}^*: \mathbf{V} \to \mathbf{E}^N$ such that $\mathbf{y}^*(\mathbf{v}) \in \operatorname{intr} \mathbf{Y}(\mathbf{v})$ for each $\mathbf{v} \in \mathbf{V}$.

Proof: $\langle \mathbf{Y}, \mathbf{V} \rangle$ lower hemicontinuous and convex implies $\langle \operatorname{intr} \mathbf{Y}, \mathbf{V} \rangle$ lower hemicontinuous. Then a theorem of Michael [see Parthasarathy (1971, Thm. 1.1)] establishes the existence of a continuous function $\mathbf{y}^*$ with $\mathbf{y}^*(\mathbf{v}) \in \mathbf{Y}(\mathbf{v})$. Q.E.D.

16. Differential Properties

In this section, we examine the relation between the curvature of the surface of a closed convex set $\mathbf{Y}$ and the curvature of its support function.

16.1. *Definition.* Consider a non-empty, closed, convex, semi-bounded set $\mathbf{Y} \subseteq \mathbf{E}^N$ and assume without loss (by translation, if necessary) $\mathbf{0} \in \mathbf{Y}$. Let $\mathbf{T}^{\mathbf{Y}}$ denote the cone spanned by $\mathbf{Y}$, and let $\mathbf{S}^{\mathbf{Y}} = \mathbf{N}\mathbf{Y}$. Define the real-valued function $\langle H^{\mathbf{Y}}, \mathbf{T}^{\mathbf{Y}} \rangle$ by $H^{\mathbf{Y}}(\mathbf{y}) = \inf\{\lambda > 0 | (1/\lambda)y \in \mathbf{Y}\}$ for $\mathbf{y} \in \mathbf{T}^{\mathbf{Y}}$. $\langle H^{\mathbf{Y}}, \mathbf{T}^{\mathbf{Y}} \rangle$ is termed the *gauge function* of $\mathbf{Y}$.

16.2. *Lemma.* If $\mathbf{Y} \subseteq \mathbf{E}^N$ is non-empty, convex, closed, and semi-bounded, and $\mathbf{0} \in \mathbf{Y}$, then (1) $\langle H^{\mathbf{Y}}, \mathbf{T}^{\mathbf{Y}} \rangle$ is convex, closed, and positively linear homogeneous; (2) for $\mathbf{y} \in \mathbf{T}^{\mathbf{Y}}$, $\lambda \geqq H^{\mathbf{Y}}(\mathbf{y}) > 0$ implies $\mathbf{y}/\lambda \in \mathbf{Y}$, $H^{\mathbf{Y}}(\mathbf{y}) > \lambda > 0$ implies $\mathbf{y}/\lambda \notin \mathbf{Y}$, and $H^{\mathbf{Y}}(\mathbf{y}) = 0$ implies $\mathbf{y} \in \mathbf{A}\mathbf{Y} \subseteq \mathbf{Y}$; (3) the sub-differential $\langle \mathbf{\Lambda}^{\mathbf{Y}}, \operatorname{intr} \mathbf{T}^{\mathbf{Y}} \rangle$ of $\langle H^{\mathbf{Y}}, \mathbf{T}^{\mathbf{Y}} \rangle$ is an upper hemicontinuous correspondence; and (4) if $\mathbf{0} \in \operatorname{intr} \mathbf{Y}$, then $\mathbf{T}^{\mathbf{Y}} = \operatorname{intr} \mathbf{T}^{\mathbf{Y}}$ is linear subspace.

Proof: It is an immediate consequence of the definition of $\langle H^{\mathbf{Y}}, \mathbf{T}^{\mathbf{Y}} \rangle$ that this function exists, is non-negative and positively linear homogeneous, and satisfies (2). For $\mathbf{y}$, $\mathbf{z} \in \mathbf{T}^{\mathbf{Y}}$, consider any $\lambda, \mu > 0$ such that $\mathbf{y}/\lambda \in \mathbf{Y}, \mathbf{z}/\mu \in \mathbf{Y}$, and let $\theta = \lambda/(\lambda + \mu)$. Then $\theta(\mathbf{y}/\lambda) + (1 - \theta)(\mathbf{z}/\mu) =$

$(\mathbf{y}+\mathbf{z})/(\lambda+\mu)\in\mathbf{Y}$ by convexity, implying $H^{\mathbf{Y}}(\mathbf{y}+\mathbf{z})\leqq\lambda+\mu$. Taking $\lambda\to H^{\mathbf{Y}}(\mathbf{y})$, $\mu\to H^{\mathbf{Y}}(\mathbf{z})$ establishes convexity of $H^{\mathbf{Y}}$. The following argument establishes $H^{\mathbf{Y}}$ closed. Suppose $\mathbf{y}^i\in\mathbf{T}^{\mathbf{Y}}$, $\mathbf{y}^i\to\mathbf{y}\neq\mathbf{0}$. For any $\epsilon>0$, there exist λ_i such that either $\mathbf{y}^i\in\mathbf{AY}$ and $\lambda_i=0$, or $\mathbf{y}^i\notin\mathbf{AY}$ and $\mathbf{y}^i/\lambda_i\notin\mathbf{Y}$, $\mathbf{y}^i/(\lambda_i+\epsilon)\in\mathbf{Y}$. If $\liminf\lambda_i=\lambda$ is finite, then $\mathbf{y}/(\lambda+\epsilon)\in\mathbf{Y}$, implying $H^{\mathbf{Y}}(\mathbf{y})\leqq\lambda+\epsilon\leqq\epsilon+\liminf H^{\mathbf{Y}}(\mathbf{y}^i)$. If $\mathbf{y}\notin\mathbf{T}^{\mathbf{Y}}$, then $+\infty=\liminf\lambda_i$, implying $\liminf H^{\mathbf{Y}}(\mathbf{y}^i)=+\infty$. At $\mathbf{y}=\mathbf{0}$, $H^{\mathbf{Y}}(\mathbf{0})=0\leqq H^{\mathbf{Y}}(\mathbf{y}^i)$. Hence, $H^{\mathbf{Y}}$ is closed, and (1) holds. Result (3) follows from 13.8, and (4) from the definition of the relative interior of a set. Q.E.D.

16.3. *Lemma.* The gauge function $\langle H^{\mathbf{Y}},\mathbf{T}^{\mathbf{Y}}\rangle$ and the support function $\langle G^{\mathbf{Y}},\mathbf{S}^{\mathbf{Y}}\rangle$ of a non-empty, closed, convex, semi-bounded set $\mathbf{Y}\subseteq\mathbf{E}^N$ with $\mathbf{0}\in\operatorname{intr}\mathbf{Y}$ are related by: (1) $\mathbf{p}\cdot\mathbf{y}\leqq G^{\mathbf{Y}}(\mathbf{p})H^{\mathbf{Y}}(\mathbf{y})$ for all $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}$, $\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}$. If $\mathbf{p}\in\operatorname{intr}\mathbf{S}^{\mathbf{Y}}$ and $\mathbf{y}\in\operatorname{intr}\mathbf{T}^{\mathbf{Y}}$, then for $G^{\mathbf{Y}}(\mathbf{p})>0$, equality holds if and only if $\mathbf{p}/G^{\mathbf{Y}}(\mathbf{p})\in\Lambda^{\mathbf{Y}}(\mathbf{y})$, and for $H^{\mathbf{Y}}(\mathbf{y})>0$, equality holds if and only if $\mathbf{y}/H^{\mathbf{Y}}(\mathbf{y})\in\Gamma^{\mathbf{Y}}(\mathbf{p})$. (2) $G^{\mathbf{Y}}(\mathbf{p})=\inf\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda H^{\mathbf{Y}}(\mathbf{y})$ for all $\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}\}$ for $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}$. (3) $H^{\mathbf{Y}}(\mathbf{y})=\inf\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda G^{\mathbf{Y}}(\mathbf{p})$ for all $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}\}$ for $\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}$. (4) For $\mathbf{p}\in\operatorname{intr}\mathbf{S}^{\mathbf{Y}}$, $\mathbf{y}\in\operatorname{intr}\mathbf{T}^{\mathbf{Y}}$, $\mathbf{p}\in\Lambda^{\mathbf{Y}}(\mathbf{y})$ if and only if $H^{\mathbf{Y}}(\mathbf{y})=\mathbf{p}\cdot\mathbf{y}$ and $G^{\mathbf{Y}}(\mathbf{p})=1$, and $\mathbf{y}\in\Gamma^{\mathbf{Y}}(\mathbf{p})$ if and only if $G^{\mathbf{Y}}(\mathbf{p})=\mathbf{p}\cdot\mathbf{y}$ and $H^{\mathbf{Y}}(\mathbf{y})=1$.

Remark: $\langle H^{\mathbf{Y}},T^{\mathbf{Y}}\rangle$ and $\langle G^{\mathbf{Y}},\mathbf{S}^{\mathbf{Y}}\rangle$ are termed *polar reciprocal functions*, and the sets $\mathbf{Y}=\{\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}|H^{\mathbf{Y}}(\mathbf{y})\leqq1\}$ and $\mathbf{P}=\{\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}|G^{\mathbf{Y}}(\mathbf{p})\leqq1\}$ are termed *polar reciprocal sets*.

Proof: If $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}$, $\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda H^{\mathbf{Y}}(\mathbf{y})$ for all $\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}\}=\{\lambda>0|\mathbf{p}\cdot(\mathbf{y}/H^{\mathbf{Y}}(\mathbf{y}))\leqq\lambda$ for all $\mathbf{y}\in\mathbf{T}^{\mathbf{Y}}$ with $H^{\mathbf{Y}}(\mathbf{y})>0\}=\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda$ for all $\mathbf{y}\in\mathbf{Y}\}$, where the first equality follows from 16.2 (2) and the second equality follows from the definition of $H^{\mathbf{Y}}$ and $\mathbf{AY}=\mathbf{PS}^{\mathbf{Y}}$. Hence, (2) holds. Letting $\lambda\to G^{\mathbf{Y}}(\mathbf{p})$ in the condition $\mathbf{p}\cdot\mathbf{y}\leqq\lambda H^{\mathbf{Y}}(\mathbf{y})$ in (2) verifies the inequality in (1). To show (3), note that $H^{\mathbf{Y}}(\mathbf{y})\leqq1$ implies $\mathbf{y}\in\mathbf{Y}=\{\mathbf{v}\in\mathbf{T}^{\mathbf{Y}}|\mathbf{p}\cdot\mathbf{v}\leqq G^{\mathbf{Y}}(\mathbf{p})$ for all $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}\}$, and hence $\inf\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda G^{\mathbf{Y}}(\mathbf{p})$ for all $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}\}\leqq1$, and that $H^{\mathbf{Y}}(\mathbf{y})>1$ implies $\mathbf{p}\cdot\mathbf{y}>G^{\mathbf{Y}}(\mathbf{p})$ for some $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}$, and hence $\inf\{\lambda>0|\mathbf{p}\cdot\mathbf{y}\leqq\lambda G^{\mathbf{Y}}(\mathbf{p})$ for all $\mathbf{p}\in\mathbf{S}^{\mathbf{Y}}\}>1$. Then, homogeneity implies (3). Next, the "if and only if" conditions in (1) will be verified.

(a) Assume $G^{\mathbf{Y}}(\mathbf{p})>0$. If $\mathbf{p}\in G^{\mathbf{Y}}(\mathbf{p})\Lambda^{\mathbf{Y}}(\mathbf{y})$, then $\mathbf{p}/G^{\mathbf{Y}}(\mathbf{p})\in\Lambda^{\mathbf{Y}}(\mathbf{y})$ implies $\mathbf{p}\cdot\mathbf{y}/G^{\mathbf{Y}}(\mathbf{p})=H^{\mathbf{Y}}(\mathbf{y})$ by 13.8 (1). Conversely, if $\mathbf{p}\cdot\mathbf{y}=G^{\mathbf{Y}}(\mathbf{p})H^{\mathbf{Y}}(\mathbf{y})$, then $\mathbf{y}\cdot\mathbf{p}/G^{\mathbf{Y}}(\mathbf{p})=H^{\mathbf{Y}}(\mathbf{y})$ and $\mathbf{z}\cdot\mathbf{p}/G^{\mathbf{Y}}(\mathbf{p})\leqq H^{\mathbf{Y}}(\mathbf{z})$ for all $\mathbf{z}\in\mathbf{T}^{\mathbf{Y}}$, implying $\mathbf{p}/G^{\mathbf{Y}}(\mathbf{p})\in\Lambda^{\mathbf{Y}}(\mathbf{y})$ by 13.8 (1). (b) Assume $H^{\mathbf{Y}}(\mathbf{y})>0$. By an argument symmetric to that in (a), $\mathbf{y}/H^{\mathbf{Y}}(\mathbf{y})\in\Gamma^{\mathbf{Y}}(\mathbf{p})$ if and only if $\mathbf{p}\cdot\mathbf{y}=G^{\mathbf{Y}}(\mathbf{p})H^{\mathbf{Y}}(\mathbf{y})$.

Result (4) follows from (1) and 13.8. Q.E.D.

16.4. *Definition.* A convex, closed, positively linear homogeneous function $\langle F, \mathbf{X}\rangle$ is *exposed* at $\mathbf{x} \in \text{intr}\,\mathbf{X}$ if $(\mathbf{x}, F(\mathbf{x}))$ is a point in an exposed ray in the set $\mathbf{A} = \{(\mathbf{x},\xi) \in \mathbf{E}^{N+1} | \mathbf{x} \in \mathbf{X},\ \ \xi \geqq F(\mathbf{x})\}$, or, equivalently, there exists $\mathbf{y} \in \boldsymbol{\Gamma}(\mathbf{x})$, where $\langle \boldsymbol{\Gamma}, \text{intr}\,\mathbf{X}\rangle$ is the sub-differential of $\langle F, \mathbf{X}\rangle$, such that $F(\mathbf{z}) - F(\mathbf{x}) > \mathbf{y}\cdot(\mathbf{z} - \mathbf{x})$ for all $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$. $\langle F, \mathbf{X}\rangle$ is *strictly quasiconvex* at $\mathbf{x} \in \text{intr}\,\mathbf{X}$ if $F(\theta\mathbf{x} + (1-\theta)\mathbf{z}) < \theta F(\mathbf{x}) + (1-\theta)F(\mathbf{z})$ for $0 < \theta < 1$ and $\mathbf{z} \in \mathbf{X}, \mathbf{z}$ not proportional to $\mathbf{x}$. When $\text{int}\,\mathbf{X} \neq \boldsymbol{\emptyset}$, $\langle F, \mathbf{X}\rangle$ is *strictly differentially quasiconvex* at $\mathbf{x} \in \text{int}\,\mathbf{X}$ if $\langle F, \mathbf{X}\rangle$ has a first and second differential at $\mathbf{x}$ and the quadratic form $Q(\mathbf{v}, \mathbf{F}''(\mathbf{x}))$ in the Hessian matrix $\mathbf{F}''(\mathbf{x})$ is positive for $\mathbf{v}$ not proportional to $\mathbf{x}$ (i.e., $\mathbf{F}''(\mathbf{x})$ is non-negative definite and of rank $N-1$).

16.5. *Lemma.* Consider a convex, closed, positively linear homogeneous function $\langle F, \mathbf{X}\rangle$ with sub-differential $\langle \boldsymbol{\Gamma}, \text{intr}\,\mathbf{X}\rangle$. (1) For $\mathbf{x} \in \text{intr}\,\mathbf{X}$, $F(\mathbf{z}) - F(\mathbf{x}) > \mathbf{y}\cdot(\mathbf{z}-\mathbf{x})$ for all $\mathbf{y} \in \text{intr}\,\boldsymbol{\Gamma}(\mathbf{x})$, $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$, if and only if $\langle F, \mathbf{X}\rangle$ is exposed at $\mathbf{x}$. (2) For $\mathbf{x} \in \text{intr}\,\mathbf{X}$, $F(\mathbf{z}) - F(\mathbf{x}) > \mathbf{y}\cdot(\mathbf{z}-\mathbf{x})$ for all $\mathbf{y} \in \boldsymbol{\Gamma}(\mathbf{x})$, $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$, if and only if $\langle F, \mathbf{X}\rangle$ is strictly quasiconvex at $\mathbf{x}$. For $\mathbf{x} \in \text{intr}\,\mathbf{X}$, $\langle F, \mathbf{X}\rangle$ strictly quasiconvex at $\mathbf{x}$ implies $\langle F, \mathbf{X}\rangle$ exposed at $\mathbf{x}$. (3) If $\text{int}\,\mathbf{X} \neq \boldsymbol{\emptyset}$ and $\langle F, \mathbf{X}\rangle$ is strictly differentially quasiconvex at $\mathbf{x} \in \text{int}\,\mathbf{X}$, then $\langle F, \mathbf{X}\rangle$ is strictly quasiconvex at $\mathbf{x}$. (4) If $\text{int}\,\mathbf{X} \neq \boldsymbol{\emptyset}$, $\langle F, \mathbf{X}\rangle$ possesses continuous first and second differentials in a neighborhood of $\mathbf{x} \in \text{int}\,\mathbf{X}$, and $\langle F, \mathbf{X}\rangle$ is strictly quasiconvex at $\mathbf{x}$, then for any neighborhood $\mathbf{Z}$ of $\mathbf{x}$, there exists a neighborhood $\mathbf{W}$ contained in $\mathbf{Z}$ such that $\langle F, \mathbf{X}\rangle$ is strictly differentially quasiconvex on $\mathbf{W}$.

Proof: (1) The "only if" condition follows from the definition of $\langle F, \mathbf{X}\rangle$ exposed at $\mathbf{x}$. To show the "if" condition, suppose $\langle F, \mathbf{X}\rangle$ exposed at $\mathbf{x}$. Then, $F(\mathbf{z}) - F(\mathbf{x}) > \mathbf{y}\cdot(\mathbf{z}-\mathbf{x})$ for $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$, and some $\mathbf{y} \in \boldsymbol{\Gamma}(\mathbf{x})$. Suppose that for some $\mathbf{v} \in \text{intr}\,\boldsymbol{\Gamma}(\mathbf{x})$ and $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$, $F(\mathbf{z}) - F(\mathbf{x}) = \mathbf{v}\cdot(\mathbf{z}-\mathbf{x})$. Then, $(1+\theta)\mathbf{v} - \theta\mathbf{y} \in \boldsymbol{\Gamma}(\mathbf{x})$ for θ small positive, implying $F(\mathbf{z}) - F(\mathbf{x}) < ((1+\theta)\mathbf{v} - \theta\mathbf{y})\cdot(\mathbf{z}-\mathbf{x})$ and contradicting the definition of $\boldsymbol{\Gamma}$.

(2) If $\langle F, \mathbf{X}\rangle$ is strictly quasiconvex at $\mathbf{x}$, but there exists $\mathbf{y} \in \boldsymbol{\Gamma}(\mathbf{x})$, $\mathbf{z} \in \mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$ such that $F(\mathbf{z}) - F(\mathbf{x}) = \mathbf{y}\cdot(\mathbf{z}-\mathbf{x})$, then $F((\mathbf{z}+\mathbf{x})/2) - F(\mathbf{x}) < (F(\mathbf{z}) - F(\mathbf{x}))/2 = \mathbf{y}\cdot((\mathbf{z}-\mathbf{x})/2)$, contradicting the definition of $\boldsymbol{\Gamma}$. Hence, the "if" implication in (2) holds, and (1) then implies the last result in (2). Next, the "only if" implication will be established. Suppose $\langle F, \mathbf{X}\rangle$ is not strictly quasiconvex at $\mathbf{x} \in \text{intr}\,\mathbf{X}$. Then, there exists $\mathbf{z} \in \mathbf{X}, \mathbf{z}$ not proportional to $\mathbf{x}$, $\alpha \in (0,1)$ such that $F(\alpha\mathbf{z} + (1-\alpha)\mathbf{x}) =$

$\alpha F(\mathbf{z})+(1-\alpha)F(\mathbf{x})$. From the proof of 13.8, paragraph (b), we have $(F(\theta\mathbf{u}+(1-\theta)\mathbf{v})-F(\mathbf{v}))/\theta$ non-decreasing in $\theta>0$ for $\mathbf{u},\mathbf{v}\in\mathbf{X}$. For $\gamma\in(0,1)$, $F(\gamma\mathbf{z}+(1-\gamma)\mathbf{x})\leqq\gamma F(\mathbf{z})+(1-\gamma)F(\mathbf{x})$ by convexity. If $\alpha<\gamma<1$ and $\theta=\gamma/\alpha>1$, then

$$\frac{F(\theta(\alpha\mathbf{z}+(1-\alpha)\mathbf{x})+(1-\theta)\mathbf{x})-F(\mathbf{x})}{\theta}\geqq F(\alpha\mathbf{z}+(1-\alpha)\mathbf{x})-F(\mathbf{x})$$
$$=\alpha[F(\mathbf{z})-F(\mathbf{x})],$$

or

$$F(\gamma\mathbf{z}+(1-\gamma)\mathbf{x})\geqq\gamma F(\mathbf{z})+(1-\gamma)F(\mathbf{x}).$$

If $0<\gamma<\alpha$ and $\theta=(1-\gamma)/(1-\alpha)$, then

$$\frac{F(\theta(\alpha\mathbf{z}+(1-\alpha)\mathbf{x})+(1-\theta)\mathbf{z})-F(\mathbf{z})}{\theta}\geqq F(\alpha\mathbf{z}+(1-\alpha)\mathbf{x})-F(\mathbf{x})$$
$$=(1-\alpha)[F(\mathbf{x})-F(\mathbf{z})],$$

or

$$F(\gamma\mathbf{z}+(1-\gamma)\mathbf{x})\geqq\gamma F(\mathbf{z})+(1-\gamma)F(\mathbf{x}).$$

Hence,

$$F(\gamma\mathbf{z}+(1-\gamma)\mathbf{x})=\gamma F(\mathbf{z})+(1-\gamma)F(\mathbf{x}),$$

for $\gamma\in(0,1)$.

Set $\mathbf{A}=\{(\mathbf{x},\xi)\in\mathbf{E}^{N+1}|\mathbf{x}\in\mathbf{X},\ \xi\geqq F(\mathbf{x})\}$ and the flat $\mathbf{B}=\{(\theta\mathbf{z}+(1-\theta)\mathbf{x},\ \theta F(\mathbf{z})+(1-\theta)F(\mathbf{x}))|\theta \text{ real}\}$. Then, $\mathbf{A}$ is convex and $\mathbf{B}\cap\operatorname{intr}\mathbf{A}=\emptyset$, implying by 10.14 the existence of a hyperplane $\mathbf{H}((\mathbf{y},-\eta),\alpha)$ with $\mathbf{B}\subseteq\mathbf{H}((\mathbf{y},-\eta),\alpha)$, $\mathbf{A}\subseteq\mathbf{H}^-((\mathbf{y},-\eta),\alpha)$, and $\operatorname{intr}\mathbf{A}\cap\mathbf{H}((\mathbf{y},-\eta),\alpha)=\emptyset$. Using the argument of paragraph (a) of the proof of 13.8, we can normalize $\eta=1$ and obtain the implication $\mathbf{y}\in\mathbf{\Gamma}(\mathbf{x})$. But $(\mathbf{z},F(\mathbf{z}))\in\mathbf{B}$ implies $F(\mathbf{z})-F(\mathbf{x})=\mathbf{y}\cdot(\mathbf{z}-\mathbf{x})$, contradicting the hypothesis in (2).

(3) As in the proof of (2), if $\langle F,\mathbf{X}\rangle$ is not strictly quasiconvex, then there exists $\mathbf{z}\in\mathbf{X}$, $\mathbf{z}$ not proportional to $\mathbf{x}$, such that $F(\theta\mathbf{z}+(1-\theta)\mathbf{x})=\theta F(\mathbf{z})+(1-\theta)F(\mathbf{x})$ for all $\theta\in(0,1)$. Letting $\mathbf{v}=\theta\mathbf{z}+(1-\theta)\mathbf{x}$, F has a second-order expansion given in 12.1, $F(\mathbf{v})=F(\mathbf{x})+\mathbf{y}\cdot(\mathbf{v}-\mathbf{x})+\frac{1}{2}\theta^2Q(\mathbf{z}-\mathbf{x},\mathbf{F}''(\mathbf{x}))+i(\theta^2|\mathbf{z}-\mathbf{x}|^2)$, where $\mathbf{y}=\mathbf{F}'(\mathbf{x})$ and $\mathbf{\Gamma}(\mathbf{x})=\{\mathbf{y}\}$. For $\theta>0$ small, $\frac{1}{2}\theta^2Q(\mathbf{z}-\mathbf{x},\mathbf{F}''(\mathbf{x}))+i(\theta^2|\mathbf{z}-\mathbf{x}|^2)>0$, implying $F(\mathbf{v})-F(\mathbf{x})>\mathbf{y}\cdot(\mathbf{v}-\mathbf{x})$. But this contradicts the supposition, and $\langle F,\mathbf{X}\rangle$ is strictly quasiconvex at $\mathbf{x}$.

(4) Consider any neighborhood $\mathbf{U}$ of $\mathbf{x},\mathbf{U}\subseteq\mathbf{X}$, in which F is twice continuously differentiable. Define $\tilde{\mathbf{x}}=(x_1,\ldots,x_{N-1})$ and $\tilde{F}(\tilde{\mathbf{z}})=F(\tilde{\mathbf{z}},x_N)$

for $\tilde{\mathbf{z}} \in \mathbf{E}^{N-1}$ such that $(\tilde{\mathbf{z}}, x_N) \in \mathbf{U}$. Since F is strictly quasiconvex at $\mathbf{x}$, $\tilde{F}(\tilde{\mathbf{x}}+\tilde{\mathbf{y}}) - F(\tilde{\mathbf{x}}) - \mathbf{F}'(\tilde{\mathbf{x}})\tilde{\mathbf{y}} > 0$ for $(\tilde{\mathbf{x}}+\tilde{\mathbf{y}}, x_N) \in \mathbf{U}$, $\tilde{\mathbf{y}} \neq 0$. By continuity, there exists a neighborhood $\tilde{\mathbf{U}}$ of $\tilde{\mathbf{x}}$ and $\alpha > 0$ such that $|\tilde{\mathbf{y}}| \leqq \alpha$ and $\tilde{\mathbf{z}} \in \tilde{\mathbf{U}}$ implies $(\tilde{\mathbf{z}}+\tilde{\mathbf{y}}, x_N) \in \mathbf{U}$ and $\tilde{F}(\tilde{\mathbf{z}}+\tilde{\mathbf{y}}) - \tilde{F}(\tilde{\mathbf{z}}) - \tilde{\mathbf{F}}'(\tilde{\mathbf{z}})\tilde{\mathbf{y}} > 0$ for $\tilde{\mathbf{y}} \neq \mathbf{0}$. Then, $\tilde{F}$ is strictly convex on $\tilde{\mathbf{U}}$, and a theorem of Bernstein and Toupin (1962) establishes that the hessian $\tilde{\mathbf{F}}''$ is positive definite on an open dense subset of $\tilde{\mathbf{U}}$. Let $\tilde{\mathbf{W}} \subseteq \tilde{\mathbf{U}}$ be a neighborhood on which $\tilde{\mathbf{F}}''$ is positive definite, and define $\mathbf{W} = \{(\tilde{\mathbf{y}}, y_N) \in \mathbf{U} | (x_N/y_N)\tilde{\mathbf{y}} \in \tilde{\mathbf{W}}\}$. The hessian matrix of F on $\mathbf{W}$ is

$$\left[\begin{array}{c|c} \frac{x_N}{y_N}\tilde{\mathbf{F}}''\left(\frac{x_N}{y_N}\tilde{\mathbf{y}}\right) & -\frac{x_N}{y_N^2}\tilde{\mathbf{F}}''\left(\frac{x_N}{y_N}\tilde{\mathbf{y}}\right)\tilde{\mathbf{y}} \\ \hline -\frac{x_N}{y_N^2}\tilde{\mathbf{y}}'\tilde{\mathbf{F}}''\left(\frac{x_N}{y_N}\tilde{\mathbf{y}}\right) & \frac{x_N}{y_N^3}\tilde{\mathbf{y}}'\tilde{\mathbf{F}}''\left(\frac{x_N}{y_N}\tilde{\mathbf{y}}\right)\tilde{\mathbf{y}} \end{array}\right].$$

By construction, this matrix is of rank $N-1$. Q.E.D.

16.6. *Definition*. Suppose $\mathbf{Y} \subseteq \mathbf{E}^N$ is closed, convex, and semi-bounded, with $\mathbf{0} \in \operatorname{int} \mathbf{Y}$, and define $\mathbf{Y}^* = \{\mathbf{y} \in \mathbf{E}^N | \mathbf{y} \in \mathbf{\Phi}(\mathbf{p})$ for some $\mathbf{p} \in \mathbf{S}\}$, where $\mathbf{S} = \operatorname{int} \mathbf{NY}$ and $\langle \mathbf{\Phi}, \mathbf{S} \rangle$ is the maximand correspondence of $\mathbf{Y}$. Note that $\mathbf{0} \in \operatorname{int} \mathbf{Y}$ implies $\mathbf{p} \cdot \mathbf{y} = G(\mathbf{p}) > 0$ for $\mathbf{p} \in \mathbf{S}$, $\mathbf{y} \in \mathbf{\Phi}(\mathbf{p})$, and hence, by 16.3 (1), $H(\mathbf{y}) = 1$.

16.7. *Lemma*. Suppose $\mathbf{Y} \subseteq \mathbf{E}^N$ is closed, convex, and semi-bounded, with $\mathbf{0} \in \operatorname{int} \mathbf{Y}$. Let $\langle G, \mathbf{NY} \rangle$ denote the support function of $\mathbf{Y}$, and $\langle \mathbf{\Gamma}, \mathbf{S} \rangle$ denote its sub-differential. Let $\langle H, \mathbf{E}^N \rangle$ denote the gauge function of $\mathbf{Y}$, and $\langle \mathbf{\Lambda}, \mathbf{E}^N \rangle$ denote its sub-differential. Then, the following conditions hold: (1) If $\mathbf{y} \in \mathbf{E}^N$, then $\mathbf{\Lambda}(\mathbf{y}) \subseteq \mathbf{NY}$. If $\mathbf{y} \in \mathbf{Y}^*$, then $\operatorname{intr} \mathbf{\Lambda}(\mathbf{y}) \subseteq \mathbf{S}$. If $\mathbf{y} \in \mathbf{Y}^*$ and $\langle H, \mathbf{E}^N \rangle$ is strictly quasiconvex at $\mathbf{y}$, then $\mathbf{\Lambda}(\mathbf{y}) \subseteq \mathbf{S}$. (2) If $\mathbf{p} \in \mathbf{S}$, then $\mathbf{\Gamma}(\mathbf{p}) \subseteq \mathbf{Y}^*$. (3) $\langle H, \mathbf{E}^N \rangle$ differentiable at $\mathbf{y} \in \mathbf{Y}^*$ implies $\langle G, \mathbf{NY} \rangle$ exposed at $\mathbf{p} \in \mathbf{\Lambda}(\mathbf{y}) \subseteq \mathbf{S}$. (4) $\langle G, \mathbf{NY} \rangle$ differentiable at $\mathbf{p} \in \mathbf{S}$ implies $\langle H, \mathbf{E}^N \rangle$ exposed at $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{p})$. (5) $\langle H, \mathbf{E}^N \rangle$ exposed at $\mathbf{y} \in \mathbf{Y}^*$ implies $\langle G, \mathbf{NY} \rangle$ differentiable at $\mathbf{p} \in \operatorname{intr} \mathbf{\Lambda}(\mathbf{y})$. (6) $\langle G, \mathbf{NY} \rangle$ exposed at $\mathbf{p} \in \mathbf{S}$ implies $\langle H, \mathbf{E}^N \rangle$ differentiable at $\mathbf{y} \in \operatorname{intr} \mathbf{\Gamma}(\mathbf{p})$. (7) $\langle H, \mathbf{E}^N \rangle$ strictly quasiconvex at $\mathbf{y} \in \mathbf{Y}^*$ implies $\langle G, \mathbf{NY} \rangle$ differentiable at $\mathbf{p} \in \mathbf{\Lambda}(\mathbf{y})$. (8) $\langle G, \mathbf{NY} \rangle$ strictly quasiconvex at $\mathbf{p} \in \mathbf{S}$ implies $\langle H, \mathbf{E}^N \rangle$ differentiable at $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{p})$. (9) $\langle H, \mathbf{E}^N \rangle$ differentiable at all $\mathbf{y} \in \mathbf{\Gamma}(\mathbf{p})$ for some $\mathbf{p} \in \mathbf{S}$ implies $\langle G, \mathbf{NY} \rangle$ strictly quasiconvex at $\mathbf{p}$. (10) $\langle G, \mathbf{NY} \rangle$ differentiable at all $\mathbf{p} \in \mathbf{\Lambda}(\mathbf{y})$ for some $\mathbf{y} \in \mathbf{Y}^*$ implies $\langle H, \mathbf{E}^N \rangle$ strictly quasiconvex at $\mathbf{y}$. (11) If $\langle H, \mathbf{E}^N \rangle$ possesses a continuous first and second differential in a neighborhood of $\mathbf{y} \in \mathbf{Y}^*$, and is strictly differentially quasiconvex at $\mathbf{y}$, then $\langle G, \mathbf{NY}\rangle$ possesses a continuous first and second differential in a neighborhood of $\mathbf{p} \in \Lambda(\mathbf{y})$, and is strictly differentially quasiconvex at $\mathbf{p}$. (12) If $\langle G, \mathbf{NY}\rangle$ possesses a continuous first and second differential in a neighborhood of $\mathbf{p} \in \mathbf{S}$, and is strictly differentially quasiconvex at $\mathbf{p}$, then $\langle H, \mathbf{E}^N\rangle$ possesses a continuous first and second differential in a neighborhood of $\mathbf{y} \in \Gamma(\mathbf{p})$, and is strictly differentially quasiconvex at $\mathbf{y}$.

Proof: (1) By 13.8, $\mathbf{p} \in \Lambda(\mathbf{y})$ implies $\mathbf{p}\cdot\mathbf{y} = H(\mathbf{y})$ and $\mathbf{p}\cdot\mathbf{z} \leqq H(\mathbf{z})$ for all $\mathbf{z} \in \mathbf{E}^N$. Hence, $G(\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{z} | \mathbf{z} \in \mathbf{Y}\} = 1$ and $\mathbf{p} \in \mathbf{NY}$. If $\mathbf{y} \in \mathbf{Y}^*$, then $\mathbf{y} \in \Gamma(\mathbf{q})$ for some $\mathbf{q} \in \mathbf{S}$ by 13.9. Since $H(\mathbf{y}) = 1$, we can scale $\mathbf{q}$ so that $\mathbf{q}\cdot\mathbf{y} = G(\mathbf{q}) = 1$, and hence $\mathbf{q} \in \Lambda(\mathbf{y})$, by 16.3 (1). If there exists $\mathbf{p} \in \operatorname{intr} \Lambda(\mathbf{y})$, $\mathbf{p} \notin \mathbf{S}$, then there exists $\mathbf{z} \in \mathbf{AY}$, $\mathbf{z} \neq \mathbf{0}$, such that $\mathbf{p}\cdot\mathbf{z} = 0$. Further, $(1+\theta)\mathbf{p} - \theta\mathbf{q} \in \Lambda(\mathbf{y})$ for θ small positive and $((1+\theta)\mathbf{p} - \theta\mathbf{q})\cdot\mathbf{z} > 0$, contradicting $\Lambda(\mathbf{y}) \subseteq \mathbf{NY}$. Hence, $\operatorname{intr} \Lambda(\mathbf{y}) \subseteq \mathbf{S}$. If H is strictly quasiconvex at $\mathbf{y} \in \mathbf{Y}^*$, then by 16.5 (2), $H(\mathbf{z}) - H(\mathbf{y}) > \mathbf{p}\cdot(\mathbf{z}-\mathbf{y})$ for $\mathbf{z} \in \mathbf{E}^N$, $\mathbf{z}$ not proportional to $\mathbf{y}$, $\mathbf{p} \in \Lambda(\mathbf{y})$. Taking $\mathbf{z} = \mathbf{y} + \mathbf{v}$ with $\mathbf{v} \in \mathbf{AY}$, $\mathbf{V} \neq \mathbf{0}$ implies $\mathbf{z} \in \mathbf{Y}$ by 10.9. For some $\mathbf{q} \in \mathbf{S}$, $\mathbf{q}\cdot\mathbf{y} = G(\mathbf{y}) > 0$, implying $\mathbf{v}$, and hence $\mathbf{z}$, not proportional to $\mathbf{y}$. Therefore, $0 \geqq H(\mathbf{z}) - H(\mathbf{y}) > \mathbf{p}\cdot\mathbf{v}$. But $\mathbf{p}\cdot\mathbf{v} < 0$ for all $\mathbf{v} \in \mathbf{AY}$, $\mathbf{v} \neq \mathbf{0}$, implies $\mathbf{p} \in \mathbf{S}$.

(2) If $\mathbf{p} \in \mathbf{S}$, then $\Gamma(\mathbf{p}) = \Phi(\mathbf{p}) \subseteq \mathbf{Y}^*$ by 13.9.

(3) $\langle H, \mathbf{E}^N\rangle$ differentiable at $\mathbf{y} \in \mathbf{Y}^*$ implies $\operatorname{intr} \Lambda(\mathbf{y}) = \Lambda(\mathbf{y}) = \{\mathbf{p}\} \subseteq \mathbf{S}$. For $\mathbf{q} \in \mathbf{S}$, $\mathbf{q}$ not proportional to $\mathbf{p}$, 16.3(1) implies $\mathbf{q}\cdot\mathbf{y} < G(\mathbf{q})H(\mathbf{y})$, and hence $(\mathbf{q}-\mathbf{p})\cdot\mathbf{y}/H(\mathbf{y}) < G(\mathbf{q}) - G(\mathbf{p})$. But this is the condition for $\langle G, \mathbf{NY}\rangle$ to be exposed at $\mathbf{p}$.

(4) $\langle G, \mathbf{NY}\rangle$ differentiable at $\mathbf{p} \in \mathbf{S}$ implies $\Gamma(\mathbf{p}) = \{\mathbf{y}\} \subseteq \mathbf{Y}^*$ and $\mathbf{p}\cdot\mathbf{z} < G(\mathbf{p})H(\mathbf{z})$ for $\mathbf{z} \in \mathbf{E}^N$, $\mathbf{z}$ not proportional to $\mathbf{y}$. Hence, $(\mathbf{z}-\mathbf{y})\cdot\mathbf{p}/G(\mathbf{p}) < H(\mathbf{z}) - H(\mathbf{y})$, and $\langle H, \mathbf{E}^N\rangle$ is exposed at $\mathbf{y}$.

(5) $\langle H, \mathbf{E}^N\rangle$ exposed at $\mathbf{y} \in \mathbf{Y}^*$ implies that for $\mathbf{p} \in \operatorname{intr} \Lambda(\mathbf{y})$, $H(\mathbf{z}) - H(\mathbf{y}) > \mathbf{p}\cdot(\mathbf{z}-\mathbf{y})$ for $\mathbf{z}$ not proportional to $\mathbf{y}$. Then, $\mathbf{p}\cdot\mathbf{y} = H(\mathbf{y}) = 1$ and $\mathbf{p}\cdot\mathbf{z} < 1$ for $\mathbf{z} \in \mathbf{Y}$, $\mathbf{z} \neq \mathbf{y}$, implying $\Gamma(\mathbf{p}) = \Phi(\mathbf{p}) = \{\mathbf{y}\}$. Hence, $\langle G, \mathbf{NY}\rangle$ is differentiable at $\mathbf{p}$.

(6) $\langle G, \mathbf{NY}\rangle$ exposed at $\mathbf{p} \in \mathbf{S}$ implies that $G(\mathbf{q}) - G(\mathbf{p}) > \mathbf{y}\cdot(\mathbf{q}-\mathbf{p})$ for $\mathbf{q} \in \mathbf{NY}$, $\mathbf{q}$ not proportional to $\mathbf{p}$, $\mathbf{y} \in \operatorname{intr} \Gamma(\mathbf{p})$, and hence $\mathbf{y} \in \mathbf{Y}^*$ and $G(\mathbf{q}) > \mathbf{y}\cdot\mathbf{q}$. If $\mathbf{q} \in \Lambda(\mathbf{y})$, then $H(\mathbf{z}) - H(\mathbf{y}) \geqq \mathbf{q}\cdot(\mathbf{z}-\mathbf{y})$, implying $G(\mathbf{q}) = \mathbf{q}\cdot\mathbf{y} \geqq \mathbf{q}\cdot\mathbf{z}$, for all $\mathbf{z} \in \mathbf{Y}$. Then, $\mathbf{q}$ must be proportional to $\mathbf{p}$, and since $H(\mathbf{y}) = \mathbf{q}\cdot\mathbf{y}$, $\mathbf{q} = \mathbf{p}/G(\mathbf{p})$. Hence, $\Lambda(\mathbf{y}) = \{\mathbf{p}/G(\mathbf{p})\}$ and $\langle H, \mathbf{E}^N\rangle$ is differentiable at $\mathbf{y}$.

(7) $\langle H,\mathbf{E}^N\rangle$ strictly quasiconvex at $\mathbf{y}\in\mathbf{Y}^*$ implies, by 16.5(2) and (1) above, that $\Lambda(\mathbf{y})\subseteq\mathbf{S}$ and the proof of (5) above holds for every $\mathbf{p}\in\Lambda(\mathbf{y})$.

(8) By 16.5 (2), the proof of (6) above holds for every $\mathbf{y}\in\Gamma(\mathbf{p})$.

(9) $\langle H,\mathbf{E}^N\rangle$ differentiable at $\mathbf{y}\in\Gamma(\mathbf{p})$, $\mathbf{p}\in\mathbf{S}$ implies, by 16.3(4), that intr $\Lambda(\mathbf{y})=\Lambda(\mathbf{y})=\{\mathbf{p}/G(\mathbf{p})\}\subseteq\mathbf{S}$. Then, $\mathbf{q}\in\mathbf{S}$, $\mathbf{q}$ not proportional to $\mathbf{p}$, and $\mathbf{y}\in\Gamma(\mathbf{p})$ implies $H(\mathbf{y})=1$ and $\mathbf{q}\cdot\mathbf{y}<G(\mathbf{q})$. Hence, $(\mathbf{q}-\mathbf{p})\cdot\mathbf{y}<G(\mathbf{q})-G(\mathbf{p})$ for all $\mathbf{y}\in\Gamma(\mathbf{p})$, and by 16.5(2), $\langle G,\mathbf{NY}\rangle$ is strictly quasiconvex at $\mathbf{p}$.

(10) The proof is the same as that for (9), with the appropriate interchange in notation.

(11) If $\Lambda(\mathbf{y})=\{\mathbf{p}\}$, then $G(\mathbf{p})=\mathbf{p}\cdot\mathbf{y}=1=\text{Max}\{\mathbf{p}\cdot\mathbf{z}|H(\mathbf{z})\leqq 1\}$. Since $H(\mathbf{0})=0<1$, the Kuhn–Tucker theorem [Karlin (1959, Thm. 7.1.1)] can be applied to establish the existence of $\lambda\geqq 0$ such that $\mathbf{p}\cdot\mathbf{z}+\lambda(1-H(\mathbf{z}))\leqq\mathbf{p}\cdot\mathbf{y}$ for all $\mathbf{z}\in\mathbf{E}^N$. Using differentiability, this implies $\mathbf{p}=\lambda\mathbf{H}'(\mathbf{y})$ and $1=\mathbf{p}\cdot\mathbf{y}=\lambda\mathbf{y}\cdot\mathbf{H}'(\mathbf{y})=\lambda$. Hence the system of $N+1$ equations $\mathbf{q}=\lambda\mathbf{H}'(\mathbf{z})$ and $1=H(\mathbf{z})$ has a solution at $\mathbf{q}=\mathbf{p}$, $\lambda=1$, $\mathbf{z}=\mathbf{y}$. Further, the Jacobian matrix of this system exists in a neighborhood of $(\mathbf{q},\lambda,\mathbf{z})=(\mathbf{p},1,\mathbf{y})$, and has the form

$$\mathbf{J}(\lambda,\mathbf{z})=\left[\begin{array}{c|c}\lambda\mathbf{H}''(\mathbf{z}) & \mathbf{H}'(\mathbf{z})\\ \hline (\mathbf{H}'(\mathbf{z}))^T & 0\end{array}\right],$$

where the transpose of any column vector $\mathbf{v}$ is denoted by $\mathbf{v}^T$. By hypothesis, $Q(\mathbf{v},\mathbf{H}''(\mathbf{y}))$ is positive for $\mathbf{v}$ not proportional to $\mathbf{y}$, and hence is positive for non-zero $\mathbf{v}$ satisfying $\mathbf{v}\cdot\mathbf{H}'(\mathbf{y})=0$. Therefore, $\mathbf{J}(1,\mathbf{y})$ is non-singular (Appendix A.1, Lemma 4). Hence, by continuity of the first and second differentials of H, $\mathbf{J}(\lambda,\mathbf{z})$ is non-singular in a neighborhood of $(1,\mathbf{y})$. The implicit function theorem [Bartle (1964, 21.11)] establishes the existence of a neighborhood of $\mathbf{p}$ and continuously differentiable functions $\lambda=L(\mathbf{q})$, $\mathbf{z}=\mathbf{Z}(\mathbf{q})$ defined on this neighborhood such that $\mathbf{q}\equiv L(\mathbf{q})\mathbf{H}'(\mathbf{Z}(\mathbf{q}))$, $1\equiv H(\mathbf{Z}(\mathbf{q}))$, and $L(\mathbf{p})=1$, $\mathbf{Z}(\mathbf{p})=\mathbf{y}$. The Kuhn–Tucker theorem then implies $G(\mathbf{q})=\text{Max}\{\mathbf{q}\cdot\mathbf{z}|H(\mathbf{z})\leqq 1\}=\mathbf{q}\cdot\mathbf{Z}(\mathbf{q})$ on the given neighborhood of $\mathbf{p}$. From the identity $1\equiv H(\mathbf{Z}(\mathbf{q}))$, the implication $\mathbf{0}\equiv\mathbf{Z}'(\mathbf{q})\mathbf{H}'(\mathbf{Z}(\mathbf{q}))$, where $\mathbf{Z}'(\mathbf{q})$ is the $N\times N$ matrix of partial derivatives of $\mathbf{Z}(\mathbf{q})$, is obtained by differentiation. Hence, $\mathbf{G}'(\mathbf{q})=\mathbf{Z}(\mathbf{q})+\mathbf{Z}'(\mathbf{q})\mathbf{q}=\mathbf{Z}(\mathbf{q})+L(\mathbf{q})\mathbf{Z}'(\mathbf{q})\mathbf{H}'(\mathbf{Z}(\mathbf{q}))=\mathbf{Z}(\mathbf{q})$ exists and is continuously differentiable, implying that the first and second differentials of G exist and are continuous in the given neighborhood of $\mathbf{p}$. Finally, $\mathbf{G}''(\mathbf{q})=\mathbf{Z}'(\mathbf{q})$ satisfies $\mathbf{Z}'(\mathbf{q})\mathbf{q}=\mathbf{0}$ and the matrix equation

$$\mathbf{J}(L(\mathbf{q}),\mathbf{Z}(\mathbf{q}))\left[\begin{array}{c}\mathbf{Z}'(\mathbf{q})\\ \hline (\mathbf{L}'(\mathbf{q}))^T\end{array}\right]=\left[\begin{array}{c}\mathbf{I}_{N\times N}\\ \hline \mathbf{0}^T_{N\times 1}\end{array}\right],$$

where $\mathbf{I}_{N\times N}$ is the N-dimensional identity matrix, $\mathbf{0}_{N\times 1}$ is the N-dimensional column vector of zeros. Since the right-hand side of this equation is of rank N, the matrix

$$\left[\frac{\mathbf{Z}'(\mathbf{q})}{(\mathbf{L}'(\mathbf{q}))^T}\right]$$

must be of rank N, and hence $\mathbf{Z}'(\mathbf{q})$ must be at least of rank $N-1$. Since G is convex, $\mathbf{Z}'(\mathbf{q})$ is non-negative definite, implying the quadratic form $Q(\mathbf{r},\mathbf{Z}'(\mathbf{q}))$ positive for r not proportional to q. This establishes $\langle G,\mathbf{NY}\rangle$ strictly differentially quasiconvex on a neighborhood of $\mathbf{p}$.

(12) Without loss, assume $G(\mathbf{p})=1$. If $\mathbf{\Gamma}(\mathbf{p})=\{\mathbf{y}\}$, then from 16.3, $H(\mathbf{y})=1=\inf\{\lambda>0|\mathbf{q}\cdot\mathbf{y}\leqq\lambda G(\mathbf{q})$ for all $\mathbf{q}\in\mathbf{S}\}=\mathrm{Max}\{\mathbf{q}\cdot\mathbf{y}|\mathbf{q}\in\mathbf{S},\ G(\mathbf{q})=1\}=\mathbf{p}\cdot\mathbf{y}$. As in the proof of (11), the Kuhn–Tucker theorem can be applied to establish the existence of λ such that $\mathbf{y}=\lambda\mathbf{G}'(\mathbf{p})$ and $1=\mathbf{p}\cdot\mathbf{y}=\lambda\mathbf{p}\cdot G'(\mathbf{p})=\lambda$. The result then follows by an argument completely symmetric with the proof of (11). Q.E.D.