Chapter I.2

SYMMETRIC DUALITY AND POLAR PRODUCTION FUNCTIONS

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1. Introduction

The Shephard–Uzawa–McFadden duality theorems[1] relating production functions to cost and profit functions, may be utilized to generate new valid functional forms for production functions and production frontiers, or equivalently, new valid cost and profit functions. To any given standard production relation, namely one which satisfies the conditions for existence and uniqueness of the dual, there corresponds at least one other standard production relation, which satisfies the same requirements, but may exhibit rather different specific patterns. This process of getting "two for the price of one" in the search for useful functional forms is made possible by reformulating the duality relations in a perfectly symmetric way. The process has been applied before to production as well as to consumer demand theory – although it seems to have never been formalized and recognized as a generally valid procedure.

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[1]See Shephard (1953), Uzawa (1964), and McFadden (Chapter I.1). More detailed presentations and modified proofs are in Diewert (1971, 1973a).

As will be clarified below, the specific formulation used in the established analysis of duality, self-duality, and related topics in utility theory,[2] has unfortunately hindered the development of analogous results in production. In particular, the choice of a utility-index representation – which is arbitrary in the context of an ordinal utility function – could not be carried over as it stood to cardinal production functions, where output is measurable and non-negative.

Heuristically, given any standard production, cost, or profit function, a standard *polar* cost, profit, or production function is obtained by a transformation from the variable quantities (prices) space into the corresponding prices (quantities) space, using the functional form of the dual relation for the new, polar, primal relation. Moreover, the fixed quantity, such as output in cost minimization analysis, is transformed into its reciprocal. In order to show this more rigorously, however, it is necessary to modify the formulation of the duality relations, so as to get a perfect symmetry between the primal and the dual – with exactly the same type of restrictions on sets and on functions appearing on both sides. This is done for cost functions in Section 2, and for profit functions and joint-production frontiers in Section 4.

Section 3 establishes the existence and uniqueness of the polar production and cost functions, as well as some special modifications for homothetic production functions and for separable production frontiers. In Section 5 similar results are stated and proved with respect to profit functions and joint production frontiers. Section 6 discusses some extensions and an application to various definitions of elasticities of substitution. Examples of specific functional forms generated by this approach are given in Chapter II.3.

2. A Symmetric Formulation of Cost and Production Functions

Suppose $y = f(\mathbf{x})$ is a standard production function, satisfying the following regularity conditions[3] for existence of a unique dual cost function $C = G(y;\mathbf{p})$:

> *Condition I.* $f(\mathbf{x})$ is defined for all $\mathbf{x} = \{x_1,...,x_n\} \geq \mathbf{0}$ $(\mathbf{x} \in \bar{\Omega}_n)$, and is real, single-valued, right-continuous, non-decreasing in $\mathbf{x}$, quasi-

[2]E.g., in Houthakker (1965), Samuelson (1947, 1965a), and Lau (1969a).
[3]For a specification of these conditions and a proof, see Diewert (1971).

concave, finite for finite $\mathbf{x}$, and unbounded for at least some unbounded sequence $\{\mathbf{x}^N\}$, with $f(\mathbf{0}) = 0$.

For any non-negative output y, the production possibilities set $\mathbf{L}(y) \subseteq \bar{\Omega}_n$ is defined as

$$\mathbf{L}(y) = \{\mathbf{x}: f(\mathbf{x}) \geq y\}, \tag{1}$$

and satisfies the following:

Condition II.[4] For $y \geq 0$, $\mathbf{L}(y)$ is a non-empty, closed, convex set, with free disposal:

$$\mathbf{x}' > \mathbf{x} \in \mathbf{L}(y) \Rightarrow \mathbf{x}' \in \mathbf{L}(y); \quad y' > y \Rightarrow \mathbf{L}(y') \subseteq \mathbf{L}(y),$$

where for all $\mathbf{x}$ there exists a $y' > 0$ such that $\mathbf{x} \notin \mathbf{L}(y')$ [$\mathbf{x}$ is not in $\mathbf{L}(y')$]; and $\mathbf{L}(0) = \bar{\Omega}_n$. If $y > 0$, then $\mathbf{0} \notin \mathbf{L}(y)$. The set $\{(y,\mathbf{x}): \mathbf{x} \in \mathbf{L}(y)\}$ (the graph of $\mathbf{L}$) is closed.

Given a positive output y, it is always possible to represent uniquely (for strictly positive vectors $\mathbf{x} \gg \mathbf{0}$) the standard production set $\mathbf{L}(y)$, and the production function equation $f(\mathbf{x}) = y$, by a normalized equation of the form: $D(1/y;\mathbf{x}) = 1$, such that the "distance function" $D(1/y;\mathbf{x})$ behaves with respect to its arguments $(1/y;\mathbf{x})$ exactly in the same manner as a standard cost function with respect to $(y;\mathbf{p})$. This statement is now formalized and proved. Define the distance function[5] as follows (where $\mathbf{\Omega}_n$ is the positive orthant, $\mathbf{\Omega}_n = \{\mathbf{x}: \mathbf{x} \gg \mathbf{0}\}$):

$$\begin{aligned} D(1/y;\mathbf{x}) &= \sup\{d: (1/d)\mathbf{x} \in \mathbf{L}(y); \mathbf{x} \in \mathbf{\Omega}_n\} \\ &= \sup\{d: f((1/d)\mathbf{x}) \geq y; \mathbf{x} \in \mathbf{\Omega}_n\}, \end{aligned} \tag{2}$$

by equation (1).

Theorem 1. If $\mathbf{L}(y)$ defined in equation (1) satisfies the conditions on standard production possibilities sets (Condition II), the function $D(1/y;\mathbf{x})$ defined in equation (2) satisfies Condition III below. The set $\mathbf{L}^*(y) = \{\mathbf{x}: D(1/y;\mathbf{x}) \geq 1; \mathbf{x} \in \mathbf{\Omega}_n\}$ coincides with the set $\mathbf{L}(y)$ for $\mathbf{x} \gg \mathbf{0}: \mathbf{L}^*(y) \equiv \mathbf{L}(y) \cap \mathbf{\Omega}_n$.

[4]These are Condition II(2.7) in Diewert (1971), with slight modifications. The notation adopted here is: $\mathbf{x}' \geq \mathbf{x}$ means $x_i' \geq x_i$ (all i); $\mathbf{x}' > \mathbf{x}$ means $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{x}' \neq \mathbf{x}$; $\mathbf{x}' \gg \mathbf{x}$ means $x_i' > x_i$ (all i). $\mathbf{x}'$ is the transpose of $\mathbf{x}$.

[5]The distance function was first used for isoquants and unit cost functions (for the differentiable case) by Shephard (1953, p. 6). However, Shephard did not show the symmetry with respect to y, $1/y$, respectively, of the distance functions.

Condition III.[6] The function $D(1/y;\mathbf{x})$ is

(a) positive, real-valued, defined, and finite for all finite $\mathbf{x} \gg \mathbf{0}$, $1/y > 0$;

(b) non-decreasing in $1/y$ and unbounded if $1/y$ is unbounded;

(c) non-decreasing in $\mathbf{x}$;

(d) positive linear homogeneous in $\mathbf{x}$, for all finite $1/y > 0$; i.e., if $\lambda > 0$, $1/y > 0$ and finite, and $\mathbf{x} \gg \mathbf{0}$, then $D(1/y;\lambda\mathbf{x}) = \lambda D(1/y;\mathbf{x})$;

(e) concave in $\mathbf{x}$, for finite $1/y > 0$.

(f) continuous from below (left-continuous) in $1/y$.

Proof: (a) If $\mathbf{x} \in \mathbf{L}(y) \cap \boldsymbol{\Omega}_n$, $y > 0$, then by definition [equation (2)] D exists and $D \geq 1$, since $(\mathbf{x}/1) \in \mathbf{L}(y)$. D is finite, since $\lim_{d\to\infty}(1/d)\mathbf{x} = \mathbf{0} \notin \mathbf{L}(y)$. If $\mathbf{0} \ll \mathbf{x} \notin \mathbf{L}(y)$, then since $\mathbf{L}(y)$ is not empty, there exists an $\mathbf{x}^0$, such that $\mathbf{0} \ll \mathbf{x}^0 \in \mathbf{L}(y)$. Let $d_0 = \mathrm{Min}_i\{x_i/x_i^0\} > 0$; then $(1/d_0)\mathbf{x} \geq \mathbf{x}^0$ and thus $(1/d_0)\mathbf{x} \in \mathbf{L}(y)$, by the free disposal assumption. Thus $D(1/y;\mathbf{x})$ exists and $D \geq d_0 > 0$. Also, $D < 1$; since if $D \geq 1$, $(1/(1-\epsilon))\mathbf{x} \geq (1/(D-\epsilon))\mathbf{x} \in \mathbf{L}(y)$, and by taking limits as $\epsilon \to 0$, the closedness of $\mathbf{L}(y)$ implies that $\mathbf{x} \in \mathbf{L}(y)$, a contradiction. We have thus proved that $D \geq 1 \Leftrightarrow \mathbf{0} \ll \mathbf{x} \in \mathbf{L}(y)$, which implies the last statement of Theorem 1:

$$\mathbf{L}^*(y) = \{\mathbf{x}: D(1/y,\mathbf{x}) \geq 1; \mathbf{x} \in \boldsymbol{\Omega}_n\} \equiv \mathbf{L}(y) \cap \boldsymbol{\Omega}_n.$$

This identity, in addition to equation (2), leads almost immediately to the proof of Conditions III(b) – III(f):

(b) Since $\mathbf{L}(0) = \bar{\boldsymbol{\Omega}}_n$, and $y' > y \Rightarrow \mathbf{L}(y') \subseteq \mathbf{L}(y)$, Condition III(b) follows from equation (2).

(c) Follows from the free disposal assumption.

(d) $D(1/y;\mathbf{x})$ is linear-homogeneous in $\mathbf{x}$, since for $\lambda > 0$, $(1/\lambda d)(\lambda\mathbf{x}) = (1/d)\mathbf{x}$.

(e) The convexity of $\mathbf{L}(y)$ implies that $D(1/y;\mathbf{x})$ is quasi-concave in $\mathbf{x}$. Since D is linear-homogeneous in $\mathbf{x}$, it is concave in $\mathbf{x}$.

(f) The continuity from below in $1/y$ follows from the closedness of the graph of $\mathbf{L}(y)$. Q.E.D.

It should be noted, that since $\mathbf{L}(y)$ is the closure of $\mathbf{L}^*(y)$, one could extend the definition of $D(1/y;\mathbf{x})$ to all non-negative $\mathbf{x}$, by assuming that $D = 0$ if $(1/d)\mathbf{x} \notin \mathbf{L}(y)$ for all $d > 0$, $y \geq 0$ [or $\lim_{d\to 0}(1/d)\mathbf{x} \in \mathbf{L}(y)$]. In this case,

[6] See Diewert (1971, Condition III, 2.13) for specification of similar conditions with respect to the cost function $c(y;\mathbf{p})$. Cf. also Shephard (1953) and Uzawa (1964).

$$\mathbf{L}'(y) = \{\mathbf{x}: D(1/y;\mathbf{x}) \geq 1; \mathbf{x} \geq \mathbf{0}\} \equiv \mathbf{L}(y).$$

We now analyze the dual relation, i.e., the cost function. The Shephard duality theorem, as extended by Uzawa and McFadden, states that the function

$$C = G(y;\mathbf{p}) = \text{Min}_{\mathbf{x}}\{\mathbf{p}'\mathbf{x}: \mathbf{x} \in \mathbf{L}(y)\} \tag{3}$$

(where $\mathbf{p}'\mathbf{x} = \sum_i p_i x_i$) is uniquely determined by $\mathbf{L}(y)$ and satisfies Condition III above, with $(y;\mathbf{p})$ substituted for $(1/y;\mathbf{x})$. The set of $\mathbf{p}$ (for given y), defined by the equation $G(y;\mathbf{p}) = 1$, is the *unit cost frontier.* Define the *unit cost set* as follows:

$$\mathbf{V}(1/y) = \{\mathbf{p}: G(y;\mathbf{p}) \geq 1\}. \tag{4}$$

Since G satisfies Condition III, the previous discussion suffices to establish that $\mathbf{V}(1/y)$ satisfies, in $\mathbf{\Omega}_n$, exactly the same condition as $\mathbf{L}(y)$ (Condition II) with $1/y$ substituted for y, and that $G(y;\mathbf{p})$ is the "distance function" corresponding to $\mathbf{V}(1/y)$; that is,

$$G(y;\mathbf{p}) = \sup\{g: (1/g)\mathbf{p} \in \mathbf{V}(1/y); \mathbf{p} \in \mathbf{\Omega}_n\}. \tag{5}$$

Due to the perfect symmetry established between the sets $\mathbf{L}(y)$ and $\mathbf{V}(1/y)$, and the functions $D(1/y;\mathbf{x})$ and $G(y;\mathbf{p})$, it is now possible to apply the duality theorem "in reverse", without changing the proof, to obtain the following theorem:

Theorem 2. $$D(1/y;\mathbf{x}) = \text{Min}_{\mathbf{p}}\{\mathbf{x}'\mathbf{p}: \mathbf{p} \in \mathbf{V}(1/y)\}, \tag{6}$$

where D and $\mathbf{V}$, are defined by equations (2) and (4), respectively.

Proof: Identical with the proof of the duality theorem on costs, with the dual variables $(y;\mathbf{p})$ substituted for the primal variables $(1/y;\mathbf{x})$, and vice-versa.[7] Q.E.D.

In addition, the functions G and D satisfy *Shephard's Lemma*;[8] that is, the first partial derivatives of G or D with respect to an input price or quantity, respectively – whenever they exist – are equal to the corresponding dual variables; i.e., if the derivatives exist,

$$\partial G(y;\mathbf{p})/\partial p_i = x_i^*, \quad \partial D(1/y;\mathbf{x})/\partial x_i = p_i^*. \tag{7}$$

[7] E.g., McFadden (Chapter I.1) or Diewert (1971, Theorem 4).
[8] Shephard (1953). Cf. also Diewert (1971).

The right-hand sides of equations (7) are the input demand and inverse demand functions, respectively (if existing), where p_i^* are the normalized (to yield unit cost) shadow prices (p_i/C):[9]

$$x_i^* = x_i^*(y;\mathbf{p}), \quad p_i^* = p_i^*(1/y;\mathbf{x}).$$

If the equation $D(1/y;\mathbf{x}) = 1$ is solved explicitly for y in terms of $\mathbf{x}$, the result $y = f(\mathbf{x})$ (for $\mathbf{x} \in \Omega_n$) is the production function, satisfying the required regularity condition (Condition I).

Formally,

$$\begin{aligned} y = f(\mathbf{x}) &= \sup\{\eta : D(1/\eta;\mathbf{x}) \geq 1\} \\ &= \sup\{\eta : \mathbf{x} \in \mathbf{L}(\eta)\}. \end{aligned} \tag{8}$$

Similarly, the unit cost equation $G(y;\mathbf{p}) = 1$ may be solved explicitly for $1/y$ $(0 < y < \infty)$,

$$\begin{aligned} 1/y = g(\mathbf{p}) &= \sup\{1/\gamma : G(\gamma,\mathbf{p}) \geq 1\} \\ &= \sup\{1/\gamma : \mathbf{p} \in \mathbf{V}(1/\gamma)\} = 1/\inf\{\gamma : \mathbf{p} \in \mathbf{V}(1/\gamma)\}. \end{aligned} \tag{9}$$

The function $g(\mathbf{p})$ satisfies the same conditions with respect to $\mathbf{p}$ as $f(\mathbf{x})$ with respect to $\mathbf{x}$ (Condition I). In analogy to the accepted terminology of consumer utility theory, the function $h(\mathbf{p}) = 1/g(\mathbf{p})$ may be denoted the "*indirect production function*", corresponding to the direct production function $y = f(\mathbf{x})$ and the function $g(\mathbf{p})$ may be denoted the "reciprocal indirect production function". The equation $g(\mathbf{p}) = 1/y$, for a given y, is the "factor price frontier". Any standard production relation may be uniquely characterized by each of these functions $g(\mathbf{p})$ or $h(\mathbf{p})$.

In the discussion of duality and "self-duality" in utility theory, the accepted formulation[10] for the dual indirect form corresponding to utility $U(\mathbf{x})$ is $-V(\mathbf{p})$ [rather than $1/V(\mathbf{p})$]. However, since utility is ordinal, one could equally choose $e^{U(\mathbf{x})}$ and $e^{-V(\mathbf{p})} = 1/e^{V(\mathbf{p})}$, in analogy to the present results, without affecting the corresponding direct and indirect demand functions, or any real behavior. A similar monotone transformation is not acceptable in production theory (unless output y is replaced by $\log y$), since y is a measurable, non-negative quantity. Equation (9)

[9]The notation $\mathbf{p}^*$ implies both the optimality property of $\mathbf{p}$ (i.e., $\mathbf{p}^*$ are shadow prices) and the normalization to yield unit cost $\mathbf{p}^* = (1/c)\mathbf{p}$.

[10]E.g., Houthakker (1960, 1965), Samuelson (1947, 1965a), Lau (1969a) and Pollak (1972). The indirect utility $V(\mathbf{p})$ referred to here, is in terms of the *n normalized* prices (i.e., per unit of expenditure) and not the alternative (homogeneous) indirect utility $V(E,\mathbf{p})$ which is homogeneous of degree zero in $(n+1)$ arguments – the non-normalized prices $\mathbf{p}$ and expenditure E.

TABLE 1
Symmetric dual relations for cost and production functions.

	Primal (production)	Dual (unit costs)
Variables:	Input quantity x_i Output y	Input price p_i $1/y$
Sets (satisfying Condition II):	Production possibilities $\mathbf{L}(y)$	Unit cost $\mathbf{V}(1/y)$
Functions:		
Distance (satisfying Condition III):	$D(1/y;\mathbf{x}) = \sup\{d:(1/d)\mathbf{x} \in \mathbf{L}(y)\}$	$G(\mathbf{y};\mathbf{p}) = \sup\{g:(1/g)\mathbf{p} \in \mathbf{V}(1/y)\}$
Explicit (satisfying Condition I):	$y = f(\mathbf{x}) = \sup\{\eta:\mathbf{x} \in \mathbf{L}(\eta)\}$	$1/y = g(\mathbf{p}) = \sup\{1/\gamma:\mathbf{p} \in \mathbf{V}(1/\gamma)\}$
Minimum property:	$D(1/y;\mathbf{x}) = \mathrm{Min}_{\mathbf{p}}\{\mathbf{x}'\mathbf{p}:\mathbf{p} \in \mathbf{V}(1/y)\}$	$G(y;\mathbf{p}) = \mathrm{Min}_{\mathbf{x}}\{\mathbf{p}'\mathbf{x}:\mathbf{x} \in \mathbf{L}(y)\}$
Partial derivatives (when existing):	$\partial D/\partial x_i = p_i^*$ (factor's inverse demand)	$\partial G/\partial p_i = x_i^*$ (factor's demand)

seems therefore to be a more natural definition of the indirect production function, since it preserves the perfect symmetry between the primal and the dual representations.

Finally, if $h(y)$ is any strictly monotone function, such that $h(0)=0$ and $h(\infty)=\infty$, one may choose the pair of dual variables to be $h(y)$ and $1/h(y)$ (rather than y and $1/y$), without affecting the results, nor the perfect symmetry of the dual relations. (The significance of this remark is clarified below, in the discussions of homothetic functions and of separable production frontiers.)

Table 1 summarizes these symmetric dual relations for the case of production and cost functions with a single output.

3. Polar Production and Cost Functions

Having demonstrated in the previous section the perfect symmetry between production and cost relations, it is now evident that new valid cost and production relations (namely the *polar* relations[11]), may be obtained by exchanging the roles of the sets $\mathbf{L}(y)$ and $\mathbf{V}(1/y)$, or equivalently the functions $D(1/y;\bar{\mathbf{x}})$ and $G(y;\mathbf{p})$, through substitution of the dual variables $(1/y;\mathbf{p})$ for the primal variables $(y;\mathbf{x})$, and vice-versa (within the positive orthant $\mathbf{\Omega}_{n+1}$). The new cost and production functions thus generated are necessarily standard, satisfying the required conditions.

If the original production function is represented *implicitly* by an identity $F(y;\mathbf{x})\equiv 0$ [where F satisfies the conditions of the implicit function theorem[12] for yielding a unique standard $y=f(\mathbf{x})$], then the unit-cost frontier of the polar production function is given by $F(1/y;\mathbf{p})\equiv 0$, and the polar total cost $D(y;\mathbf{p})$ is defined implicitly by the identity $F(1/y;(1/D)\mathbf{p})\equiv 0$. Conversely, if the original cost function G is given implicitly by $H(y;(1/G)\mathbf{p})\equiv 0$ (G being linear homogeneous in $\mathbf{p}$), the polar production function is given implicitly by $H(1/y;\mathbf{x})\equiv 0$, provided H satisfies the required conditions. Similar substitutions of variables would appear if either the cost or the production function is represented by a

[11]The term *polar* was adopted in accordance with Shephard's geometric interpretation (1953), Ch.5), where the isoquant and the unit cost surfaces are shown to be polar reciprocal to each other with respect to the unit sphere $\Sigma x_i^2=1$.

[12]E.g., in Hadley (1964), Courant (1936). The implicit function theorem for a single function (identity) could be made somewhat stronger, to apply to non-differentiable cases (with only strict monotonicity at 0) such as the general case discussed here.

set of parametric equations. The polar relations are now stated formally:

Theorem 3. Given a standard production function $y = f(\mathbf{x})$ which may be represented uniquely by any one of the following equivalent relations:[13]

Primal	Dual	Satisfying Conditions
(1) PF: $y = f(\mathbf{x})$	(a) InPF: $1/y = g(\mathbf{p})$	I
(2) ImPF: $F(y;\mathbf{x}) \equiv 0$	(b) IIPF: $H(1/y;\mathbf{p}) \equiv 0$	
(3) UDE: $D(1/y;\mathbf{x}) = 1$	(c) UCE: $G(y;\mathbf{p}) = 1$	III
(4) FS: $\mathbf{L}(y)$	(d) UCS: $\mathbf{V}(1/y)$	II

there exists a unique Polar Production Function $y = g(\mathbf{x})$ which is standard, and may be represented uniquely by any one of the following equivalent relations:

Primal	Dual	Satisfying Conditions
(1′) PF: $y = g(\mathbf{x})$	(a′) InPF: $1/y = f(\mathbf{p})$	I
(2′) ImPF: $H(y;\mathbf{x}) \equiv 0$	(b′) IIPF: $F(1/y;\mathbf{p}) \equiv 0$	
(3′) UDE: $G(1/y;\mathbf{x}) = 1$	(c′) UCE: $D(y;\mathbf{p}) = 1$	III
(4′) FS: $\mathbf{V}(y)$	(d′) UCS: $\mathbf{L}(1/y)$	II

Proof: By construction and by the results of Section 2, the Conditions I, II, or III are satisfied with respect to $g(\mathbf{x})$, $\mathbf{V}(y)$ or $D(y;\mathbf{p})$, respectively, for finite $y > 0$, $\mathbf{x} \gg \mathbf{0}$ [that is, $(y;\mathbf{x})$ and $(1/y;\mathbf{p})$ in $\mathbf{\Omega}_{n+1}$]. In order to extend these relations to all $\mathbf{x} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{0}$, $y \geq 0$, one should put $g(\mathbf{0}) = 0$, and extend the definition of $g(\mathbf{x})$ to the (right-hand) limits as $\mathbf{x}$ approaches the boundaries of $\bar{\mathbf{\Omega}}_n$. Equivalently, the sets $\mathbf{V}(y)$ are to be replaced by their closures, i.e., include their boundary points in $\bar{\mathbf{\Omega}}_n$. Similar modifications apply to the other relations in Theorem 3.

[13] PF = production function; ImPF = implicit production function; UDE = unit distance equation; FS = feasible sets; InPF = indirect production function; IIPF = indirect implicit production function; UCE = unit cost equation; UCS = unit cost sets.

The uniqueness of the polar functions follows from their construction. The minimum properties appearing in Theorem 2 and in Table 1, assure that $D(y;\mathbf{p})$ is indeed the cost function corresponding to the production distance function $G(1/y;\mathbf{x})$ by the duality theorem; that is

$$D(y;\mathbf{p}) = \underset{\mathbf{x}}{\text{Min}}\{\mathbf{p}'\mathbf{x}: G(1/y;\mathbf{x}) \geq 1\}, \tag{10}$$

and also

$$G(1/y;\mathbf{x}) = \underset{\mathbf{p}}{\text{Min}}\{\mathbf{x}'\mathbf{p}: D(y;\mathbf{p}) \geq 1\}. \tag{11}$$

This completes our proof. A few additional remarks are in order. First, differentiability of $f(\mathbf{x})$ does not necessarily imply differentiability of the polar function $g(\mathbf{x})$, or vice-versa. This is best demonstrated by the fact that non-differentiable standard production functions may have differentiable dual cost functions, and therefore differentiable polar production functions (such as the Leontief fixed-coefficient function, with a linear differentiable cost function; see Chapter II.3). A separate duality theorem applies to the restricted class of smooth neo-classical production and cost functions,[14] such that both the original and the polar functions belong to this class, and yield everywhere continuously differentiable demand functions. It may be shown, however, that the polar transformation defined here for a more general class, transforms each isoquant surface so that any smooth, strictly convex part is transformed into a smooth counterpart; every planar section into a vertex, and every vertex into a planar section [see Shephard (1953, p. 11)].

Let us now examine the special case of *homothetic functions*. If $f(\mathbf{x})$ is homothetic, it may be written in the form $h(y) = f^*(\mathbf{x})$, where $f^*(\mathbf{x})$ is linear homogenous, and $h(y)$ strictly increases with y from 0 to ∞. The dual unit cost function is separable in this case into $G(y;\mathbf{p}) = h(y)\cdot g^*(\mathbf{p}) = 1$, or $1/h(y) = g^*(\mathbf{p})$, where $g^*(\mathbf{p})$ is the cost of producing the output $y = h^{-1}(1)$, and where the elasticity of total cost with respect to output [see Hanoch (1975b)] is $\eta_{cy} = \partial \log G/\partial \log y = yh'(y)/h(y)$, and is independent of prices! Applying our polar transformation yields the new production function given by $1/h(1/y) = g^*(\mathbf{x})$, with a corresponding unit cost equation, $1/h(1/y)\cdot f^*(\mathbf{p}) = 1$. The new output elasticity of cost is $\eta^*_{cy} = (1/y)(h'(1/y)/h(1/y))$, and the polar function is also homothetic.

[14]Cf. Lau (Chapter I.3) for a presentation and a proof of this restricted duality theorem.

However, if one wishes to impose on the polar function the same behavior of cost with respect to output as in the original – transforming only the form of any given isoquant-surface, but preserving the behavior of output along any given ray $(\lambda \mathbf{x})$ – one can modify the polar transformation so as to make $1/h(y)$ the dual variable to $h(y)$ (that is, $h^*(y) = h^{-1}[1/h(y)]$ dual to y), as explained in Section 2 above. In this case, the *homothetic-polar* transformation yields the production function $h(y) = g^*(\mathbf{x})$, with the corresponding cost function $c^* = h(y) \cdot f^*(\mathbf{p})$. These two transformations are identical, if and only if $f(\mathbf{x})$ is *homogeneous* of some degree $\mu > 0$. In this case it may be shown that $h(y) = y^{1/\mu}$, hence $h(1/y) = y^{-1/\mu} = 1/h(y)$, and thus $h^*(y) = 1/y$.

A similar approach allows extension of the polar transformation through cost functions to the case of joint production with multiple outputs, *if outputs are separable from inputs.* That is, if the production frontier is of the form $F[h(\mathbf{y});\mathbf{x}] \equiv 0$, which may be solved for $h(\mathbf{y})$ ($\mathbf{y}$ a vector of order m),

$$h(\mathbf{y}) = f(\mathbf{x}), \tag{12}$$

where $f(\mathbf{x})$ is standard, and $h(\mathbf{y})$ increasing in $\mathbf{y}$, such that $h(\mathbf{0}) = 0$; $\mathbf{y}' > \mathbf{y}^0 \Rightarrow h(\mathbf{y}') > h(\mathbf{y}^0)$ [an increase in at least one output requires an increase in $f(\mathbf{x})$, and therefore of $\mathbf{x}$, since $f(\mathbf{x})$ is single-valued and non-decreasing] and $h(\mathbf{y}^N)$ is unbounded if $\mathbf{y}^N$ is unbounded. The corresponding cost function is $G[h(\mathbf{y});\mathbf{p}]$, and the unit cost frontier is separable into $1/h(\mathbf{y}) = g(\mathbf{p})$. Hence, the previous analysis is carried over entirely, with $h(\mathbf{y})$ substituted for y as the primal variable, and $1/h(\mathbf{y})$ replacing $1/y$ as the dual variable. The cost-polar transformation now yields a new separable production frontier, of the form $h(\mathbf{y}) = g(\mathbf{x})$, with $g(\mathbf{x})$ standard, satisfying Condition I, and a new separable unit cost frontier $1/h(\mathbf{y}) = f(\mathbf{p})$.

4. A Symmetric Formulation of Profit Functions and Production Frontiers

The analysis of duality relations between profit functions and production frontiers for the case of joint production with multiple outputs and inputs, may be carried out along lines similar to the cost–production analysis of Section 2, so as to yield a perfectly symmetric formulation for the primal and the dual relations.

Suppose $\mathbf{z} = (\mathbf{y};\mathbf{x})$ are non-negative[15] vectors of m outputs $\mathbf{y}$ ($m \geq 1$), and k inputs $\mathbf{x}$ ($k \geq 1; m + k = n$). The corresponding price vector is denoted by $\mathbf{q} = (\mathbf{p};\mathbf{w}) \in \bar{\Omega}_n$. The set of feasible input–output combinations for a given production process is denoted by $\mathbf{T}(\subseteq \bar{\Omega}_n)$. The conditions for $\mathbf{T}$ being *regular*, namely for the existence of a unique non-negative dual profit function,

$$\pi = Q(\mathbf{p};\mathbf{w}) = \sup_{(\mathbf{y};\mathbf{x})} \{\mathbf{p}'\mathbf{y} - \mathbf{w}'\mathbf{x}:(\mathbf{y};\mathbf{x}) \in \mathbf{T} \subseteq \bar{\Omega}_n\}, \tag{13}$$

are as follows:[16]

Condition B. $\mathbf{T}$ is a closed, convex set in $\bar{\Omega}_n$, with free disposal:

$(\mathbf{y};\mathbf{x}') \geq (\mathbf{y};\mathbf{x}) \in \mathbf{T} \Rightarrow (\mathbf{y};\mathbf{x}') \in \mathbf{T}$,
$(\mathbf{0};\mathbf{x}) \leq (\mathbf{y}';\mathbf{x}) \leq (\mathbf{y};\mathbf{x}) \in \mathbf{T} \Rightarrow (\mathbf{y}';\mathbf{x}) \in \mathbf{T}$.

$(\mathbf{0};\mathbf{0}) \in \mathbf{T}$, and $(\mathbf{y}^0;\mathbf{x}^0) \in \mathbf{T}$ for some $\mathbf{y}^0 \gg \mathbf{0}$.

Bounded inputs $\mathbf{x}$ imply bounded outputs $\mathbf{y}$ in $\mathbf{T}$.

By McFadden's duality theorem, the profit function defined in equation (13) exists uniquely and satisfies the following:

Condition A.

(1) $Q(\mathbf{p};\mathbf{w})$ is a real, non-negative function of $\mathbf{q} = (\mathbf{p};\mathbf{w}) \geq \mathbf{0}$. $Q(\mathbf{0};\mathbf{0}) = 0$ and $Q(\mathbf{q}^0) > 0$ for some $\mathbf{q}^0 \gg \mathbf{0}$. [$Q$ may be infinite for finite $\mathbf{q}$.]
(2) Q is non-increasing in $\mathbf{w}$ and non-decreasing in $\mathbf{p}$.
(3) If $\mathbf{w} \gg \mathbf{0}$, $\lim_{d \to 0} Q(\mathbf{p};(1/d)\mathbf{w}) \leq \mathbf{p}'\mathbf{a}$, where $\mathbf{a} > \mathbf{0}$ is a vector of fixed, finite values.
(4) $Q(\mathbf{q})$ is a convex, closed function for $\mathbf{q} \geq \mathbf{0}$.
(5) Q is positive linear homogeneous in $\mathbf{q}$: $\mathbf{q} > \mathbf{0}, \lambda > 0 \Rightarrow Q(\lambda\mathbf{q}) = \lambda Q(\mathbf{q})$.

Define the *unit profit set* $\mathbf{V}$ as follows:

$$\mathbf{V} = \{\mathbf{q}: Q(\mathbf{q}) \leq 1; \mathbf{q} \geq \mathbf{0}\}; \tag{14}$$

[15]It is more convenient for our purposes to define the quantity vectors $(\mathbf{y};\mathbf{x})$ with all the arguments non-negative, rather than the "net outputs" notation $(\mathbf{y};-\mathbf{x})$. In our notation, outputs are mathematically distinguished from inputs by the direction of change of the frontier functions, rather than by their sign. Cf. McFadden (Chapter I.1).

[16]These are the conditions in Diewert (1973a), modified to imply non-negative (but not identically zero) profits.

then the following theorem holds:

Theorem 4. If $Q(\mathbf{p};\mathbf{w})$ satisfies Condition A, the unit profit set $\mathbf{V}$ satisfies Condition B, where input prices $\mathbf{w}$ are to be substituted for inputs $\mathbf{x}$; output prices $\mathbf{p}$ for outputs $\mathbf{y}$, and $\mathbf{V}$ for $\mathbf{T}$.

Proof: By Condition A(4), $\mathbf{V}$ is convex and closed, since $Q(\mathbf{q})$ is a convex, closed function over the domain $\{\mathbf{q}:0 \leq Q(\mathbf{q}) \leq 1\}$.

By Condition A(2), the free disposal conditions follow immediately.

To show that $(\mathbf{p}^0;\mathbf{w}^0) \in \mathbf{V}$ for some $\mathbf{p}^0 \gg \mathbf{0}$, note that by Condition A(3), there exists a $(\mathbf{p}';\mathbf{w}') \gg \mathbf{0}$ such that $Q(\mathbf{p}';\mathbf{w}') = Q_0 < \infty$. If $Q_0 \leq 1$, choose $(\mathbf{p}^0;\mathbf{w}^0) = (\mathbf{p}';\mathbf{w}') \in \mathbf{V}$; and if $Q_0 > 1$ then $\mathbf{0} \ll (\mathbf{p}^0;\mathbf{w}^0) = ((1/Q_0)\mathbf{p}';(1/Q_0)\mathbf{w}') \in \mathbf{V}$, by Condition A(5).

It remains to be shown that bounded $\mathbf{w}$ imply bounded $\mathbf{p}$ in $\mathbf{V}$. Suppose $\{\mathbf{w}^M\} \leq \mathbf{B}$ ($\mathbf{B}$ a finite vector), but $\{\mathbf{p}^M\}$ unbounded in $\mathbf{V}$. Since there exists by Condition A(1) a strictly positive vector $(\mathbf{p}^0;\mathbf{w}^0) \gg \mathbf{0}$ such that $Q(\mathbf{p}^0;\mathbf{w}^0) > 0$, we may choose a partial sequence $\{\mathbf{p}^N;\mathbf{w}^N\}$ such that $\mathbf{p}^N \geq N\mathbf{p}^0;\mathbf{w}^N \leq \mathbf{B} \leq N\mathbf{w}^0$ for all $N \geq N_0$. Hence we get by Conditions A(2) and A(5): $\lim_{N\to\infty} Q(\mathbf{p}^N;\mathbf{w}^N) \geq \lim_{N\to\infty} Q(N\mathbf{p}^0;N\mathbf{w}^0) = \infty$, and $(\mathbf{p}^N;\mathbf{w}^N)$ cannot be all in $\mathbf{V}$, a contradiction. Q.E.D.

Theorem 4 establishes a complete symmetry between the regular production set $\mathbf{T}$ and the regular corresponding unit profit set $\mathbf{V}$, which is the dual of $\mathbf{T}$. Equipped with the duality theorem and this result, we may now state without further proof all the other symmetric results which follow by the transformation from the prices space into the quantities space and conversely.

First, applying the transformation to the duality theorem we may define the "*gauge function*" $H(\mathbf{y};\mathbf{x})$, by the following maximum property:

$$H(\mathbf{y};\mathbf{x}) = \sup_{(\mathbf{p},\mathbf{w})} \{\mathbf{y}'\mathbf{p} - \mathbf{x}'\mathbf{w}:(\mathbf{p};\mathbf{w}) \in \mathbf{V} \subseteq \bar{\Omega}_n\}, \tag{15}$$

which is the dual counterpart of equation (13). The set $\mathbf{T}$ is then derivable from $H(\mathbf{y};\mathbf{x})$ by

$$\mathbf{T} = \{(\mathbf{y};\mathbf{x}):H(\mathbf{y};\mathbf{x}) \leq 1;(\mathbf{y};\mathbf{x}) \geq \mathbf{0}\}, \tag{16}$$

which is equivalent to equation (14). The function H defined here satisfies Condition A, with the appropriate substitution of variables. In

particular, H is linear homogeneous in $(\mathbf{y};\mathbf{x})$ for any production frontier which is regular (but not necessarily homogeneous).

Given a regular unit profit set $\mathbf{V}$ satisfying Condition B, the profit function $Q(\mathbf{p};\mathbf{w})$ is derivable from $\mathbf{V}$ by the following relation:

$$Q(\mathbf{p};\mathbf{w}) = \inf\left\{\theta: \left(\frac{1}{\theta}\mathbf{p};\frac{1}{\theta}\mathbf{w}\right) \in \mathbf{V};0 < \theta \leq \infty\right\}. \tag{17}$$

This is shown as follows:

$$\inf\left\{\theta: \left(\frac{1}{\theta}\mathbf{q}\right) \in \mathbf{V};0 < \theta \leq \infty\right\} = \inf\left\{\theta: Q\left(\frac{1}{\theta}\mathbf{q}\right) \leq 1;0 < \theta \leq \infty\right\}$$
$$= \inf\{\theta: Q(\mathbf{q}) \leq \theta;0 < \theta \leq \infty\} = Q(\mathbf{q}).$$

Q is the "gauge function" of the set $\mathbf{V}$, representing the distance from the origin of a point $(\mathbf{p};\mathbf{w}) \geq 0$, divided by the furthest distance from $(\mathbf{0})$ of points $((1/\theta)\mathbf{p};(1/\theta)\mathbf{w})$ which lie on the efficient boundary of the set $\mathbf{V}$.[17] In an exactly analogous derivation for the dual case, we have

$$H(\mathbf{y};\mathbf{x}) = \inf\left\{h: \left(\frac{1}{h}\mathbf{y};\frac{1}{h}\mathbf{x}\right) \in \mathbf{T};0 < h \leq \infty\right\}, \tag{18}$$

where $H \leq 1$ in the production set $\mathbf{T}$, and $H = 1$ on the production frontier (when it exists). In general, if the frontier exists and is given by an implicit function $F(\mathbf{y};\mathbf{x}) \equiv 0$, where F satisfies the following:

> *Condition C.* The set $\mathbf{T} = \{(\mathbf{y};\mathbf{x}): F(\mathbf{y};\mathbf{x}) \leq 0;(\mathbf{y};\mathbf{x}) \geq \mathbf{0}\}$ satisfies Condition B above;[18]

then this frontier could equally be represented by $H(\mathbf{y};\mathbf{x}) = 1$ where H is linear homogeneous and satisfies Condition A. Symmetrically, maximum profits π may be represented by an implicit equation in the variables $((1/\pi)\mathbf{p};(1/\pi)\mathbf{w})$ [since π is linear homogeneous]: $R((1/\pi)\mathbf{p};(1/\pi)\mathbf{w}) \equiv 0$, and the *unit profit frontier* by $R(\mathbf{p}^*;\mathbf{w}^*) \equiv 0$ (R is generally *not* linear homogeneous; $\mathbf{p}^*, \mathbf{w}^*$ are prices normalized to yield maximum profits equal to unity).

When the partial derivatives of Q and H exist, they satisfy the relations[19] $\partial Q/\partial p_i = y_i^*(\mathbf{q})$; $\partial Q/\partial \mathbf{w}_j = -x_j^*(\mathbf{q})$, where y_i^* and x_j^* are the

[17]This includes $Q = 0$ if $(1/\theta)\mathbf{q} \in \mathbf{V}$ for all $\theta > 0$; and $Q = \infty$, if $(1/\theta)\mathbf{q} \notin \mathbf{V}$ for all $\theta > 0$.

[18]For specification of direct conditions on the function F (rather than the set $\mathbf{T}$), see Diewert (1973a). He assumes, however, that F is normalized, i.e., solved for one argument as a dependent variable. More general conditions on F could also be given, but are omitted here.

[19]E.g., in Diewert (1973a), Lau (Chapter I.3), and others.

TABLE 2
Symmetric dual relations for regular profit and production functions.

	Primal (production)	Dual (unit profits)
Variables (non-negative):	Inputs x_i Outputs y_i	Input prices w_i Output prices p_i
Sets (satisfying Condition B):	Production **T**	Unit profit **V**
Functions: *Gauge* (satisfying Condition A):	$H(\mathbf{y};\mathbf{x}):\inf\left\{h:\left(\frac{1}{h}\mathbf{y};\frac{1}{h}\mathbf{x}\right)\in\mathbf{T}\right\}$	$Q(\mathbf{p};\mathbf{w})=\inf\left\{\theta:\left(\frac{1}{\theta}\mathbf{p};\frac{1}{\theta}\mathbf{w}\right)\in\mathbf{V}\right\}$
Implicit (satisfying Condition C):	$F(\mathbf{y};\mathbf{x})\equiv 0$ ($F\leq 0$ in **T**)	$R(\mathbf{p};\mathbf{w})\equiv 0$ ($R\leq 0$ in **V**)
Maximum property:	$H=\sup\{\mathbf{p}'\mathbf{y}-\mathbf{w}'\mathbf{x}:(\mathbf{p};\mathbf{w})\in\mathbf{V}\}$	$Q=\sup\{\mathbf{y}'\mathbf{p}-\mathbf{x}'\mathbf{w}:(\mathbf{y};\mathbf{x})\in\mathbf{T}\}$
Partial derivatives (when existing):	$\partial H/\partial y_i=p_i^*;\ \partial H/\partial x_j=-w_j^*$	$\partial Q/\partial p_i=y_i^*;\ \partial Q/\partial w_j=-x_j^*$

output supply and factor demand functions, respectively. Similarly, $\partial H/\partial y_i = p_i^*(\mathbf{z})$; $\partial H/\partial x_i = -w_i^*(\mathbf{z})$ (if existing), where p_i^* and w_i^* are the optimal *inverse* supply and demand functions, determining the normalized shadow price variables $(1/\pi)\mathbf{p}$,$(1/\pi)\mathbf{w}$.

The Table 2 summarizes these symmetric relations.

The foregoing analysis rests heavily on the assumption that zero variable outputs and inputs are feasible; i.e., $(\mathbf{0};\mathbf{0}) \in \mathbf{T}$, since in this case maximum variable profits are non-negative, and the profit function is completely determined by the unit profit set. If, however, some minimum positive inputs are always required – either due to indivisibilities, or because some outputs are fixed or are bounded below through exogeneously determined restrictions – then $(\mathbf{0},\mathbf{0})$ is not feasible, and maximum variable profits assume negative values. The symmetric duality relations in terms of the original variables break down. However, as shown by McFadden in Chapter I.1, the variables may be translated to be measured from a point $(\boldsymbol{\eta},\boldsymbol{\xi})$ in $\mathbf{T}$, and the symmetric duality applies with respect to the translated variables $(\mathbf{y}-\boldsymbol{\eta},\mathbf{x}-\boldsymbol{\xi})$ and with respect to the corresponding modified profit and production functions. (Details and proofs of these general statements are omitted here.)

5. The Polar Profit and Production Functions

In analogy to the derivation of cost polar production functions, the perfect symmetry of the dual production–profit relations exhibited above leads to the definition, for any regular production set $\mathbf{T}$, of another regular production set $\mathbf{T}^*$, which coincides with the original unit profit set $\mathbf{V}$, if prices are transformed to the respective quantities. That is,

$$\mathbf{T}^* = \{(\mathbf{y};\mathbf{x}):(\mathbf{y};\mathbf{x}) = (\mathbf{p};\mathbf{w}) \in \mathbf{V}\}. \tag{19}$$

The set $\mathbf{T}^*$ is the *profit polar production set*, determined uniquely by $\mathbf{T}$, and satisfying the same conditions specified in Condition B. Similar results apply to all equivalent representations of $\mathbf{T}^*$, as summarized by the following theorem:

Theorem 5. Given a regular production set $\mathbf{T}$, which may be represented uniquely by any one of the following equivalent relations:[20]

[20]PS = production set; PFr = production frontier; GF = gauge function; UPS = unit profit set; IPF = implicit profit function; PFn = profit function.

Primal	Dual	Satisfying Conditions
(1) PS: $\mathbf{T}$	(a) UPS: $\mathbf{V}$	B
(2) PFr: $F(\mathbf{y};\mathbf{x}) \equiv 0$	(b) IPF: $R\left(\frac{1}{\pi}\mathbf{p};\frac{1}{\pi}\mathbf{w}\right) \equiv 0$	C
(3) GF: $H(\mathbf{y};\mathbf{x})$	(c) PFn: $Q(\mathbf{p};\mathbf{w})$	A

there exists a unique polar production set $\mathbf{T}^*$, defined by equation (19), which is regular, and may be represented uniquely by any one of the following equivalent relations:

Primal	Dual	Satisfying Conditions
(1′) PS: $\mathbf{T}^* = \mathbf{V}$	(a′) UPS: $\mathbf{V}^* = \mathbf{T}$	B
(2′) PFr: $R(\mathbf{y};\mathbf{x}) \equiv 0$	(b′) IPF: $F\left(\frac{1}{\pi}\mathbf{p};\frac{1}{\pi}\mathbf{w}\right) \equiv 0$	C
(3′) GF: $Q(\mathbf{y};\mathbf{x})$	(c′) PFn: $H(\mathbf{p};\mathbf{w})$	A

Proof: The proof is immediate, using McFadden's Theorem 24 in Chapter I.1, Theorem 4 above, and the results of Section 4, summarized in Table 2 above. Q.E.D.

The results cited in Table 2 with respect to the maximum properties of the profit and gauge functions, as well as to partial derivatives of Q and H (i.e., factor demand and output supply functions), are applicable to the new polar production frontier, if proper substitutions are made throughout. However, the specific behavior of the polar production relation may be quite different from that of the original relation, as indicated by some of the examples in Chapter II.3 and in Hanoch (1975a).

Let us examine now a few special cases. For the case of a single output with a *concave* production function $y = f(\mathbf{x})$ (such that the profit function exists), the profit polar production function defined here generally yields a different production function from the cost polar function defined in Section 3. However, if $y = f(\mathbf{x})$ is *homogeneous* of degree μ $(0 < \mu < 1)$, the two functions coincide, except for a constant

scale factor. The cost function dual to $f(\mathbf{x})$ is separable in this case in the form $C = y^{1/\mu}G(\mathbf{w})$, where $G(\mathbf{w})$ is linear homogeneous. Hence the cost polar production function is given by $y = g^c(\mathbf{x}) = [G(\mathbf{x}))]^{\mu}$. The profit function dual to $f(\mathbf{x})$ is derived as follows (assuming $\partial C/\partial y$ exists): $\partial C/\partial y = C/\mu y = p$, equating output price to marginal cost. The profit maximizing costs $\bar{c}^*$ then satisfies

$$\bar{c}^* = (\bar{c}^*/\mu p)^{1/\mu}G(\mathbf{w}),$$

or

$$\bar{c}^* = \{\mu p[G(\mathbf{w})]^{-\mu}\}^{1/(1-\mu)};$$

and profits are given by

$$\pi = py - \bar{c}^* = ((1-\mu)/\mu)\bar{c}^* = (1-\mu)\mu^{\mu/(1-\mu)}\{p[G(\mathbf{w})]^{-\mu}\}^{1/(1-\mu)}.$$

The profit polar production frontier function $y = g^{\pi}(\mathbf{x})$ is derived by substitution of $(1,y,\mathbf{x})$ for $(\pi,p,\mathbf{w})$, respectively, in the above expression, and solving for y gives

$$g^{\pi}(\mathbf{x}) = A[G(\mathbf{x})]^{\mu} = A \cdot g^c(\mathbf{x}),$$

where

$$A = [\mu^{\mu}(1-\mu)^{1-\mu}]^{-1} > 1. \tag{20}$$

Similarly, if $y = f(\mathbf{x})$ is *homothetic*, its cost function is separable in the form $C = h(y)G(\mathbf{w})$, where $(\partial \log h(\mathbf{y}))/(\partial \log y) > 1$. Similar manipulations give the regular cost polar production function as $y = g^c(\mathbf{x}) = H^{(c)}[G(\mathbf{x})]$; the *homothetic cost polar* function of Section 3 is of the form $y = H^{(hc)}[G(\mathbf{x})]$; and the profit polar function is $y = H^{(\pi)}[G(\mathbf{x})]$, where the functions $H^{(c)}$, $H^{(hc)}$ and $H^{(\pi)}$ are different functions of a single variable. Therefore, the family of isoquant surfaces given by each of the three polar transformations is the same, but their output-denominations are different. (However, all three polar functions are also homothetic.)

In the case of multiple outputs with *separability of outputs from inputs*, the original production frontier is given by $h(\mathbf{y}) = f(\mathbf{x})$. The unit profit frontier of the profit polar function is given by $h(\mathbf{p}) = f(\mathbf{w})$, and is also separable. (The polar production frontier is generally not separable, however.) Hence "direct separability" implies "indirect separability" of the polar function.[21] Clearly, the converse is also true, due to the uniqueness of the profit polar function. That is, (indirect) separability of

[21]On direct and indirect separability and related concepts, see Houthakker (1960, 1965), Sameulson (1965a, 1969b), Gorman (1968b), Goldman and Uzawa (1964), Pollak (1972), and Lau (1969a and Chapter I.3 in this volume).

the original unit profit frontier, implies (direct) separability of the polar production frontier.

The separable cost polar function defined in Section 3 [namely a production frontier with costs $c = G[h(\mathbf{y});\mathbf{w}]$, where c is implicitly given by $1/h(\mathbf{y}) = f((1/c)\mathbf{w})$], is generally different from the profit polar frontier in the direct separability case. An analysis similar to the foregoing shows that these two polar transformations coincide, except for a scale factor, if and only if both $h(\mathbf{y})$ and $f(\mathbf{x})$ are *homogeneous*. The analysis of additional special cases may be carried out along similar lines.[22]

6. Some Extensions and an Application

The process of polar transformation of single-output production functions through cost functions, may be generalized further to joint-production frontiers, under two cases of *short-run profit maximization*:

(i) If either a single output $z_0 = \bar{y}_0$, or a single input $z_0 = \bar{x}_0$ is fixed. The polar transformation of the *variable profits function* $Q(\mathbf{p};\mathbf{w};z_0)$ yields then a polar production frontier $Q(\mathbf{y};\mathbf{x};1/z_0) \equiv 1$, and conversely.

(ii) If the production frontier is separable as between the variable elements $(\mathbf{y};\mathbf{x})$ and the fixed elements $\mathbf{z}_0$ (either inputs or outputs or both). That is, $F(\mathbf{y};\mathbf{x}) = h(\mathbf{z}_0)$. The *polar variable profit* function $\bar{\pi}$ is then given by $F(\mathbf{p}/\bar{\pi};\mathbf{w}/\bar{\pi}) = 1/h(\mathbf{z}_0)$.

The *Factor Requirement Function*[23] defined for the case of a single input, is an obvious special case of (i), and may yield a revenue polar transformation, in complete analogy to the cost polar analysis. Proofs of the above cursory statement are analogous to those given in Sections 2–5.

As a final example of an application of the polar transformation, consider two alternative definitions of the elasticity of substitution which are different from each other, and from the widely used Allen–Uzawa elasticities of substitution,[24] if three or more variable factors are present:

(1) *The Direct Elasticity of Substitution* D_{ij}, for a (twice continuously

[22]For additional results on relations between production and profit functions, see Lau (Chapter I.3). Modifications of such results, so as to apply to polar relations, are straightforward.

[23]See McFadden (Chapter I.1) and Diewert (1974b) for duality theorems with respect to the factor requirement function.

[24]See McFadden (1963) for definitions of these concepts. The D_{ij} were defined by Hicks (1946). See Hanoch (Chapter II.3).

differentiable) production function $y = f(\mathbf{x})$, is defined by

$$D_{ij}(\mathbf{x}) = \left(\frac{1}{x_i f_i} + \frac{1}{x_j f_j}\right) \Big/ \left(-\frac{f_{ii}}{f_i^2} + \frac{2f_{ij}}{f_i f_j} - \frac{f_{jj}}{f_j^2}\right), \tag{21}$$

where D_{ij} is interpreted as $\mathrm{d}\log(x_i/x_j)/\mathrm{d}\log(p_j/p_i)$, for constant output and other input quantities.

(2) *McFadden's Shadow Elasticity of Substitution* S_{ij}, defined through the cost function $c = G(y;\mathbf{p})$. If the unit cost function is given in the "indirect reciprocal production function" form $1/y = g(\mathbf{p})$, it may be shown that $S_{ij}(\mathbf{p})$ is given by

$$S_{ij}(\mathbf{p}) = \left(-\frac{g_{ii}}{g_i^2} + \frac{2g_{ij}}{g_i g_j} - \frac{g_{jj}}{g_j^2}\right) \Big/ \left(\frac{1}{p_i g_i} + \frac{1}{p_j g_j}\right), \tag{22}$$

where S_{ij} is interpreted as $\mathrm{d}\log(x_i/x_j)/\mathrm{d}\log(p_j/p_i)$, for other prices, output and unit cost held constant. Thus,

$$S_{ji} = S_{ij} = \frac{\mathrm{d}\log(G_j/G_i)}{\mathrm{d}\log(p_i/p_j)} = \frac{\mathrm{d}\log(g_j/g_i)}{\mathrm{d}\log(p_i/p_j)},$$

which is analogous to

$$\frac{1}{D_{ij}} = \frac{\mathrm{d}\log(f_j/f_i)}{\mathrm{d}\log(x_i/x_j)}.$$

Applying the cost polar transformation, the polar direct and indirect production functions are $y = g(\mathbf{x})$ and $1/y = f(\mathbf{p})$, respectively; hence the elasticities $\bar{D}_{ij}$, $\bar{S}_{ij}$ of the polar function satisfy $\bar{D}_{ij}(\mathbf{x}) = 1/S_{ij}(\mathbf{x})$; $\bar{S}_{ij}(\mathbf{p}) = 1/D_{ij}(\mathbf{p})$, where $S_{ij}(\ \ )$ and $D_{ij}(\ \ )$ are the functions defined by (22) and (21), respectively.

For example, the D_{ij} for CRESH [Hanoch (1971)] are given by [25]

$$D_{ij} = a_i a_j/(s_i a_i + s_j a_j),$$

where $s_i = p_i x_i / \sum p_l x_l$ are the cost shares.

Thus, the $\bar{S}_{ij}$ for the CDE polar function (Chapter II.3) are given by

$$\bar{S}_{ij} = (s_i a_i + s_j a_j)/a_i a_j = s_i(1/a_j) + s_j(1/a_i)$$

(since the cost shares s_l are symmetric in $\mathbf{x}$ and $\mathbf{p}$).

Similar applications may be used for the generalized elasticities of transformation and the profit polar production frontier. Examples of a number of particular polar pairs of functional forms are presented and

[25] See Hanoch (1971, p. 12, n. 2), and Hanoch (Chapter II.3).

discussed in Chapter II.3. Other widely used polar pairs of production function are: Diewert's (1971) Generalized Linear and Generalized Leontief Production Functions; the Transcendental-logarithmic (Translog) models of Christensen, Jorgenson and Lau (1973, 1975), and the polar pair of Quadratic functions [e.g., Lau (1974)]. The results of the present analysis, however, imply the existence and validity of the polar function generated by *any* functional form used previously, either in the direct or in the indirect mode. Thus the available choice of functional forms in production models is considerably enriched.