Suggested Solutions for Problem Set #2
Reminder: 13 points were possible but the max we count toward your grade is 10 points.

1. *(2 points total, 1 point each)*

A. *Starting from the national income identity (Y = C + I + G + GX - IM), derive the expression sY = I.*

$$Y = C + I + G + GX - IM$$
$$Y + TR - TA = C + I + G + TR - TA + GX - IM$$
$$(Y^D - C) + (TA - G - TR) + (IM - GX) = I$$
$$S^H + S^G + S^F = I$$
$$S = I$$
$$s \cdot Y = I$$

Where the saving rate, s, is defined as $s = \frac{S}{Y}$

B. *Starting from the expression* $\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{1-\alpha}$ *, derive the expression* $\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} E$

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{1-\alpha}$$
$$\frac{Y}{L} = \frac{K^{\alpha}}{L^{\alpha}} \cdot \frac{Y^{\alpha}}{Y^{\alpha}} \cdot E^{1-\alpha}$$
$$\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\alpha} \left(\frac{Y}{L}\right)^{\alpha} E^{1-\alpha}$$
$$\frac{\frac{Y}{L}}{\left(\frac{Y}{L}\right)^{\alpha}} = \left(\frac{K}{Y}\right)^{\alpha} E^{1-\alpha}$$
$$\left(\frac{Y}{L}\right)^{1-\alpha} = \left(\frac{K}{Y}\right)^{\alpha} E^{1-\alpha}$$
$$\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\left(\frac{\alpha}{1-\alpha}\right)} E$$

2. *(1 point total, ½ point each)*

Assume the production function is the usual Cobb-Douglas production function $\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{1-\alpha}$

A. *What is the value of output per worker per year when K = $90,000,000 million, L = 150 million workers, E = $75,000 per year, and α = 0.33? Put a box around your numerical answer.*

$$\frac{Y}{L} = \left(\frac{90{,}000{,}000}{150}\right)^{0.33} (75{,}000)^{1-0.33} = (600{,}000)^{0.33}(75{,}000)^{0.67} = 80.68 \cdot 1846.25 \approx \$150{,}000 \; per\ worker\ per\ year$$

B. *Suppose instead the value of α = 0.4. Now what is the value of output per worker per year? Put a box around your numerical answer.*

$$\frac{Y}{L} = \left(\frac{90{,}000{,}000}{150}\right)^{0.4} (75{,}000)^{1-0.4} = (600{,}000)^{0.4}(75{,}000)^{0.6} = 204.77 \cdot 841.5 \approx \$170{,}000 \; per\ worker\ per\ year$$

3. (2 points; 1 point each)

Starting from the Cobb-Douglas production function $\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{1-\alpha}$ *answer these two questions.*

A. In which case, when α = 0.33 or when α = 0.4, are the returns to investment greater?

The returns to investment are greater when $\alpha = 0.4$. Returns to investment refer to how much more output per worker we get from adding more capital per worker to the existing stock of capital per worker. When we add more capital per worker to the existing stock of capital per worker, we get a greater increase in output per worker the higher the value of α.

B. Explain, in words that would make sense to someone who doesn't yet understand, what it means for the returns to investment to be greater. (Remember this pedagogical trick: when someone doesn't understand an abstract concept, it often helps to start with an example.)

Suppose you have two different economies: one that is very developed and one that is still developing. Now suppose each economy experiences some net investment in capital that increases each economy's capital stock per worker (machines and buildings per worker) by the same dollar amount. Will both enjoy the same increase in output?

Probably not. Probably the developed economy, which already has lots of capital (machines) for each of its workers to use, will gain less output than will the developing economy. The developing economy gains more output – has a greater "return to investment" – than does the developed economy.

4. (2 points; 1 point each)

A. The value of α measures capital's share of total income, Y_K/Y. The value of 1-α measures labor's share of income, Y_L/Y. Starting from the Cobb-Douglas production function in this format $Y = A*K^{\alpha}L^{(1-\alpha)}$, where A stands for $E^{1-\alpha}$, and assuming perfectly competitive markets in which the factors of production K and L are paid their marginal product, and assuming that capital and labor are the only two inputs (so $Y_L + Y_K = Y$), derive the expression that shows labor's share of income equals 1-α.

Labor's share of income is $\frac{Y_L}{Y}$. Total labor income Y_L is what's paid to each worker (the wage) times the number of workers (L). In perfectly competitive markets, the wage is equal to the marginal product of labor. The marginal product of labor is the derivative of output with regard to labor. So we need to do a bit of calculus to figure out the expression for the wage, then a bit of algebra to compute labor's share of income.

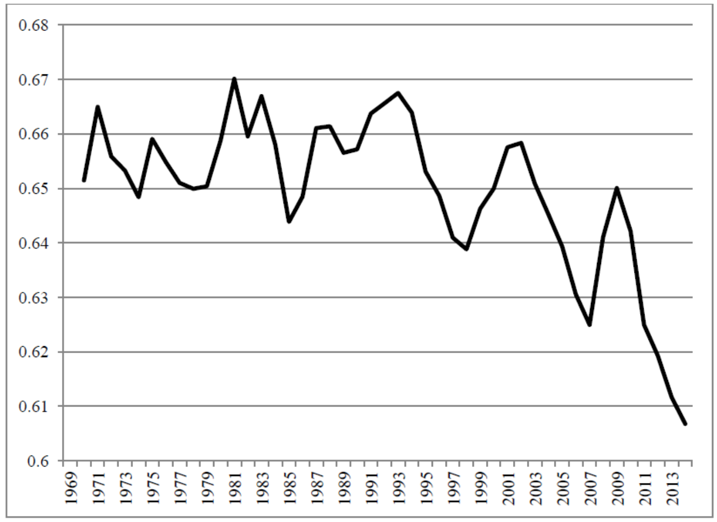
$$w = MP_L = \frac{\partial Y}{\partial L} = A \cdot K^{\alpha} \cdot (1-\alpha)L^{1-\alpha-1} = A \cdot K^{\alpha} \cdot (1-\alpha)L^{-\alpha}$$

$$Y_L = w \cdot L = A \cdot K^{\alpha} \cdot (1-\alpha) \cdot L^{-\alpha} \cdot L = A \cdot K^{\alpha} \cdot (1-\alpha) \cdot L^{1-\alpha}$$

$$\frac{Y_L}{Y} = \frac{(1-\alpha) \cdot A \cdot K^{\alpha} \cdot L^{1-\alpha}}{A \cdot K^{\alpha} \cdot L^{1-\alpha}} = 1 - \alpha$$

B. In an NBER working paper by Robert Lawrence (https://www.nber.org/papers/w21296), Lawrence shows in Figure 1 (repeated here) that labor's share of income in the U.S. has declined from about 66% to about 60% over the last two decades. Comparing today with the early 1990s, does this fact tell you that α is larger or smaller today than it was in the early 1990s? Does it tell you that returns to investment are larger or smaller today than they were in the early 1990s?

Figure 1: Share of labor compensation in US national income 1969 to 2013



A lower share for labor in total income means that $1 - \alpha$ has decreased, which means α has increased over the last two decades.

A higher value of α is consistent with larger returns to investment today as compared with the early 1990s.

Based on the graph, the standard assumption that $\alpha = 0.33$ should be replaced with an assumption that $\alpha = 0.4$.

5. **(4 points total; ½ point each)**

a. The definition of the balanced growth equilibrium is that what is balanced with what? What is the equation that expresses the balanced growth equilibrium condition?

Definition: Balanced growth equilibrium occurs when the growth rate of output per worker (Y/L) is balanced with (equal to) the growth rate of capital stock per worker (K/L).

Equation: The balanced growth equilibrium condition: g(Y/L) = g(K/L).

b. From the equation expressing the balanced growth equilibrium condition, derive the implication: K/Y is constant in BGE.

g(Y/L) = g(K/L)	definition of BGE
g(Y/L) – g(K/L) = 0	algebra
g[(Y/L) / (K/L)] = 0	growth rules (see PS 1, #2)
g(Y/K) = 0	algebra, L's cancel
Y/K constant	meaning of zero growth

c. From the equation expressing the balanced growth equilibrium condition and using a Cobb-Douglas production function, derive the implication: in BGE, the growth rate of the standard of living (Y/L) equals the growth rate of efficiency.

$g(Y/L) = g(K/L)$, where $Y/L = (K/L)^{\alpha}E^{1-\alpha}$	
$g(Y/L) = g((K/L)^{\alpha}E^{1-\alpha})$	substitution
$g(Y/L) = g((K/L)^{\alpha}) + g(E^{1-\alpha})$	growth rules
$g(Y/L) = (\alpha)g(K/L) + (1-\alpha)g(E)$	growth rules
$g(Y/L) = (\alpha)g(Y/L) + (1-\alpha)g(E)$	substitution from BGE condition
$(1-\alpha)g(Y/L) = (1-\alpha)g(E)$	algebra, gathering of like terms
$g(Y/L) = g(E)$	algebra, cancelling out $(1-\alpha)$

d. Suppose n = 0.01, g = 0.02, δ = 0.04, s = 0.21, α = ⅓ and E =8,000. What is the value of the capital-output ratio when the economy is in balanced growth equilibrium? What is the balanced-growth-equilibrium value of output per worker when E=8,000?

To find the value of the capital-output ratio when the economy is in balanced growth equilibrium, we use the equation for the value of K/Y in balanced growth equilibrium condition: $\frac{K}{Y} = \frac{s}{n+g+\delta}$

$$\frac{K}{Y} = \frac{s}{n+g+\delta} = \frac{0.21}{0.01+0.02+0.04} = \frac{.21}{.07} = 3$$

In balanced growth equilibrium, the value of the capital-output ratio K/Y = 3

To determine the value of output per worker Y/L when the economy is in balanced growth equilibrium, we use this form of the production function: $\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} E$. Plug in the values of K/Y, α, and E to get:

$$\frac{Y}{L} = (3)^{\frac{1/3}{2/3}} \cdot 8{,}000 = 3^{\frac{1}{2}} \cdot 8{,}000 = 1.73 \cdot 8{,}000 = \$13{,}856\ per\ worker\ per\ year$$

Note that we have assumed that g = 0.02, which means that efficiency *E* will increase from one year to the next. So in the next year, efficiency (E) will equal 8,000*(1.02) = 8,160. And so next year, the balanced-growth-equilibrium output per worker will be $14,134 per worker. In the year after that, BGE Y/L will equal 14,416. So the value of BGE output per worker of $13,856 is not "what Y/L will be forever, and ever" but "what Y/L will be in BGE when E equals 8,000."

e. When E = 8,000 and K/L = 4,000, is this economy in balanced growth equilibrium? Give two different mathematical conditions, both of which support your answer.

No. When K/L=4,000, the economy is not in balanced growth equilibrium.

Condition 1: Check whether Y/L is equal to balanced growth equilibrium value when K/L is 4,000.

$$\frac{Y}{L} = (4{,}000)^{\frac{1}{3}} \cdot (8{,}000)^{\frac{2}{3}} = 6{,}350 \neq 13{,}856$$

When K/L=4,000, Y/L=6,350, which is not equal to balanced growth equilibrium value, which is 13,856. Therefore, the economy is not in balanced growth equilibrium.

Condition 2: Check whether K/Y is equal to balanced growth equilibrium value when K/L is 4,000. This requires calculating Y/L so replicates much of the work in "condition 1."

$$\frac{K}{Y} = \frac{K/L}{Y/L} = \frac{4{,}000}{6{,}350} = 0.63 \neq 3$$

When K/L = 4,000, K/Y = 0.63, which is not equal to the balanced growth equilibrium value of K/Y which is 3. Therefore, the economy is not in balanced growth equilibrium.

f. Provide expressions that show, for the economy described in part d, the BGE values of Y/L and K/L after 10 years.

We know that when the economy is in BGE, $g(E) = g\left(\frac{Y}{L}\right) = g\left(\frac{K}{L}\right)$. From the prompt we know that g(E) = 0.02. Therefore in BGE, Y/L and K/L will also grow by 2 percent per year.

Therefore, $\left(\frac{Y}{L}\right)_{t+10} = \left(\frac{Y}{L}\right) \cdot (1 + growth\ rate)^{10} = \left(\frac{Y}{L}\right) \cdot (1.02)^{10}$

And $\left(\frac{K}{L}\right)_{t+10} = \left(\frac{K}{L}\right) \cdot (1 + growth\ rate)^{10} = \left(\frac{K}{L}\right) \cdot (1.02)^{10}$.

6. (2 points total, ½ point each)

(From the textbook, page 117) Suppose that Mexico in 2000 was on its balanced-growth path. Output per worker in Mexico in the year 2000 was about $10,000 per year. Labor-force growth was 2.5 percent per year. The depreciation rate was 3 percent per year. The rate of growth of the efficiency of labor was 2.5 percent per year. The saving rate was 16 percent of GDP. The diminishing-returns-to-investment parameter α is 0.5

A. What is Mexico's equilibrium capital-output ratio? What was the value of efficiency in 2000?

First, solve for the BGE value of K/Y, then use that to solve for E.

$$\frac{K}{Y} = \frac{0.16}{0.025 + 0.025 + 0.03} = \frac{0.16}{0.08} = \frac{16}{8} = 2$$

Now use the K/Y form of the production function to solve for E.

$$\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}} E$$

$$10{,}000 = 2^{\frac{1/2}{1/2}} \cdot E$$

$$10{,}000 = 2 \cdot E$$

$$\mathbf{5{,}000 = \boldsymbol{E}}$$

B. What is your forecast of output per worker in Mexico for 2040?

In BGE, Y/L grows at the same rate as efficiency. From the prompt, g(Y/L) therefore equals 0.025. Plugging in gives us

$$\left(\frac{Y}{L}\right)_{2040} = \left(\frac{Y}{L}\right)_{2000} \cdot (1.025)^{40} = 10{,}000 \cdot 2.685 = \$26{,}850 \; per\ worker\ per\ year$$

C. Suppose instead the labor-force growth rate fell to 1 percent per year in 2000 and remained at 1 percent. What is your new forecast of output per worker in Mexico for 2040?

Be sure you read the clarification of this question offered on Piazza and repeated in the 2/7/2019 email from Prof. Olney: assume that the economy is in a new BGE in 2000. Use the value of E you calculated in part a.

$$new\ BGE \frac{K}{Y} = \frac{0.16}{0.01 + 0.025 + 0.03} = \frac{16}{6.5} = 2.46$$

$$new\ BGE \left(\frac{Y}{L}\right)_{2000} = 2.46^{\frac{1/2}{1-1/2}} \cdot 5{,}000 = 2.46 \cdot 5{,}000 = \$12{,}300 \; per\ worker\ per\ years$$

$$new\ BGE \left(\frac{Y}{L}\right)_{2040} = \left(\frac{Y}{L}\right)_{2000} \cdot (1+g)^{40} = 12{,}300 \cdot (1.025)^{40} = \$33{,}025 \; per\ worker\ per\ year$$

D. In words, not equations, explain why your forecasted level of the standard of living changed as a result of the lower labor-force growth rate.

A lower rate of labor force growth (1 percent in part c, versus 2.5 percent in part a) means that the annual gains in efficiency are benefitting a smaller number of workers relative to the counterfactual (that is, relative to part a). This greater benefit per worker is reflected in higher levels of output and income per worker: higher standard of living.

7. (1 point total)

Start from the Cobb-Douglas production function $\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{\beta}$ where the parameters α and β are constant, 0 < α < 1 and α + β = 1.2. Derive the expression for the growth rate of output per worker in balanced growth equilibrium. Show all your steps. In this case, will the growth rate of output per worker be equal to, less than, or greater than the growth rate of efficiency in balanced growth equilibrium?

In BGE, g(Y/L) = g(K/L).

$$g\left(\frac{Y}{L}\right) = \alpha \cdot g\left(\frac{K}{L}\right) + \beta \cdot g(E) \qquad now, substitute\ using\ BGE\ def$$

$$g\left(\frac{Y}{L}\right) = \alpha \cdot g\left(\frac{Y}{L}\right) + \beta \cdot g(E)$$

$$g\left(\frac{Y}{L}\right) - \alpha \cdot g\left(\frac{Y}{L}\right) = \beta \cdot g(E)$$

$$(1-\alpha) \cdot g\left(\frac{Y}{L}\right) = \beta \cdot g(E)$$

$$g\left(\frac{Y}{L}\right) = \frac{\beta}{1-\alpha} g(E)$$

Now, what can we say about $\frac{\beta}{1-\alpha}$? In particular, is it greater than, equal to, or less than 1?

We know from the prompt that $\alpha + \beta = 1.2$ and therefore $\beta = 1.2 - \alpha$.

$$\frac{\beta}{1-\alpha} = \frac{1.2-\alpha}{1-\alpha}$$

$$Is \frac{\beta}{1-\alpha} > 1?$$

$$Is \frac{1.2-\alpha}{1-\alpha} > 1?$$

$$Is\ 1.2 - \alpha > 1 - \alpha?$$

$$Is\ 1.2 > 1?$$

$$Yes!$$

$$Therefore\ \frac{\beta}{1-\alpha} > 1.$$

And therefore

$$g\left(\frac{Y}{L}\right) > g(E)\ when\ \alpha + \beta > 1$$