Chapter II.3

POLAR FUNCTIONS WITH CONSTANT TWO FACTORS – ONE PRICE ELASTICITIES*

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1. Introduction

If more than two variable factors are involved in a production process, the degree of substitutability between factors, measured by the Elasticity of Substitution (ES), may de defined in a variety of ways. Mundlak (1968b) has shown that the different concepts of ES are different combinations (constrained or unconstrained) of elements of the underlying Hessian matrix. Following his classification, we distinguish between one factor-one price ES concepts (such as Allen–Uzawa partial ES, denoted here by A_{ij}), two factors–one price ES (TOES), and two factors–two prices ES (TTES) (such as Hicks' Direct ES) (DES_{ij}), or McFadden's Shadow ES (SES_{ij}).[1]

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[1]See Allen (1938), Hicks (1963), McFadden (1963), and Mundlak (1968b).

Mundlak argued correctly, that the choice of the relevant ES measure is independent of assumptions regarding constancy of any particular measure, and generally of the choice of functional forms and methods of estimation of the production relation. However, traditionally the search for econometrically convenient functional forms with a limited set of parameters was attempted by imposing constancy on some ES concept. This yielded disappointing results for both the one factor–one price and the TTES concepts for more than two variable factors. Uzawa (1962) and McFadden (1963) have shown, that constant A_{ij}, DES_{ij}, or SES_{ij} yield functional forms which are too restrictive, and generally unacceptable, for more than two or three factors. For example, constant A_{ij} implies that all A_{ij} are equal for pairs of factors within the same group, and $A_{ij} = 1$ for factors belonging to different groups. (For two factors, all ES concepts coincide, and constant A_{ij} yield the well-known two-factor CES model.[2])

This chapter presents and analyzes several useful polar pairs of functional forms for production functions and joint production frontiers, the common feature of which is the *constancy of* some *TOES* concept.[3] All these functions are true generalizations of the CES model, which yield, in many cases, less restrictive but manageable estimation equations for factor–demand and output–supply relations under competitive markets.

As shown below, the two TOES concepts held constant in these models are simply related to the more basic ES concepts A_{ij}. The first, R^k_{ij}, equals the ratio A_{ik}/A_{jk}, the constancy of which yields the family of Constant-Ratio-ES (CRES) functions defined by Gorman (1965), and the specific subfamily analyzed in Hanoch (1975a). Two noted special cases of CRES are: (1) the homothetic case of CRESH, defined and analyzed in Hanoch (1971); and (2) the non-homothetic Mukerji (1963) function, used also by Dhrymes and Kurz (1964).

The second TOES concept, D^k_{ij}, equals the difference $A_{ik} - A_{jk}$. The major focus of the present analysis are functional forms with constant R_{ij}, and their *polar* functions, which turn out to yield constant D_{ij}, and to have some additional desirable properties.

The family of implicitly additive models with a single output, given in

[2]See Arrow et al. (1961). The statement refers to ES defined for constant output. See Mundlak (1968b, p. 231).

[3]On polar functions, see Chapter I.2. Another common feature of these models is their *Implicit Additivity*, as defined in Hanoch (1975a), where constancy of TOES concepts is shown to be equivalent to implicit additivity.

Hanoch (1975a), yields many well-known special cases of polar pairs of production functions, when equality of certain parameters is imposed.

In joint production situations with multiple outputs, the ES between outputs is defined in an analogous manner to A_{ij}, R^k_{ij}, and D^k_{ij}, substituting maximum revenue (at fixed inputs) for minimum costs (at fixed outputs). To each constant TOES production function model, there corresponds a similar constant TOES factor requirement function (with modified parameter restrictions to assure convexity instead of concavity). Equating two such functions to each other, clearly yields a frontier which exhibits separability of outputs from inputs, and constant R^k_{ij} or D^k_{ij} for both outputs and inputs.

The concept of *Elasticity of Transformation*[4] (T_{ij}) is defined as a generalization of A_{ij} to the situation of competitive profit maximization with multiple variable outputs and inputs. Again, we define two quantities–one price ET (TOET),

$$RT^k_{ij} = T_{ik}/T_{jk} \quad \text{and} \quad DT^k_{ij} = T_{ik} - T_{jk}.$$

Finally, polar pairs of production relations with *constant TOET* are presented, generalizing the single-output non-homothetic CRES and CDE models, through the profit-polar transformation suggested in Hanoch (1975a).

Section 2 below defines and interprets various ES and ET concepts used here. Section 3 summarizes, in the most part, previous results concerning CRES and CDE models, for production functions with a single output, and their corresponding various special cases. Finally, Section 4 includes generalizations of both the CDE and the CRES models to joint production relations with many outputs, under constant ratios or differences of elasticities of substitution or transformation.

2. Elasticities of Substitution and Transformation

Let y_j and x_i denote output and input quantities, respectively, with p_j and w_i the corresponding prices. Assume first that a firm produces efficiently a single output y, minimizing costs $\sum x_i w_i$, under competitive factor markets (exogenous w_i). The production function $y = f(x_1,x_2,\ldots,x_n) = f(\mathbf{x})$ is assumed to be a strictly increasing, twice continuously differentiable, quasi-concave function, possessing a unique

[4]For the definition of T_{ij}, see Diewert (1973a).

dual cost function $C(\mathbf{w};y)$, which is twice continuously differentiable, concave and linear-homogeneous in $\mathbf{w}$, and increasing with y from 0 to ∞.[5]

Following the notation in Mundlak (1968b), let

$$\hat{z}_i = \mathrm{d}\log z_i = \frac{1}{z_i}\mathrm{d}z_i.$$

The elasticity of demand for factor x_i with respect to w_j, at constant output, is

$$E_{ij} = \left.\frac{\hat{x}_i}{\hat{w}_j}\right|_y = \frac{w_j}{x_i}K_{ij} = s_j A_{ij}, \tag{1}$$

where K_{ij} is the element of the inverse bordered Hessian matrix,

$$[K] = \begin{bmatrix} 0 & f_1 \ldots f_n \\ f_1 & f_{11} \ldots f_{1n} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ f_n & f_{n1} \ldots f_{nn} \end{bmatrix}^{-1},$$

and s_j is the optimal share of factor x_j in total variable costs, or the elasticity of C with respect to w_j,

$$s_j = \frac{w_j x_j}{C} = \left.\frac{\hat{C}}{\hat{w}_j}\right|_y \quad \text{since} \quad \frac{\partial C}{\partial w_j} = x_j.$$[6]

A useful interpretation of Allen's A_{ij} is as follows:

$$A_{ij} = \frac{E_{ij}}{s_j} = \frac{\hat{x}_i/\hat{w}_j}{\hat{C}/\hat{w}_j} = \left.\frac{\hat{x}_i}{\hat{C}}\right|_{(\mathrm{d}w_j)}, \tag{2}$$

namely, A_{ij} is the elasticity of x_i with respect to C, for a change in another price w_j (output and all w_i constant). It is also the demand cross-elasticity E_{ij}, "normalized" by the relative change in C. Also,

$$A_{ij} = A_{ji} = \left.\frac{\hat{x}_j}{\hat{C}}\right|_{(\mathrm{d}w_i)},$$

due to the symmetry of $[K]$.

Generalizing this particular interpretation to two factors–one price

[5]See Chapter I.2 for specifications of the conditions for the general (non-differentiable) case, and for references with respect to this particular case.

[6]See Shephard (1953, p. 11).

elasticities,[7] we get

$$D_{ij}^k = \frac{\widehat{(x_i/x_j)}}{\hat{C}}\bigg|_{(\mathrm{d}w_k)} = \frac{\hat{x}_i}{\hat{C}}\bigg|_{(\mathrm{d}w_k)} - \frac{\hat{x}_j}{\hat{C}}\bigg|_{(\mathrm{d}w_k)} = A_{ik} - A_{jk}, \tag{3}$$

that is, D_{ij}^k is the elasticity of the factors ratio x_i/x_j with respect to costs C, for a change in another price w_k. It is also the cross-elasticity of *relative* demand,

$$\frac{\widehat{(x_i/x_j)}}{\hat{w}_k}\bigg|_y,$$

normalized by $\hat{C}/\hat{w}_k = s_k$.

Another TOES concept is defined as follows:

$$R_{ij}^k = \frac{\hat{x}_i}{\hat{x}_j}\bigg|_{(\mathrm{d}w_k)} = \frac{\hat{x}_i/\hat{C}}{\hat{x}_j/\hat{C}}\bigg|_{(\mathrm{d}w_k)} = \frac{A_{ik}}{A_{jk}}; \tag{4}$$

hence R_{ij}^k is the elasticity of x_i with respect to x_j, for a change in another factor's price w_k, output and other prices w_i $(i \neq k)$ constant. Also, by equation (1),

$$R_{ij}^k = \frac{\hat{x}_i/\hat{w}_k}{\hat{x}_j/\hat{w}_k}\bigg|_y = \frac{E_{ik}}{E_{jk}}.$$

Thus, both D_{ij}^k and R_{ij}^k defined in equations (3) and (4) are subject to relatively simple and intuitive economic interpretations. In addition these TOES concepts are free from a basic flaw common to all TTES concepts, as pointed out by Mundlak; namely, that the relative magnitude of two price changes, $\hat{w}_i$ and $\hat{w}_j$, has to be restricted. Any particular such restriction (equivalent to picking a particular directional change in the prices space) is essentially arbitrary, and yields a different TTES concept.[8]

The polar pairs of CDE and CRES models presented here, have constant D_{ij}^k and R_{ij}^k, respectively – with the additional (necessary) property, that both are independent of k, for any i,j $(i \neq k \neq j)$.

For the case of multiple outputs $\mathbf{y} = (y_1,\ldots,y_m)$, assume the joint production frontier to be given by $F(\mathbf{y};\mathbf{x}) \equiv 0$, where F is increasing in $\mathbf{y}$, decreasing in $\mathbf{x}$, continuously twice differentiable and convex in $(\mathbf{y};\mathbf{x})$,

[7]The "pure" TOES concept defined by Mundlak (1968b, p. 229) is $H_{ij}^k = \widehat{(x_i/x_j)}/\hat{w}_k|_y$, hence $H_{ij}^k = s_k D_{ij}^k$, analogous to $E_{ij} = s_j A_{ij}$.

[8]See Mundlak (1968b, Sect. 3.3).

with $F(\mathbf{0};\mathbf{0})=0$.[9] A competitive firm maximizes profits ($\sum y_j p_j - \sum x_i w_i$), under exogenous (positive) prices ($\mathbf{p};\mathbf{w}$). The unique dual profit function $\pi(\mathbf{p};\mathbf{w})$, is increasing in $\mathbf{p}$ and decreasing in $\mathbf{w}$, and is also twice continuously differentiable; π is non-negative, positive linear homogeneous in ($\mathbf{p};\mathbf{w}$), and convex over the domain where π is finite.[10] The partial derivatives of $\pi(\mathbf{p};\mathbf{w})$ yield the factor–demands and output–supplies, as follows:[11]

$$\frac{\partial \pi}{\partial w_i} = -x_i^* \quad \text{and} \quad \frac{\partial \pi}{\partial p_j} = y_j^*.$$

The ES between inputs may be generalized to this joint-production case, by simply substituting in their definition the fixed vector of outputs $\mathbf{y}$ for the single output y. Analogously, the corresponding ES between outputs are defined so that maximization of revenue at fixed input quantities $\mathbf{x}$ is substituted for minimization of costs at fixed $\mathbf{y}$, with the revenue function $R(\mathbf{p};\mathbf{x})$ being linear homogeneous and convex in $\mathbf{p}$. At any point ($\mathbf{y}^*,\mathbf{x}^*$) which yields maximum profits π at given prices ($\mathbf{p};\mathbf{w}$), costs are minimized, subject to fixed $\mathbf{y}^*$; revenue maximized for fixed $\mathbf{x}^*$, and π satisfies $\pi = R(\mathbf{p};\mathbf{x}^*) - C(\mathbf{w};\mathbf{y}^*)$. The output–output ES are then defined as

$$A_{ik} = \frac{\hat{y}_i/\hat{p}_k}{\hat{R}/\hat{p}_k} = \frac{\hat{y}_i}{\hat{R}}\bigg|_{(\mathrm{d}p_k)},$$

$$D_{ij}^k = \frac{(\widehat{y_i/y_j})}{\hat{R}}\bigg|_{(\mathrm{d}p_k)} = A_{ik} - A_{jk},$$

and

$$R_{ij}^k = \frac{\hat{y}_i}{\hat{y}_j}\bigg|_{(\mathrm{d}p_k)} = \frac{A_{ik}}{A_{jk}},$$

where i,j,k refer to outputs, and $\mathbf{x}^*$ constant.

Generalized separable CRES and CDE models presented in Section 4, exhibit constant TOES between inputs as well as between outputs.

[9] $F(\mathbf{y},\mathbf{x})$ as given may not satisfy these conditions, although some transformation $F^* = h(F)$ does. In this case, choose $F^*(\mathbf{y},\mathbf{x}) = 0$ to represent the production frontier. The differentiability requirements restrict this relative to the general (non-differentiable) case of duality discussed in Chapter I.2.

[10] See McFadden Chapter I.1 and Diewert (1973a) for proofs of this version of the duality theorem and a more complete specification of the properties of π and F.

[11] See Diewert (1973a) for this extension of Shephard's Lemma, and Hanoch in Chapter I.2.

A one quantity–one price Elasticity of Transformation (ET) is the following generalization of A_{ij}: $T_{ij} = -\pi\pi_{ij}/\pi_i\pi_j$, where $i,j = 1,\ldots,m+n$, and subscripts of π denote partial derivatives.[12] In analogy to equation (2), we may interpret T_{ij} as (minus) the elasticity of a variable quantity with respect to profits π, for a change in one price (holding constant other prices, including the variable's own price); that is, for inputs x_i and x_j:

$$-T_{ij} = \frac{\hat{x}_i}{\hat{\pi}}\bigg|_{(dw_j)} = \frac{\hat{x}_j}{\hat{\pi}}\bigg|_{(dw_i)},^{13}$$

for outputs y_i and y_j:

$$-T_{ij} = \frac{\hat{x}_i}{\hat{\pi}}\bigg|_{(dp_j)} = \frac{\hat{y}_j}{\hat{\pi}}\bigg|_{(dp_i)}, \tag{5}$$

and for output y_j and input x_i:

$$-T_{ij} = \frac{\hat{y}_j}{\hat{\pi}}\bigg|_{(dw_i)} = \frac{\hat{x}_i}{\hat{\pi}}\bigg|_{(dp_j)}.$$

Equations (5) are easily derived from the definition of T_{ij} above, noting that the elasticity of π with respect to a price is the corresponding share in profits,

$$\frac{\hat{\pi}}{\hat{w}_i} = -\frac{x_i w_i}{\pi} = \bar{s}_i, \qquad \frac{\hat{\pi}}{\hat{p}_j} = \frac{y_j p_j}{\pi} = \bar{s}_j, \qquad \sum_{k=1}^{m+n} \bar{s}_k = 1,$$

where $\bar{s}_k$ is negative for inputs and positive for outputs.

Extending the analogy, define two concepts of two quantities–one price ET (TOET), in analogy to equation (3),

$$DT_{ij}^k = \frac{\widehat{(\pi_j/\pi_i)}}{\hat{\pi}}\bigg|_{(dw_k)\text{ or }(dp_k)} = T_{ik} - T_{jk}, \tag{6}$$

and, in analogy to equation (4),

$$RT_{ij}^k = \frac{\hat{\pi}_i}{\hat{\pi}_j}\bigg|_{(dw_k)\text{ or }(dp_k)} = \frac{T_{ik}}{T_{jk}}. \tag{7}$$

CDET and CRET models, which generalize the corresponding single-output CDE and CRES models, are production frontiers with constant

[12] See Diewert (1973a). This is in analogy to $A_{ij} = CC_{ij}/C_iC_j$ with respect to the cost function.

[13] The last equality follows from the symmetry of $[\pi_{ij}]$, in the continuously twice differentiable case. Note that $\hat{\pi}/\hat{w}_i < 0$.

DT^k_{ij} and RT^k_{ij}, respectively, which are again independent of k ($i \neq k \neq j$; $n+m>3$).

3. A Summary of CRES and CDE (Implicitly Additive) Models

This section presents without proofs functional forms with constant R^k_{ij} or D^k_{ij} between inputs, for production functions with one output. These are discussed in more detail in Gorman (1965), Hanoch (1971, 1975a) and others. The polar functions are derived through the cost-polar transformation (Chapter I.2, Theorem 3). Profit-polar transforms (Theorem 5) may also be derived, if the production functions are restricted to be concave, but are omitted. Note, however, that in the homothetic cases, the cost-polar and the profit-polar functions have identical isoquant-maps (see Chapter I.2). Observe first, that constant A_{ij}/A_{kl} ($i \neq j$; $k \neq l$) for all pairs of factors is equivalent to constant $R^k_{ij} = A_{ik}/A_{jk}$ for all $i \neq k \neq j$, since

$$\frac{A_{ij}}{A_{kl}} = \frac{A_{ij}}{A_{kj}} \cdot \frac{A_{jk}}{A_{lk}} = R^j_{ik} \cdot R^k_{jl}.$$

Gorman (1965) proved, that the general form of a CRES production function is the function $y(\mathbf{x})$ defined implicitly by the equation[14]

$$\sum D_i(y) x_i^{d_i(y)} \equiv 1, \tag{8}$$

where

$$d_i(y) = 1 - \frac{1}{\theta(y) a_i},$$

assuming that a_i, $D_i(y)$ and $\theta(y)$ preserve the required conditions on $y(\mathbf{x})$;[15] if $d_i(y) = 0$, $\log x_i$ is substituted for $x_i^{d_i(y)}$. The ES are given by

$$A_{ij} = \theta(y) \frac{a_i a_j}{\sum s_k a_k}, \tag{9}$$

where

[14] The function $\phi(y)$ appearing on the right-hand side in Gorman (1965) may be absorbed in the functions $D_i(y)$, as in equation (8).

[15] Namely, the conditions for yielding a positive, finite, quasi-concave and increasing function $y = f(\mathbf{x})$.

$$s_k = \frac{x_k^* w_k}{\sum x_l^* w_l}$$

is the (variable) share of costs at optimal factor combinations. Thus, $R_{ij}^k = a_i/a_j$ is constant everywhere, and is *independent of k*.

The specific CRES and CDE models. Any econometric application of CRES production functions requires specification of the general functions $D_i(y)$ and $\theta(y)$ appearing in equation (8).

The specific form of CRES presented in Hanoch (1975a) is sufficiently flexible and generalizes many other well-known functions. It is derived by specifying $D_i(y) = D_i y^{-e_i d_i}$ and $\theta(y) \equiv 1$, where D_i (the "distribution parameters") and e_i (the "expansion parameters") are positive constants.[16]

Applying the polar transformation[17] yields another model defined through the dual cost function, in which the ES have *constant differences*, $D_{ij}^k = d_j - d_i$.

The following defines implicitly a function $f(\mathbf{z})$ of n variables z:

$$\sum D_i f^{-e_i d_i} z_i^{d_i} \equiv 1, \tag{10}$$

where $D_i > 0$, $e_i > 0$, $d_i < 1$, and all d_i of same sign; i.e., either $0 < d_i < 1$, or $d_i \leq 0$, $i = 1,...,n$. $\text{Log}[f^{-e_i} z_j]$ replaces $[f^{-e_j d_j} z_j^{d_j}]$ if $d_j = 0$.[18]

The CRES production function is defined by

$$y = f(\mathbf{x}). \tag{11}$$

The polar *CDE cost function* C^* is defined implicitly by applying the cost-polar transformation,

$$\frac{1}{y} = f\left(\frac{1}{C^*}\mathbf{w}\right),$$

which is in the form of "reciprocal indirect production function", defined in Chapter I.2, and $f(\quad)$ is defined in equation (10).

[16]The parameter restrictions given here may be relaxed, if $f(\mathbf{z})$ defined in equation (10) is to be valid only *locally*, rather than for all $\mathbf{x} \gg \mathbf{0}$ (globally). In particular, d_1 (say) may satisfy $d_1 > 1$ $(a_1 < 0)$, if $n \geq 3$ and Z_1 is large enough, relative to other Z_i. See Hanoch (1971, 1975a) for specification and analysis of the weaker local conditions.

[17]I.e., substituting $1/y$ and $\mathbf{w}/C^*$ for y and $\mathbf{x}$, respectively. See Hanoch's Theorem 3 in Chapter I.2.

[18]The conditions for *local* validity are weaker, allowing one d_i to be larger than 1, and some D_i and d_i to be of different sign (if $D_i d_i$ are all of the same sign, and $\max_i D_i > 0$). See footnote 16.

More specifically, the equation defining $C^*(\mathbf{w},y)$ implicitly is

$$\sum D_i y^{e_i d_i}(w_i/C^*)^{d_i} \equiv 1, \tag{12}$$

where $\log[y^{e_j}(w_j/C^*)]$ replaces $[y^{e_j d_j}(w_j/C^*)^{d_j}]$ for $d_j = 0$.

The ES corresponding to equations (11) and (12) are given by $(i \neq j; i,j = 1,\ldots,n)$[19]

$$A_{ij} = \frac{a_i a_j}{\sum s_k a_k}, \qquad R^k_{ij} = \frac{a_i}{a_j} = \frac{1-d_j}{1-d_i}, \tag{13}$$

$$A^*_{ij} = \frac{1}{a_i} + \frac{1}{a_j} - \sum s^*_k \cdot \frac{1}{a_k}, \qquad D^{k*}_{ij} = \frac{1}{a_i} - \frac{1}{a_j} = d_j - d_i, \tag{14}$$

where $a_i = 1/(1-d_i) > 0$, and the corresponding cost shares are given by

$$s_k = \frac{D_k d_k y^{-e_k d_k} x_k^{d_k}}{\sum_j D_j d_j y^{-e_j d_j} x_j^{d_j}}, \qquad s^*_k = \frac{D_k d_k y^{e_k d_k} w_k^{d_k}}{\sum_j D_j d_j y^{e_j d_j} w_j^{d_j}}, \tag{15}$$

where x_k are the cost minimizing quantities for output y, and y is defined as a function of $\mathbf{x}$ or $\mathbf{w}$, respectively, in equations (11) or (12).

As shown in Hanoch (1975a), this pair of polar functions may be estimated by log-linear equations, and yield various well-known models as special cases, by assuming equality restrictions on various parameters. Thus, many special cases are testable within the more general framework of these models.

The following summarizes some of these special cases. [For more details, see Hanoch (1975a).]

The homogeneous and homothetic cases (CRESH and HCDE). If $e_i = e = 1/\mu$, all i, both production functions are *homogeneous* of degree μ.

Let a function $H(\mathbf{z})$ be defined as

$$\sum D_i (z_i/H)^{d_i} \equiv 1, \tag{16}$$

specializing equation (10) for this case, with $H = f^e$. $H(\mathbf{z})$ is clearly linear homogeneous. The CRESH homogeneous production function is

$$y = [H(\mathbf{x})]^{\mu}, \tag{17}$$

[19]See Hanoch (1971, 1975a).

and its polar CDEH function is defined by its cost function,

$$C^* = y^{1/\mu} H(\mathbf{w}), \tag{18}$$

where equation (18) implies that the polar production function is also homogeneous of degree μ.[20] Replacing $y^{1/\mu}$ by any function $h(y)$, where h is strictly increasing in y from 0 to ∞ gives the general *homothetic* CRES and CDE models.

If the expansion parameters e_i are not equal to each other, the functions defined in equations (11) and (12) are non-homothetic. A sufficient condition for *concavity* of the production functions for all $\mathbf{x} \gg \mathbf{0}$ is $e_i > 1$, $i = 1,\dots,n$, for both models.

Explicitly additive models. If the products $\{e_i d_i\}$ are constant,

$$e_i d_i = d/\mu, \qquad i = 1,\dots n,$$

the corresponding production and cost functions are *explicitly additive.*

The CRES case:

$$y = \left(\sum D_i x_i^{d_i}\right)^{\mu/d}, \tag{19}$$

is the Mukerji (1963) and Dhrymes–Kurz (1964) production function, analyzed also by Hanoch (1971).

The polar CDE function exhibits "indirect explicit additivity", with the "indirect production function" given by

$$y = \left(\sum D_i (w_i/C^*)^{d_i}\right)^{-\mu/d}. \tag{20}$$

Both equations (19) and (20) yield constant TOES, as given in equations (13) and (14), and have simplified log-linear estimation equations as compared with equations (11) and (12). However, as shown in Hanoch (1975a), their usefulness is limited, due to the built-in restrictions connecting expansion behavior to substitution behavior in any explicitly additive (direct or indirect) models.

Direct and indirect addilog. The special cases, $d = \mu$ in equation (19) and $d = -\mu$ in equation (20), yield the functional forms applied in consumer demand analysis by Houthakker (1965).

[20]See Chapter I.2.

"*Direct Addilog*":

$$y = \sum D_i x_i^{d_i}, \qquad 0 < d_i < 1, \tag{21}$$

and

"*Indirect Addilog*":

$$y = \sum D_i (w_i/C^*)^{d_i}, \qquad d_i < 0. \tag{22}$$

These functions, however, are not polar to each other. Since $e_i > 0$, all i, d_i must be positive in equation (21) and negative in equation (22), for these equations to yield valid (positive monotone and quasi-concave) production functions, respectively.[21]

Non-homothetic CES. If e_i are not all equal, but $d_i = d$, all i, then both equations (11) and (12) correspond to the non-homothetic CES model, with constant and equal ES,

$$A_{ij} = \frac{1}{1-d} = a \quad \text{and} \quad A_{ij}^* = 1 - d = \frac{1}{a} = a^*, \tag{23}$$

by equations (13) and (14). In this case, the polar pair of production functions $f(\mathbf{x})$ and $f^*(\mathbf{x})$ may both be expressed (implicitly) by similar forms in the direct mode, namely,

$$\sum D_i (f^{-e_i} x_i)^d \equiv 1 \quad \text{and} \quad \sum D_i^* (f^{*-e_i} x_i)^{d^*} \equiv 1, \tag{24}$$

where

$$D_i^* = D_i^{1/(1-d)} \quad \text{and} \quad d^* = -d/(1-d).$$

Thus, these functions are "self-polar", in the weak sense defined by Houthakker (1965), having the same functional form but different parameters for the polar function. But the subfamily with $0 < a < 1$ has significantly distinct features from its polar sub-family with $a^* = 1/a > 1$. If $a < 1$, all factors are "absolutely essential", since the isoquant surfaces do not intersect any of the axes; whereas if $a^* > 1$, the isoquant surfaces intersect all axes, and no factor is essential.

[21] These restrictions are ignored in the discussions of consumer-demand applications, since utility is ordinal, and may assume negative values.

Limiting CES cases: Linear and Leontief production functions (non-homothetic). If $d = 1$ in equation (24) [or $d_i = 1$, all i, in equation (11)], the ES are the limiting cases: $a = \infty$ and $a^* = 0$ in equation (23). The first case corresponds to the (non-homothetic) production function,

$$\sum D_i y^{-e_i} x_i \equiv 1, \tag{25}$$

with *linear isoquant-surfaces* for each y. The polar model exhibits a linear cost function,

$$C^* = \sum D_i y^{e_i} w_i, \tag{26}$$

and the polar direct production function corresponding to equation (26) is the (non-homothetic) *Leontief fixed coefficients production function,*

$$y = f^*(\mathbf{x}) = \min_i \{(x_i/D_i)^{1/e_i}\}. \tag{27}$$

The optimal quantities (for given y,$\mathbf{w}$) under equation (27) then satisfy

$$(x_i^*/D_i)^{1/e_i} = y \quad \text{or} \quad x_i^* = D_i y^{e_i}, \qquad i = 1,\ldots,n,$$

and thus $\mathbf{x}^*$ is independent of factor prices $\mathbf{w}$, with $C^* = \sum x_i^* w_i$, as in equation (26).

Homogeneous CES. If both $e_i = e = 1/\mu$ and $d_i = d$ in equation (11), the well-known homogeneous CES function is obtained:[22]

$$y = f(\mathbf{x}) = \left(\sum D_i x_i^d\right)^{\mu/d}, \tag{28}$$

where equation (28) is clearly a special case of both equation (19) (explicitly additive) and equation (24) (CES). The cost function dual to equation (28) is

$$C = y^{1/\mu} \sum D_i^{1/(1-d)} w_i^{-d/(1-d)}. \tag{29}$$

Applying the cost-polar transformation to equation (29) gives the polar production function,

$$y = f^*(\mathbf{x}) = \left(\sum D_i^* x_i^{d^*}\right)^{\mu/d^*},$$

[22]See McFadden (1963) and Uzawa (1962) for this n factor case, and the derivation of the cost function. See also Hanoch (1975a).

as in equation (24), with

$$A^*_{ij} = a^* = \frac{1}{a} = \frac{1}{A_{ij}}, \qquad i \neq j.$$

The *homogeneous* limiting cases corresponding to $d = 1$ in equation (28) are the

Linear production function:

$$y = \left(\sum D_i x_i\right)^{\mu},$$

and the

Leontief production function:

$$y = f^*(\mathbf{x}) = [\min_i \{x_i / D_i\}]^{\mu},$$

both homogeneous of degree μ.

Cobb–Douglas. If $d_i = 0$, all i, in equation (11), the (direct) function is

$$\sum D_i \log(x_i / y^{e_i}) \equiv 1,$$

which yields, after some manipulations,

$$y = f(\mathbf{x}) = A \Pi x_i^{b_i \mu}, \tag{30}$$

where

$$b_i = \frac{D_i}{\sum D_k}, \quad \mu = \frac{\sum D_k}{\sum e_k D_k}, \quad A = \exp\left\{-\frac{1}{\sum e_k D_k}\right\},$$

with $\sum b_k = 1$.

This is the well-known homogeneous Cobb–Douglas production function, with ES all equal to 1, and degree of homogeneity μ. Its corresponding (dual) cost function is

$$C = (y/A)^{1/\mu} \Pi[(w_i/b_i)^{b_i}] = A^* y^{1/\mu} \Pi(w_i^{b_i}), \tag{31}$$

where

$$A^* = A^{-1/\mu} \Pi(b_i^{-b_i}).$$

Applying the polar transformation to equation (31) gives the original function (except for the scale constant),

$$y = f^*(\mathbf{x}) = (A^*)^{\mu}\Pi(x_i^{b_i\mu}).$$

This production function is thus "self-polar" (in the strong sense), and is exactly self-polar (i.e., $f \equiv f^*$) if $(A^*)^{\mu} = A$, or equivalently if

$$A = (\Pi b_i^{b_i\mu})^{-1/2}.$$

[The function (30) could also be obtained as a limiting case of the homogeneous CES case (28), for $d \to 0$.]

The *non-homothetic Cobb–Douglas* function is defined as a special case of the general CRES function (8), i.e.,

$$\sum D_i(y) \log x_i \equiv 1. \tag{32}$$

However, the previous discussion shows that the specialized CRES model in equation (11) yields a homogeneous function in this case even if e_i are not equal for all i.

The models summarized here, as well as all their special cases, could be generalized somewhat by substituting an arbitrary increasing function $h(y)$ for y. This will generalize any *homogeneous* case to a *homothetic* function, and will allow some additional flexibility in estimating the non-homothetic models as well, if the functional form of $h(y)$ is specified with a small number of unknown parameters. For example, if $h(y) = e^{y/y_0}$, the average cost function in the homogeneous cases will be U-shaped, with a minimum at y_0. Substituting y/y_0 for $\log y$ in the estimation equations will then preserve their linear properties, and allow estimation of y_0.[23]

4. Multiproduct Production Frontiers with Constant TOES or TOET

This section presents several generalizations of the single-output production functions presented above, for joint-production with multiple outputs. Some of these frontiers exhibit CRES and CDE for elasticities of substitution between inputs (at constant outputs), and between outputs (at constant inputs); others have constant TOET, i.e., constant $RT_{ij}^k = T_{ik}/T_{jk}$ (CRET), or constant $DT_{ij}^k = T_{ik} - T_{jk}$ (CDET).

Special cases of these, which are generalizations to m outputs of the

[23]See Hanoch (1971).

CES and the Cobb–Douglas production functions, are given at the end. Additional generalizations or specializations may be derived along similar lines. Proofs of monotonicity and convexity conditions, as well as details regarding demand and supply relations and estimation equations are similar in nature to the corresponding single output cases, and are generally omitted.

Separable frontiers with constant TOES: constant TOES between inputs. A production frontier $F(\mathbf{y},\mathbf{x}) \equiv 0$, where $\mathbf{y}$ is a vector of m outputs, is denoted *separable* (between outputs and inputs), if $\mathbf{y}$ is weakly separable in F, i.e., there exists a function $g(\mathbf{y})$ such that

$$F(\mathbf{y},\mathbf{x}) = F[g(\mathbf{y}),\mathbf{x}] \equiv 0. \tag{33}$$

with $\partial F/\partial g > 0$. This implies strong separability as well,[24] since $g(\mathbf{y}) = f(\mathbf{x})$ may be solved from equation (33), applying the implicit function theorem.

An immediate implication of this, is that the aggregate $y = g(\mathbf{y})$ may be substituted for the single output y in all the CRES and CDE models presented in Section 3. The cost function is defined for a fixed vector $\mathbf{y}$, hence a fixed scalar $y = g(\mathbf{y})$, and the cost-polar transformation is applicable in complete analogy to the one-output case. Specifying an arbitrary (but valid) $g(\mathbf{y})$ thus generates the production models of Section 3, with constant R^k_{ij} or D^k_{ij} *between inputs.*

Constant TOES between outputs. The function $g(\mathbf{y})$ may itself exhibit constant R^k_{ij} or D^k_{ij} *between outputs* (given constant input quantities $\mathbf{x}$), if it is of similar form to the models discussed above, with appropriate modifications.

Specifically, the aggregate $x = f(\mathbf{x})$, in the separable frontier $g(\mathbf{y}) = f(\mathbf{x})$, is substituted for the single input x in the *revenue function* $R(\mathbf{p},x)$. The revenue function is analogous to the cost function $C(\mathbf{w},y)$, with the exception that R is *convex* (and linear homogeneous) in $\mathbf{p}$, whereas $C(\mathbf{w},y)$ is concave in $\mathbf{w}$. Similarly, $g(\mathbf{y})$ must be quasi-convex, and the transformation surfaces $g(\mathbf{y}) = g_0$ are concave [whereas $f(\mathbf{x})$ is quasi-concave, with convex isoquant surfaces].

The functional form (10) is modified to yield the function $g(\mathbf{y})$ with CRES between outputs as follows:

[24] Weak and strong separability are known to be identical concepts with respect to a partition into two groups. See Goldman and Uzawa (1964).

$$\sum b_j g^{-h_j b_j} y_j^{b_j} \equiv 1, \tag{34}$$

where

$$b_j > 1, \qquad B_j, h_j > 0, \qquad j = 1, \ldots, m.$$

The ES between outputs are

$$A_{ij} = \frac{a_i a_j}{\sum a_k s_k} < 0 \quad \text{and} \quad R_{ij}^k = \frac{a_i}{a_j} > 0, \tag{35}$$

where

$$a_j = \frac{1}{1 - b_j} < 0,$$

and s_k are shares of revenues,

$$s_k = \frac{y_k p_k}{\sum y_j p_j} > 0,$$

and A_{ij} are therefore negative (corresponding to concave transformation surfaces).

The corresponding *polar transformation* function $g^*(\mathbf{y})$ exhibits, by analogy, CDE between outputs.

The polar revenue function $R^*(\mathbf{p},x)$ [dual to $g^*(\mathbf{y})$] is defined implicitly by

$$\sum B_j x^{h_j b_j} (p_j / R^*)^{b_j} \equiv 1, \qquad b_j > 1, \tag{36}$$

with

$$A_{ij}^* = -b_i - b_j + \left(\sum s_k^* b_k - 1\right) = \frac{1}{a_i} + \frac{1}{a_j} - \sum s_k^* \left(\frac{1}{a_k}\right),$$

and

$$D_{ij}^{k*} = A_{ik}^* - A_{jk}^* = \frac{1}{a_i} - \frac{1}{a_j} = b_j - b_i,$$

in analogy to equation (14). Note, that the CDE models (12) or (36) allow some A_{ij}^* (but not all) to be of opposite sign to A_{ij}, if more than two inputs or outputs exist, and thus allow cases of *complementarity* between outputs ($A_{ij}^* > 0$) or inputs ($A_{ij}^* < 0$), if $|1/a_i + 1/a_j|$ is small relative to the weighted mean $|\sum s_k^*(1/a_k)|$. Thus, CDE models are generally more flexible in this respect, than their polar CRES models.

Reductions of the general CRES and CDE models to various special cases, by imposing equality restrictions on various parameters, are entirely analogous to all the cases discussed in Section 3, and thus yield a variety of special functional forms for output transformation functions $g(\mathbf{y})$ and their corresponding dual revenue functions $R(\mathbf{p},x)$.

Constant TOES between inputs and between outputs. If $n > 2$ and $m > 2$, the models with CRES and CDE for outputs and for inputs may be combined to yield a polar pair of CRES/CDE frontiers for both inputs and outputs. The following three equations define (simultaneously) the CRES frontier:

$$\begin{aligned} &\sum D_i f^{-e_i d_i} x_i^{d_i} \equiv 1, \qquad d_i < 1, \\ &\sum B_j g^{-h_j b_j} y_j^{b_j} \equiv 1, \qquad b_j > 1, \end{aligned} \tag{37}$$

and

$$f(\mathbf{x}) = g(\mathbf{y}).$$

The polar CDE frontier applies both the cost (for inputs) and the revenue (for outputs) polar transformations to equation (37), to give

$$\begin{aligned} &\sum D_i f^{e_i d_i} (w_i/C^*)^{d_i} \equiv 1, \\ &\sum B_j g^{h_j b_j} (p_j/R^*)^{b_j} \equiv 1, \end{aligned} \tag{38}$$

and

$$f(\mathbf{w}/C^*) = g(\mathbf{p}/R^*).$$

It follows, that the direct form of the polar frontier [equation (38)] is also separable, of the general form $f^*(\mathbf{x}) = g^*(\mathbf{y})$. Various special cases of either f or g or both are analogous to the single-output cases of Section 3.

The profit-polar frontier. The condition for existence of a *profit function* dual to the frontier [equation (37)], is that the feasible set $\mathbf{T} = \{(\mathbf{y},\mathbf{x}): f(\mathbf{x}) \geq g(\mathbf{y})\}$ is *convex*. A sufficient (but not necessary) condition for this is $\max_j h_j \leq \min_i e_i$. For example, if all $e_i > 1$, the function $f(\mathbf{x})$ in equation (37) is concave; if all $h_j < 1$, $g(\mathbf{y})$ is convex, implying convexity of the feasible set $\mathbf{T}$, and of the frontier function $F = g(\mathbf{y}) - f(\mathbf{x})$.

If the profit function exists, the *profit polar* transformation may be applied to equation (37), in accordance with Chapter I.2, Theorem 5. That is, a polar profit function $\pi^*(\mathbf{p},\mathbf{w})$ is defined by the following pair of equations, after elimination of the common parameter f,

$$\sum D_i f^{-e_i d_i}(w_i/\pi^*)^{d_i} \equiv 1,$$
$$\sum B_j f^{-h_j b_j}(p_j/\pi^*)^{b_j} \equiv 1. \tag{39}$$

However, the (direct) frontier corresponding to equation (39) is generally not separable, not homothetic, and *does not* in general exhibit constant TOES.

Constant TOES separable homogeneous and homothetic frontiers. Specializing the case given by equation (37) to equality of the expansion parameters, yields a *homogeneous* frontier.[25] That is: if $h_j = h$, all j, and $e_i = e$, all i, the frontier is homogeneous of degree $\mu = h/e$.

If $\mu < 1$, the frontier is convex, the profit function exists, and the profit-polar transformation is defined, and yields a *separable* special case of equation (39).

However, as shown in Chapter I.2, the cost-polar and the profit-polar transformations yield the same function $f(\mathbf{x})$ (except for a constant proportionality factor) in the homogeneous case. By analogy, the revenue-polar and profit-polar transformation are the same. Consequently, in this special case equations (39) and (38) are virtually identical, with the CDE property $D^k_{ij} = d_j - d_i$ between inputs, and $D^k_{ij} = b_j - b_i$ between outputs. These results may be generalized somewhat, using a similar reasoning. If $f(\mathbf{x}) = h[g(\mathbf{y})]$ is substituted for $g(\mathbf{y})$ in equation (37), where $h(g)$ is a positive and strictly increasing function, then the frontiers (37), (38) and (39) are all *homothetic and separable.* The two alternative polar transformations [i.e., the cost/revenue-polar of equation (38) and the profit-polar of equation (39)] give rise to frontiers with identical maps of input-isoquants and output-transformation surfaces, all with constant TOES. (Detailed proofs of these brief statements are left as an exercise to the interested reader).

A CRET non-homothetic, separable frontier. A non-homothetic separable frontier is derived from equation (37) under the special *explicitly*

[25]A frontier $F(\mathbf{y},\mathbf{x}) \equiv 0$ is μ *homogeneous*, if it satisfies $F(\lambda^\mu \mathbf{y},\lambda \mathbf{x}) \equiv 0$ for all $\lambda > 0$. See Hanoch (1970).

additive case, $h_j b_j = e_i d_i = c > 0$ ($j = 1,...,m; i = 1,...,n$), in analogy to case (19) for the one-output model. Applying these restrictions to equation (37), this yields

$$f^c = \sum D_i x_i^{d_i}, \quad g^c = \sum B_j y_j^{b_j}, \quad f = g.$$

Hence the frontier is

$$\sum B_j y_j^{b_j} - \sum D_i x_i^{d_i} \equiv 0, \tag{40}$$

where sufficient parameter restrictions for validity are again $B_j > 0$, $D_i > 0$, $b_j > 1$, $0 \le d_i < 1$ *or* $d_i \le 0$, all i,j (with $\log x_i$ replacing $x_i^{d_i}$ if $d_i = 0$).

The frontier (40) is directly additive separable, but is non-homothetic in either outputs (unless all $b_j = b$) or inputs (unless all $d_i = d$).

Derivation of the ET in an analogous manner to the single-output case[26] yields

$$T_{ij} = -\frac{a_i a_j}{\sum \delta_k a_k}, \qquad i,j = 1,...,m+n, \tag{41}$$

where $a_i = 1/(1 - d_i) > 0$ for inputs, $a_j = 1/(1 - b_j) < 0$ for outputs. δ_k are the profit shares, given by $\delta_k = -w_k x_k^* / \pi < 0$ for inputs, and $\delta_k = p_k y_k^* / \pi > 0$ for outputs, where ($\mathbf{y}^*, \mathbf{x}^*$) are optimal input and output quantities, respectively, yielding positive maximum profits

$$\pi = \sum p_j y_j^* - \sum w_i x_i^* > 0.$$

Note that

$$\sum^{m+n} \delta_k = 1 \quad \text{and} \quad \delta_k a_k < 0, \qquad \text{all } k.$$

Thus, T_{ij} of equation (41) are positive between outputs or inputs, and negative between an output y_j and an input x_i, under the sufficient parameter restrictions given above. Since this is a special case of equation (37), it also exhibits CRES for ES between inputs (under $\mathbf{y}$ constant) and between outputs (under $\mathbf{x}$ constant). This specializes, of course, to the Mukerji function of equation (19), in the case of a single output, with $B_1 = 1$ and $b_1 = d$.

[26]We omit the details of this derivation, which may be worked out by the reader.

The polar frontier: non-separable, non-homothetic CDET. By analogy, a *CDET Frontier* is obtained through the profit-polar transformation, such that the polar unit-profit frontier has the same form as equation (40) in the price variables (**p**;**w**). The profit function $\bar{\pi}(\mathbf{p};\mathbf{w})$ is then given implicitly by

$$\sum B_j(p_j/\bar{\pi})^{b_j} - \sum D_i(w_i/\bar{\pi})^{d_i} = 0. \tag{42}$$

Equation (42) yields a valid profit function, and hence a valid production frontier, under exactly the same parameter restrictions as in equation (41).

The ET for this frontier are given, in analogy to previous results, by

$$T_{ij} = -\alpha_i - \alpha_j + \sum \delta_l \alpha_l, \tag{43}$$

where $\alpha_i = 1 - b_i < 0$ for outputs, $\alpha_i = 1 - d_i > 0$ for inputs, and δ_l are profit shares, with $\delta_l \alpha_l < 0$, and $\sum \delta_l = 1$. It follows immediately, that T_{ij} between pairs of inputs are negative, in sharp contrast to the CRET frontier. This reflects dominance of expansion effects over substitution effects, for this particular frontier (under the sufficient conditions $b_j > 1 > d_i$). For output–output and output–input pairs, T_{ij} may be of either sign, depending on the relative magnitudes of α_i, α_j, and $\bar{\alpha} = \sum \delta_l \alpha_l$.

The polar frontier corresponding to equation (42) is not directly separable, and is non-homothetic in either outputs or inputs. As a result, the corresponding short-run ES, for fixed outputs or inputs, do not exhibit CDES nor any other simple relation to one another or to the corresponding T_{ij}.

This CDET frontier yields convenient estimation equations for relative demands or supplies, under competitive profit maximization, in analogy to the results corresponding to equation (20). That is, (y_j/y_1) is log-linear in the two variables, $(p_j/\bar{\pi})$ and $(p_1/\bar{\pi})$, $j \geq 2$; and (x_i/y_1) is log-linear in $(w_i/\bar{\pi})$, and $(p_1/\bar{\pi})$. Alternatively, $\bar{\pi}$ could be eliminated, to yield $(m + n - 2)$ equations, each including two quantity-ratios and two price-ratios.[27]

The generalized CES–CET frontier. The homogeneous separable models discussed above have *constant ES* between inputs as well as between

[27] See Hanoch (1975a) for the one-output analogous case.

outputs, under the special case: $d_i = d$, $b_j = b$, $e_i = e$ and $h_j = h$, for all i,j, in equation (37).

Denoting $\mu = h/e$, this frontier is given as follows:

$$\left(\sum B_j y_j^b\right)^{1/b} = \left(\sum D_i x_i^d\right)^{\mu/d}, \tag{44}$$

where the ES are $A_{ij} = 1/(1-d) > 0$ for inputs, and $A_{ij} = 1/(1-b) > 0$ for outputs, and the frontier is homogeneous of degree μ.

By analogy to the CES production function, the polar frontier has a similar form, but different parameters [see equation (46)]. The ES of the polar frontier are given by $A_{ij}^* = 1-d$ between inputs, and $A_{ij}^* = 1-b$ between outputs. (The polar frontier is also μ homogeneous.)

A homothetic generalization of equation (44), with the same CES property, is given by

$$\left(\sum B_j y_j^b\right)^{1/b} = h\left[\left(\sum D_i x_i^d\right)^{1/d}\right], \tag{45}$$

where h is an arbitrary positive increasing function.

In the homogeneous CES case, however, *the elasticities of transformation are also constant* (hence the name CES–CET). This may be shown directly, by spelling out the explicit form of the profit function corresponding to equation (44),[28]

$$\pi = \left(\sum B_j^* p_j^{b^*}\right)^{1/b^*(1-\mu)} \left(\sum D_i^* w_i^{d^*}\right)^{-\mu/d^*(1-\mu)}, \tag{46}$$

where

$$B_j^* = [(1-\mu)^{1-\mu}\mu^{\mu}]^{b/(b-1)} B_j^{-1/(b-1)},$$
$$D_i^* = D_i^{1/(1-d)},$$
$$d^* = -d/(1-d),$$
$$b^* = b/(b-1).$$

Computing the ET for this frontier by direct differentiation of the profit function (46), we get

$$T_{ij} = \frac{b}{b-1}(1-\mu) - 1,$$

for all pairs of outputs;

[28]The proof is omitted. See McFadden (1963) and Chapter I.1.

$$T_{ij} = \frac{d}{1-d} \cdot \frac{1-\mu}{\mu} - 1,$$

for all pairs of inputs; and

$$T_{ij} = -1,$$

for all input–output pairs (as in all separable, homogeneous frontiers).

A special case of this CET frontier is obtained if $d > 0$ and $\mu = d/b$, which is readily seen to be also *a special case of the additive CRET frontier* (40), derived by putting $b_j = b$, $d_i = d$ in (40). The ET are then given by

$$T_{ij} = \frac{b-1}{1-d},$$

for inputs, and their reciprocal

$$T_{kl} = \frac{1-d}{b-1},$$

for outputs, and are all positive.

The generalized Cobb–Douglas frontier. A further specialization of the homogeneous CET frontier (44) may be called the *Generalized Cobb–Douglas* frontier, which is the limiting case of equation (44) if $d \to 0$, $b = 2$. The frontier is given by

$$\sum B_j y_j^2 = \Pi x_i^{2\mu D_i}, \tag{47}$$

where $\Sigma D_i = 1$, $B_j, D_i > 0$. It yields *unitary* ES, $A_{ij} = 1$ between inputs, and $A_{ij} = -1$ between outputs. The constant ET are given by $T_{ij} = -1$ between inputs, $T_{ij} = 2(1-\mu) - 1$ between outputs, and $T_{ij} = -1$ between an output and an input.

Finally, the special case of equation (47), with $B_j = B = \mu^{\mu}(1-\mu)^{1-\mu}\Pi_i D_i^{D_i}$, all j, may be shown to be exactly self-polar, yielding a unit-profit frontier $B \Sigma p_j^2 = \Pi w_i^{2\mu D_i}$, which is exactly of the same functional form as equation (47) with $B_j = B$.[29]

The well-known CES and Cobb–Douglas production functions are obviously special cases of equations (44) and (47), respectively, in the presence of one single output.

[29]On self-duality (self-polarity) in the context of consumer demand, see Houthakker (1965) and Samuelson (1969b). The frontier (47) is viewed as a generalization of Cobb–Douglas, due to its self-polarity, as well as due to its unitary ES.