Sensitivity to Missing Data Assumptions: Theory and An Evaluation of the U.S. Wage Structure

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The Problem

Missing data is ubiquitous in modern economic research

- Roughly one quarter of earnings observations in CPS and Census.
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- Justification for imputation procedures in CPS and Census.
- And for ignoring missingness altogether...

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- And for ignoring missingness altogether...

Question: How can we evaluate sensitivity of conclusions to MAR?

- Want to consider plausible deviations from MAR without presuming much about selection mechanism.
- And to enable study of sensitivity at different points in conditional distribution (tails likely more sensitive).

The Model

Consider a triplet (Y, X, D) with $Y \in \mathbf{R}$, $X \in \mathbf{R}^l$, $D \in \{0, 1\}$.

$$Y = X'\beta(\tau) + \epsilon \qquad\qquad P(\epsilon \leq 0|X) = \tau$$

and $D = 1$ if Y is observable, and $D = 0$ if Y is missing.

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Without Missing Data

- Quantile regression as a summary of conditional distribution.
- Under misspecification $\Rightarrow$ best approximation to true model.

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Without Missing Data

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- Under misspecification $\Rightarrow$ best approximation to true model.

Without MAR

- Point identification fails.
- Need to consider misspecification and partial identification.

Missing Data

Define the conditional distribution functions,

$$F_{y|x}(c) \equiv P(Y \leq c|X = x) \qquad F_{y|d,x}(c) \equiv P(Y \leq c|D = d, X = x)$$

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Nonparametric Bounds

- Manski (1994). Exploit that $0 \leq F_{y|0,x}(c) \leq 1$ for all $c \in \mathbf{R}$.
- Often bounds are uninformative.
- And typically overly conservative.

Between these extremes lie a continuum of selection mechanisms...

Deviation from MAR

Characterize selection as distance between $F_{y|1,x}$ and $F_{y|0,x}$:

$$d(F_{y|1,x}, F_{y|0,x}) \leq k$$

- A way of nonparametrically indexing set of selection mechanisms:
  - Missing at random corresponds to imposing $k = 0$.
  - Manski Bounds corresponds to imposing $k = \infty$.

- Allows study of sensitivity to deviations from MAR
  (e.g. what level of k is necessary to overturn conclusions regarding $\beta(\tau)$?)

- And in some cases k may be estimated using validation data.

General Outline

Nominal Identified Set

- Find possible quantiles under restriction $d(F_{y|1,x}, F_{y|0,x}) \leq k$.
- Bound $\beta(\tau)$ as a function of τ, k allowing for misspecification.

General Outline

Nominal Identified Set

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- Exploit distribution as a function of (τ, k) for sensitivity analysis.

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Changes in Wage Structure

- Examine changes in wage structure across Decennial Censuses.
- Measure departures from MAR in matched CPS-SSA.

Literature Review

Missing Data:

Rubin (1974), Greenlees, Reece, & Zieschang (1982), Lillard, Smith & Welch (1986), Manski (1994,2003), Dinardo, McCrary, & Sanbonmatsu (2006), Lee (2008)

Sensitivity Analysis:

Altonji, Elder, and Taber (2005); Rosenbaum and Rubin (1983); Rosenbaum (1987, 2002).

Misspecification

White (1980, 1982), Chamberlain (1994), Angrist, Chernozhukov & Fernandez-Val (2006).

Misspecification and Partial Identification

Horowitz & Manski (2006), Stoye (2007), Ponomareva & Tamer (2009), Bugni, Canay & Guggenberger (2010).

Bounds on Conditional Quantiles

Define true conditional quantile $q(\tau|x)$ and non-missing probability $p(x)$:

$$F_{y|x}(q(\tau|x)) = \tau \qquad p(x) \equiv P(D = 1|X = x)$$

Goal: Obtain identified set for $q(\tau|x)$ under hypothetical $d(F_{y|1,x}, F_{y|0,x}) \leq k$.

For the distance metric we use Kolmogorov-Smirnov, which is given by:

$$\mathcal{S}(F) \equiv \sup_{x \in \mathcal{X}} KS(F_{y|1,x}, F_{y|0,x}) = \sup_{x \in \mathcal{X}} \sup_{c \in \mathbf{R}} |F_{y|1,x}(c) - F_{y|0,x}(c)|$$

Comments

- KS provides control over maximal distance between $F_{y|1,x}$ and $F_{y|0,x}$.
- Nests a wide nonparametric class of potential selection mechanisms.

Choice of Metric

Information necessarily lost with scalar index of selection, but ...

- Not ruling out selection mechanisms as done in parametric approaches.
- Different levels of selection can be considered at each quantile.
- Easy extension to different weights on covariate realizations.
- Scalar metric well suited to sensitivity analysis.

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Example: Suppose the data generating process is given by:

$$(Y, v) \sim N(0, \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)) \qquad D = 1\{\mu + v > 0\} .$$

where μ is chosen so that the missing probability is 25% to match data.

ρ	$\mathcal{S}(F)$
0.10	0.0672
0.20	0.1355
0.30	0.2069

ρ	$\mathcal{S}(F)$
0.40	0.2778
0.50	0.3520
0.60	0.4304

ρ	$\mathcal{S}(F)$
0.70	0.5165
0.80	0.6158
0.90	0.7377

Figure: Missing and Observed Outcome CDFs

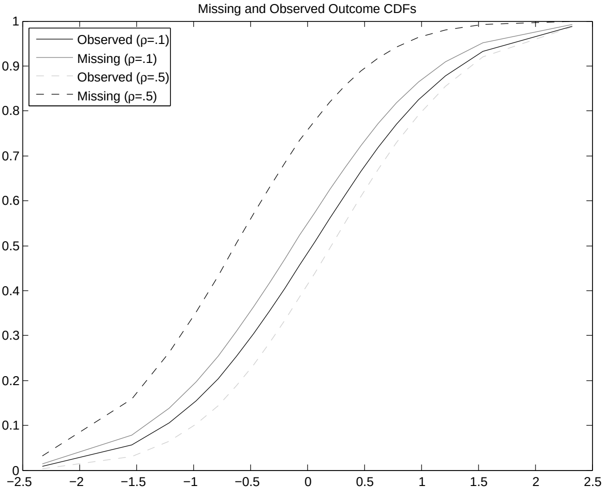
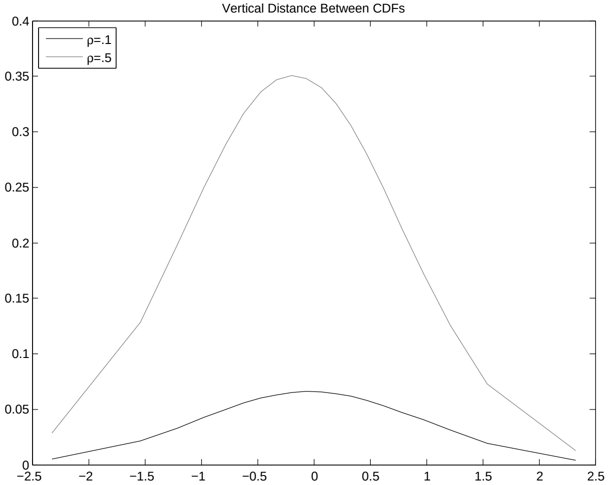


Figure: Distance Between Missing and Observed Outcome CDFs



Mixture Interpretation

Suppose a fraction k of the missing population is distributed according to an arbitrary CDF $\tilde{F}_{y|x}$, while the remaining fraction $1-k$ of that population are missing at random in the sense that they are distributed according to $F_{y|1,x}$. Then:

$$F_{y|0,x}(c) = (1-k)F_{y|1,x}(c) + k\tilde{F}_{y|x}(c) \ ,$$

where $\tilde{F}_{y|x}$ is unknown, and the above holds for all $x \in \mathcal{X}$ and any $c \in \mathbf{R}$. Now:

$$\begin{aligned}\mathcal{S}(F) &= \sup_{x\in\mathcal{X}} \sup_{c\in\mathbf{R}} \left|F_{y|1,x}(c) - k\tilde{F}_{y|x}(c) - (1-k)F_{y|1,x}(c)\right| \\ &= k \times \sup_{x\in\mathcal{X}} \sup_{c\in\mathbf{R}} \left|F_{y|1,x}(c) - \tilde{F}_{y|x}(c)\right| \ .\end{aligned}$$

Worst Case: $\mathcal{S}(F) = k$. Thus, k gives bound on the fraction of the missing sample that is not well represented by the observed data distribution.

Nominal Identified set for $q(\tau|\cdot)$

Assumption (A)

(i) $X \in \mathbf{R}^l$ has finite support $\mathcal{X}$.

(ii) $F_{y|d,x}(c)$ is continuous, strictly increasing $\forall c$ with $0 < F_{y|d,x}(c) < 1$.

(iii) D equals one if Y is observable and zero otherwise.

Lemma Under (A), if $\mathcal{S}(F) \leq k$, then the identified set for $q(\tau|\cdot)$ is:

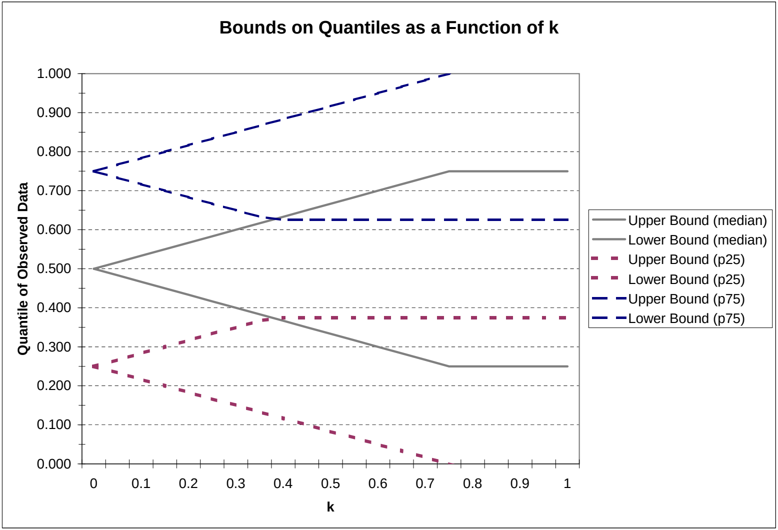
$$\mathcal{C}(\tau, k) \equiv \{\theta : \mathcal{X} \to \mathbf{R} : q_L(\tau, k|x) \leq \theta(x) \leq q_U(\tau, k|x)\}$$

where the bounds $q_L(\tau, k|x)$ and $q_U(\tau, k|x)$ are given by:

$$q_L(\tau, k|x) \equiv F^{-1}_{y|1,x}\Big(\frac{\tau - \min\{\tau + kp(x), 1\}\{1 - p(x)\}}{p(x)}\Big)$$

$$q_U(\tau, k|x) \equiv F^{-1}_{y|1,x}\Big(\frac{\tau - \max\{\tau - kp(x), 0\}\{1 - p(x)\}}{p(x)}\Big)$$

Example – No Covariates and $p(x) = 2/3$



Sensitivity Example 1 (Pointwise Analysis)

Suppose X is binary so that $X \in \{0, 1\}$, and write:

$$Y = q(\tau|X) + \epsilon \qquad\qquad P(\epsilon \leq 0|X) = \tau$$

Suppose that under MAR we have $q(\tau_0|X = 1) \neq q(\tau_0|X = 0)$ for some τ_0.

We can evaluate sensitivity of this conclusion to MAR by defining:

$$k_0 \equiv \inf k : q_L(\tau_0, k|X{=}1) - q_U(\tau_0, k|X{=}0) \leq 0 \leq q_U(\tau_0, k|X{=}1) - q_L(\tau_0, k|X{=}0)$$

Comment

- k_0 is the minimal level for overturning $q(\tau_0|X = 1) \neq q(\tau_0|X = 0)$.
- Large k_0 indicates robust conclusion.

Example 2 (Distributional Analysis)

We want to know if $F_{y|x=1}$ first order stochastically dominates $F_{y|x=0}$.

Suppose that under MAR we find $q(\tau|X=1) > q(\tau|X=0)$ at all τ.

We evaluate sensitivity of FOSD conclusion by examining:

$$k_0 \equiv \inf k : q_L(\tau, k|X=1) \leq q_U(\tau, k|X=0) \quad \text{for some } \tau \in (0,1)$$

Comment

- k_0 is the minimal level of selection under which the conclusion of FOSD may be undermined.

Example 3 (Breakdown Analysis)

$$Y = q(\tau|X) + \epsilon \qquad\qquad P(\epsilon \le 0|X) = \tau$$

Suppose that under MAR we have $q(\tau|X=1) \neq q(\tau|X=0)$ for multiple τ.

More nuanced analysis can consider the quantile specific critical level:

$$\kappa_0(\tau) \equiv \inf k : q_L(\tau, k|X{=}1) - q_U(\tau, k|X{=}0) \le 0 \le q_U(\tau, k|X{=}1) - q_L(\tau, k|X{=}0)$$

Comment

- Changes in τ map out a "breakdown function" $\tau \mapsto \kappa_0(\tau)$.
- Reveals differential sensitivity of the entire conditional distribution.

Adding Parametric Structure

With lots of covariates, convenient to assume a linear parametric model:

$$q(\tau|X) = X'\beta(\tau)$$

Identified set for $\beta(\tau)$ is intersection of $\mathcal{C}(\tau, k)$ with parametric models:

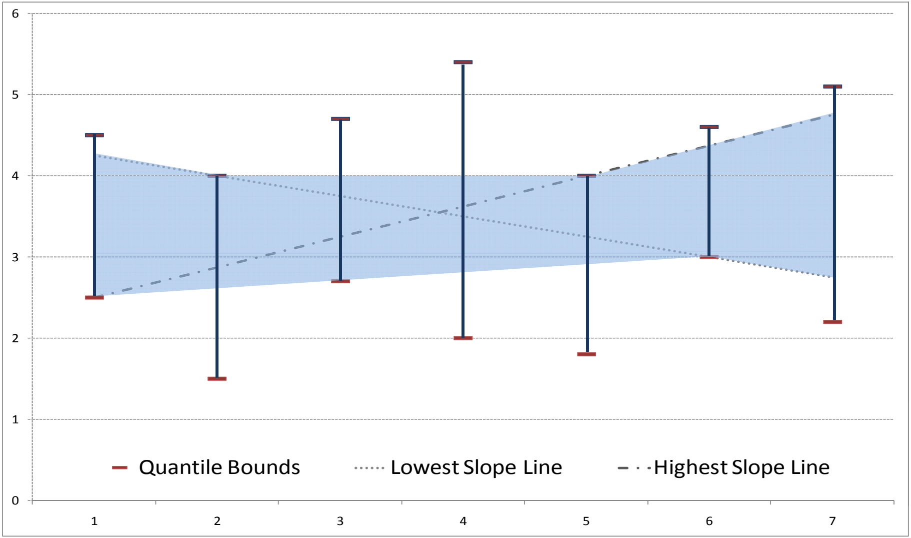
$$\left\{\beta(\tau) \in \mathbf{R}^l : q_L(\tau, k|X) \leq X'\beta(\tau) \leq q_U(\tau, k|X)\right\}$$

Comments:

- Set of functions in identified set may be severely restricted.
- Inadvertently rewards misspecification.

Identification by misspecification

Figure: Linear Conditional Quantile Functions as a Subset of the Identified Set



Adding Parametric Structure

Instead allow for misspecification in the linear quantile model

$$Y = X'\beta(\tau) + \eta$$

- If identified, misspecification as "pseudo true" approximation

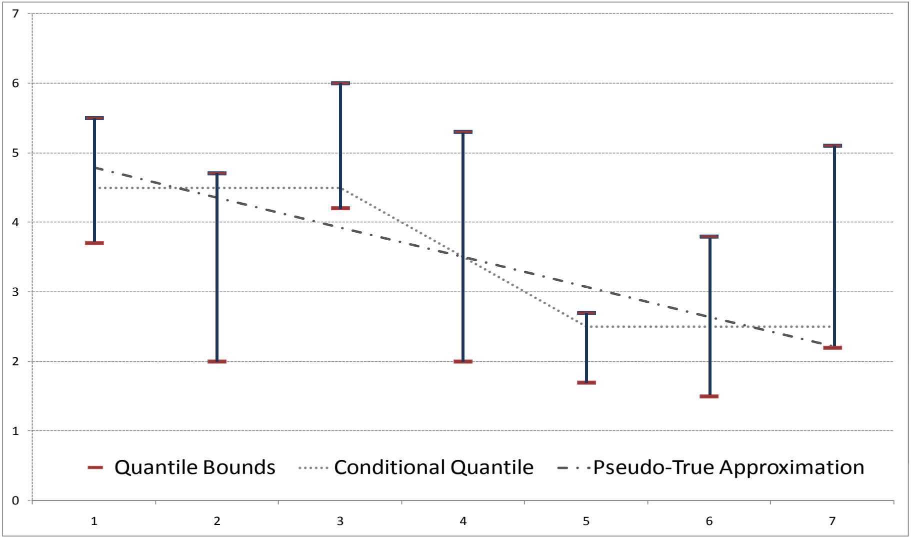
$$\beta(\tau) \equiv \arg\min_{\gamma \in \mathbf{R}^l} \int (q(\tau|x) - x'\gamma)^2 dS(x)$$

- If partially identified, each $\theta \in \mathcal{C}(\tau, k)$ implies a pseudo true vector $\beta(\tau)$

$$\mathcal{P}(\tau, k) \equiv \left\{ \beta \in \mathbf{R}^l : \beta = \arg\min_{\gamma \in \mathbf{R}^l} \int (\theta(x) - x'\gamma)^2 dS(x) \text{ for some } \theta \in \mathcal{C}(\tau, k) \right\}$$

$\Rightarrow$ i.e. consider $\beta \in \mathbf{R}^l$ that are best approximation to some $\theta \in \mathcal{C}(\tau, k)$.

Figure: Conditional Quantile and its Pseudo-True Approximation



Misspecification

Choice of quadratic loss allows for simple characterization of $\mathcal{P}(\tau, k)$.

Lemma: Under (A), if $\mathcal{S}(F) \leq k$ and $\int xx' dS(x)$ is invertible:

$$\mathcal{P}(\tau, k) = \left\{ \beta = \left[\int xx' dS(x) \right]^{-1} \int x\theta(x) dS(x) \; : q(\tau, k|x) \leq \theta(x) \leq q(\tau, k|x) \right\}$$

Note: It follows that $\mathcal{P}(\tau, k)$ is convex.

One more assumption: We will assume the measure S is known.

- Analogous to having a known loss function.

Parameter of Interest

Inference on parameters of the form $\lambda'\beta(\tau)$ for some $\lambda \in \mathbf{R}^l$.

Corollary: The identified set for $\lambda'\beta(\tau)$ is $[\pi_L(\tau,k), \pi_U(\tau,k)]$, where:

$$\pi_L(\tau,k) \equiv \inf_\theta \lambda' \left[\int xx'dS \right]^{-1} \int x\theta(x)dS(x) \text{ s.t. } q_L(\tau,k|x) \leq \theta(x) \leq q_U(\tau,k|x)$$

$$\pi_U(\tau,k) \equiv \sup_\theta \lambda' \left[\int xx'dS \right]^{-1} \int x\theta(x)dS(x) \text{ s.t. } q_L(\tau,k|x) \leq \theta(x) \leq q_U(\tau,k|x)$$

Comments:

- Examples: individual coefficients and fitted values.
- Bounds sharp for fixed τ and k.
- Bounds become wider with k and change across τ.

Bounds on the Process

Previous corollary implies that if $\mathcal{S}(F) \leq k$, then $\lambda'\beta(\cdot)$ belongs to:

$$\mathcal{G}(k) \equiv \left\{ g : [0,1] \to \mathbf{R} : \pi_L(\tau, k) \leq g(\tau) \leq \pi_U(\tau, k) \quad \text{for all } \tau \right\}$$

Unfortunately, $\mathcal{G}(k)$ is not a sharp identified set of the process $\lambda'\beta(\cdot)$.

However ...

- The bounds $\pi_L(\cdot, k)$ and $\pi_U(\cdot, k)$ are in identified set where finite.
- The bounds of $\mathcal{G}(k)$ are sharp at every point of evaluation τ.
- If $\theta \notin \mathcal{G}(k)$, then the function $\theta(\cdot)$ cannot equal $\lambda'\beta(\cdot)$.
- Ease of analysis and graphical representation.

Example 1 (cont)

$$Y = \alpha(\tau) + X'\beta(\tau) + \eta$$

Suppose that under MAR we have $\beta(\tau_0) \neq 0$ for some specific quantile τ_0.

We can evaluate sensitivity of this conclusion to MAR by defining:

$$k_0 \equiv \inf k : \pi_L(\tau_0, k) \leq 0 \leq \pi_U(\tau_0, k)$$

Comment

- k_0 is the minimal level of selection necessary to overturn $\beta(\tau_0) \neq 0$.

Example 2 (cont)

$$Y = \alpha(\tau) + X'\beta(\tau) + \eta$$

Suppose that under MAR we have $\beta(\tau) > 0$ for multiple τ.

We evaluate sensitivity to conclusion of $F_{y|x}$ being increasing at *some* τ:

$$k_0 \equiv \inf k : \pi_L(\tau, k) \leq 0 \quad \text{for all } \tau \in [0,1]$$

Comment

- k_0 is the minimal level of selection that overturns $\beta(\tau) > 0$ for *some* τ.
- $\pi_L(\cdot, k_0)$ is in identified set for $\beta(\cdot)$ under $\mathcal{S}(F) \leq k$.

Example 3 (cont)

$$Y = \alpha(\tau) + X'\beta(\tau) + \eta$$

Suppose that under MAR we have $\beta(\tau) \neq 0$ for multiple τ.

More nuanced analysis can consider the quantile specific critical level:

$$\kappa_0(\tau) \equiv \inf k : \pi_L(\tau, k) \leq 0 \leq \pi_U(\tau, k)$$

Comment

- Changing τ maps out a "breakdown function" $\tau \mapsto \kappa_0(\tau)$.
- Reveals differential sensitivity of the entire conditional distribution.

Estimating Bounds

- Study estimators for bound functions $\pi_L(\tau,k)$, $\pi_U(\tau,k)$ given by:

$$\hat{\pi}_L(\tau,k) \equiv \inf_{\theta} \lambda' \Big[\int xx' dS(x) \Big]^{-1} \int x\theta(x) dS(x) \text{ s.t. } \hat{q}_L(\tau,k|x) \leq \theta(x) \leq \hat{q}_U(\tau,k|x)$$

$$\hat{\pi}_U(\tau,k) \equiv \sup_{\theta} \lambda' \Big[\int xx' dS(x) \Big]^{-1} \int x\theta(x) dS(x) \text{ s.t. } \hat{q}_L(\tau,k|x) \leq \theta(x) \leq \hat{q}_U(\tau,k|x)$$

- Need distribution as processes on $L^\infty(\mathcal{B})$, where for $0 < 2\epsilon < \inf_x p(x)$:

$$\mathcal{B} \equiv \left\{ (\tau,k) : \begin{array}{ll} \text{(i) } kp(x)(1-p(x)) + 2\epsilon \leq \tau p(x) & \text{(iii) } k \leq \tau \\ \text{(ii) } kp(x)(1-p(x) + 2\epsilon \leq (1-\tau)p(x) & \text{(iv) } k \leq 1-\tau \end{array} \right\}$$

Comments:

- The bounds $\pi_L(\tau,k)$ and $\pi_U(\tau,k)$ are finite everywhere on $\mathcal{B}$.
- Large or small values of τ must be accompanied by small values of k.

Estimating Bounds

- Recall $q_L(\tau, k|x)$ and $q_U(\tau, k|x)$ were defined as quantiles of $F_{y|1,x}$:

$$q_L(\tau, k|x) = \arg\min_{c \in \mathbf{R}} Q_x(c|\tau, \tau + kp(x)) \quad q_U(\tau, k|x) = \arg\min_{c \in \mathbf{R}} Q_x(c|\tau, \tau - kp(x))$$

where the family of criterion functions $Q_x(c|\tau, b)$ is given by:

$$Q_x(c|\tau, b) \equiv (P(Y \leq c, X = x, D = 1) + bP(D = 0, X = x) - \tau P(X = x))^2$$

- This suggests an extremum estimation approach given by:

$$\hat{q}_L(\tau, k|x) = \arg\min_{c \in \mathbf{R}} Q_{x,n}(c|\tau, \tau + k\hat{p}(x)) \quad \hat{q}_U(\tau, k|x) = \arg\min_{c \in \mathbf{R}} Q_{x,n}(c|\tau, \tau - k\hat{p}(x))$$

where the criterion function $Q_{x,n}(c|\tau, b)$ is the immediate sample analogue.

Asymptotic Distribution

Assumptions (B)

(i) $F_{y|1,x}$ has a continuous bounded derivative $f_{y|1,x}$

(ii) $f_{y|1,x}$ has a continuous bounded derivative $f'_{y|1,x}$

(iii) The matrix $\int xx' dS(x)$ is invertible.

(iv) $f_{y|1,x}$ is positive "over relevant range".

Theorem Under Assumptions (A) and (B), if $\{Y_i, X_i, D_i\}_{i=1}^{n}$ is IID, then:

$$\sqrt{n}\begin{pmatrix} \hat{\pi}_L - \pi_L \\ \hat{\pi}_U - \pi_U \end{pmatrix} \xrightarrow{\mathcal{L}} G \; ,$$

where G is a gaussian process on $L^{\infty}(\mathcal{B}) \times L^{\infty}(\mathcal{B})$.

Proof Outline

Step 1: Study distribution of minimizers of $Q_{x,n}(c|\tau, b)$ as a function of (τ, b).

- Obtain uniform asymptotic expansions for the minimizers.
- $Q_{x,n}(c|\tau, b)$ has enough structure to establish equicontinuity.

Step 2: Find distribution $(\hat{q}_L, \hat{q}_U)$ in $L^\infty(\mathcal{B} \times \mathcal{X}) \times L^\infty(\mathcal{B} \times \mathcal{X})$.

- Simply a restriction of the process derived in Step 1.

Step 3: Establish the distribution of $(\hat{\pi}_L, \hat{\pi}_U)$ on $L^\infty(\mathcal{B})$.

- Straightforward due to linear program.

Example 1 (cont)

Suppose under MAR we find that $\beta(\tau_0) \neq 0$ for some specific quantile τ_0. Minimal level of selection necessary to undermine this conclusion is:

$$k_0 \equiv \inf k : \pi_L(\tau_0, k) \leq 0 \leq \pi_U(\tau_0, k)$$

Let $r^{(i)}_{1-\alpha}(k)$ be the $1-\alpha$ quantile of $G^{(i)}(\tau_0, k)$ and define:

$$\hat{k}_0 \equiv \inf k : \hat{\pi}_L(\tau_0, k) - \frac{r^{(1)}_{1-\alpha}(k)}{\sqrt{n}} \leq 0 \leq \hat{\pi}_U(\tau_0, k) + \frac{r^{(2)}_{1-\alpha}(k)}{\sqrt{n}}$$

Then $k_0 \in [\hat{k}_0, 1]$ with asymptotic probability greater than or equal to $1-\alpha$.

Comments

- One sided confidence interval (rather than two sided) is natural.
- Relevant critical value depends on transformation of G.

Example 2 (cont)

Suppose under MAR we find that $\beta(\tau) > 0$ for multiple τ and recall that

$$k_0 \equiv \inf k : \pi_L(\tau, k) \leq 0 \quad \text{for all } \tau \in [0,1]$$

Let $r_{1-\alpha}(k)$ be the $1-\alpha$ quantile of $\sup_\tau G^{(1)}(\tau,k)/\omega_L(\tau,k)$ and define:

$$\hat{k}_0 \equiv \inf k : \sup_\tau \hat{\pi}_L(\tau,k) - \frac{r_{1-\alpha}(k)}{\sqrt{n}} \omega_L(\tau,k) \leq 0$$

Then $k_0 \in [\hat{k}_0, 1]$ with asymptotic probability greater than or equal to $1-\alpha$.

Comments

- Weight function ω_L allows to adjust for different asymptotic variances.
- Result exploits uniformity in τ but not in k.

Example 3 (cont)

Suppose under MAR we find that $\beta(\tau) \neq 0$ for multiple τ and recall that:

$$\kappa_0(\tau) \equiv \inf k : \pi_L(\tau, k) \leq 0 \leq \pi_U(\tau, k)$$

For (ω_L, ω_U) positive weight functions, let $r_{1-\alpha}$ be the $1-\alpha$ quantile of:

$$\sup_{\tau,k} \max\left\{\frac{|G^{(1)}(\tau,k)|}{\omega_L(\tau,k)}, \frac{|G^{(2)}(\tau,k)|}{\omega_U(\tau,k)}\right\}$$

Then with asymptotic probability at least $1-\alpha$ for all τ, $\kappa_0(\tau)$ lies between:

$$\hat{\kappa}_L(\tau) \equiv \inf k : \hat{\pi}_L(\tau,k) - \frac{r_{1-\alpha}}{\sqrt{n}}\omega_L(\tau,k) \leq 0 \text{ and } 0 \leq \hat{\pi}_U(\tau,k) + \frac{r_{1-\alpha}}{\sqrt{n}}\omega_U(\tau,k)$$

$$\hat{\kappa}_U(\tau) \equiv \sup k : \hat{\pi}_L(\tau,k) + \frac{r_{1-\alpha}}{\sqrt{n}}\omega_L(\tau,k) \geq 0 \text{ or } 0 \geq \hat{\pi}_U(\tau,k) - \frac{r_{1-\alpha}}{\sqrt{n}}\omega_U(\tau,k)$$

Weighted Bootstrap

Question How do we obtain a consistent estimator for $r_{1-\alpha}$?

Answer Perturb the objective function and recompute (weighted bootstrap).

In all examples, $r_{1-\alpha}$ is quantile of $L(G_\omega)$ where L is Lipschitz and

$$G_\omega(\tau,k) = \begin{pmatrix} G^{(1)}(\tau,k)/\omega_L(\tau,k) \\ G^{(2)}(\tau,k)/\omega_U(\tau,k) \end{pmatrix}$$

In Particular

- In Example 1 $\theta \mapsto L(\theta)$ is $L(G_\omega) = G_\omega^{(i)}(\tau_0,k)$.
- In Example 2 $\theta \mapsto L(\theta)$ is $L(G_\omega) = \sup_\tau G_\omega^{(1)}(\tau,k)$.
- In Example 3 $\theta \mapsto L(\theta)$ is $L(G_\omega) = \sup_{\tau,k} \max\{|G_\omega^{(1)}(\tau,k)|, |G_\omega^{(2)}(\tau,k)|\}$.

Goal Construct a general bootstrap procedure for quantiles of $L(G_\omega)$.

Weighted Bootstrap

Step 1 Generate a random sample of weights $\{W_i\}$ and define the criterion:

$$\tilde{Q}_{x,n}(c|\tau,b) \equiv \Big(\frac{1}{n}\sum_{i=1}^{n} W_i\{1\{Y_i \le c, X_i = x, D_i = 1\} + b1\{D_i = 0, X_i = x\} - \tau 1\{X_i = x\}\}\Big)^2$$

Using $\tilde{Q}_{x,n}$ instead of $Q_{x,n}$ obtain analogues to $\hat{q}_L(\tau,k|x)$ and $\hat{q}_U(\tau,k|x)$

$$\tilde{q}_L(\tau,k|x) = \arg\min_{c\in\mathbf{R}} \tilde{Q}_{x,n}(c|\tau,\tau+k\tilde{p}(x)) \qquad \tilde{q}_U(\tau,k|x) = \arg\min_{c\in\mathbf{R}} \tilde{Q}_{x,n}(c|\tau,\tau-k\tilde{p}(x)$$

where $\tilde{p}(x) \equiv (\sum_i W_i 1\{D_i = 1, X_i = x\})/(\sum_i W_i 1\{X_i = x\})$.

Step 2 Using the bounds $\tilde{q}_L(\tau,k|x)$ and $\tilde{q}_U(\tau,k|x)$ from **Step 1**, obtain:

$$\tilde{\pi}_L(\tau,k) \equiv \inf_{\theta} \lambda'\Big[\int xx'dS(x)\Big]^{-1}\int x\theta(x)dS(x) \ \text{ s.t. } \tilde{q}_L(\tau,k|x) \le \theta(x) \le \tilde{q}_U(\tau,k|x)$$

$$\tilde{\pi}_U(\tau,k) \equiv \sup_{\theta} \lambda'\Big[\int xx'dS(x)\Big]^{-1}\int x\theta(x)dS(x) \ \text{ s.t. } \tilde{q}_L(\tau,k|x) \le \theta(x) \le \tilde{q}_U(\tau,k|x)$$

Weighted Bootstrap

Step 3 Using the bounds $\tilde{\pi}_L(\tau, k)$ and $\tilde{\pi}_U(\tau, k)$ from **Step 2**, define:

$$\tilde{G}_\omega(\tau, k) = \sqrt{n} \begin{pmatrix} (\tilde{\pi}_L - \hat{\pi}_L)/\hat{\omega}_L \\ (\tilde{\pi}_U - \hat{\pi}_U)/\hat{\omega}_U \end{pmatrix}$$

where $\hat{\omega}_L(\tau, k)$ and $\hat{\omega}_U(\tau, k)$ are estimators for $\omega_L(\tau, k)$ and $\omega_U(\tau, k)$.

Step 4 Estimate $r_{1-\alpha}$, the $1-\alpha$ quantile of $L(G_\omega)$ by $\tilde{r}_{1-\alpha}$ defined as:

$$\tilde{r}_{1-\alpha} \equiv \inf \left\{ r : P\left(L(\tilde{G}_\omega) \geq r \,\middle|\, \{Y_i, X_i, D_i\}_{i=1}^n \right) \geq 1-\alpha \right\}$$

Comments

- Notice probability is conditional on $\{Y_i, X_i, D_i\}_{i=1}^n$ but not on $\{W_i\}_{i=1}^n$.
- In practice $\tilde{r}_{1-\alpha}$ can be obtained through simulations

Weighted Bootstrap

Assumptions (C)

(i) ω_L and ω_U are strictly positive and continuous on $\mathcal{B}$.

(ii) $\hat{\omega}_L$ and $\hat{\omega}_U$ are uniformly consistent on $\mathcal{B}$.

(iii) W is positive a.s. independent of (Y, X, D).

(iv) W satisfies $E[W] = 1$ and $Var(W) = 1$.

(v) The transformation L is Lipschitz continuous.

(vi) The cdf of $L(G_\omega)$ is strictly increasing and continuous at $r_{1-\alpha}$.

Theorem Under Assumptions (A)-(C), if $\{Y_i, X_i, D_i, W_i\}_{i=1}^n$ are IID, then:

$$\tilde{r}_{1-\alpha} \xrightarrow{p} r_{1-\alpha}$$

Roadmap

Goal: Revisit results of Angrist, Chernozhukov and Fernandez-Val (2006) regarding changes across Decennial Censuses in quantile specific returns to schooling.

- Assess sensitivity of results to deviations from MAR.

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- Assess sensitivity of results to deviations from MAR.

Then... How worried should we be?

- Investigate nature of deviations from MAR in matched CPS-SSA data
- Test for and measure departures from ignorability using KS metric.

Quantile Specific Returns

Like Angrist, Chernozhukov and Fernandez-Val (2006) we estimate:

$$Y_i = X_i'\gamma(\tau) + E_i\beta(\tau) + \epsilon_i \qquad P(\epsilon_i \leq 0|X_i, E_i) = \tau$$

where Y_i is log average weekly earnings, E_i is years of schooling, and X_i consists of intercept, black dummy and quadratic in potential experience.

Sample Restrictions

- 1% Unweighted Extracts of 1980, 1990, 2000 PUMS Samples.
- Black and white men age 40-49 with education ≥ 6 years.
- Y_i treated as missing for all obs with allocated earnings or weeks worked.

Data quality is deteriorating

Table: Fraction of Observations in Estimation Sample with Missing Weekly Earnings

Census Year	Total Number of Observations	Allocated Earnings	Allocated Weeks Worked	Fraction of Total Missing
1980	80,128	12,839	5,278	19.49%
1990	111,070	17,370	11,807	23.09%
2000	131,265	26,540	17,455	27.70%
Total	322,463	56,749	34,540	23.66%

Figure: Worst Case Nonparametric Bounds on 1990 Medians and Linear Model Fits for Two Experience Groups of White Men.

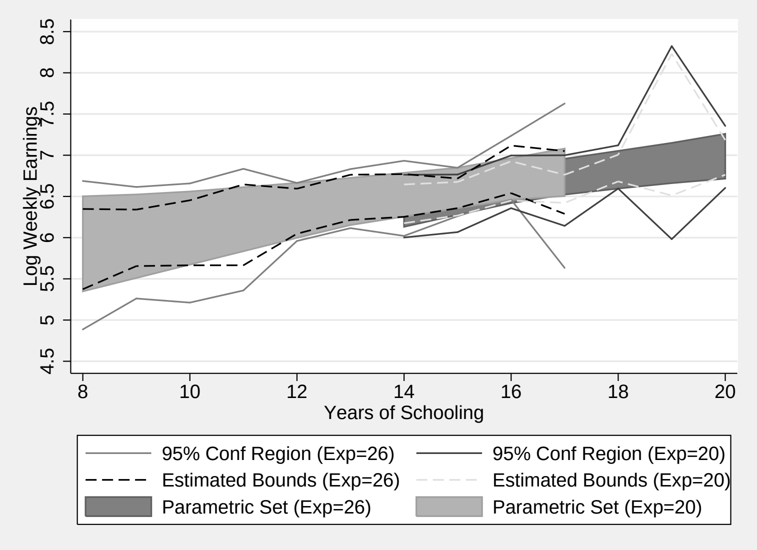


Figure: Nonparametric Bounds on 1990 Medians and Best Linear Approximations for Two Experience Groups of White Men Under $\mathcal{S}(F) \leq 0.05$.

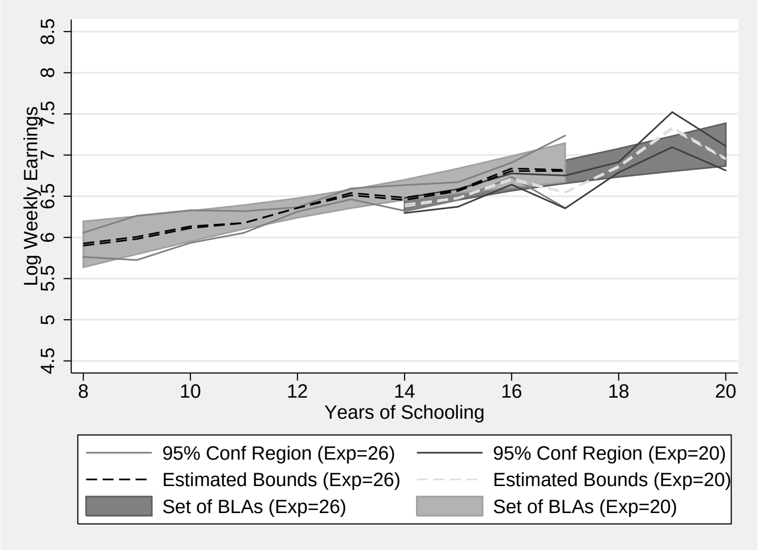


Figure: Uniform Confidence Regions for Schooling Coefficients by Quantile and Year Under Missing at Random Assumption ($\mathcal{S}(F) = 0$).

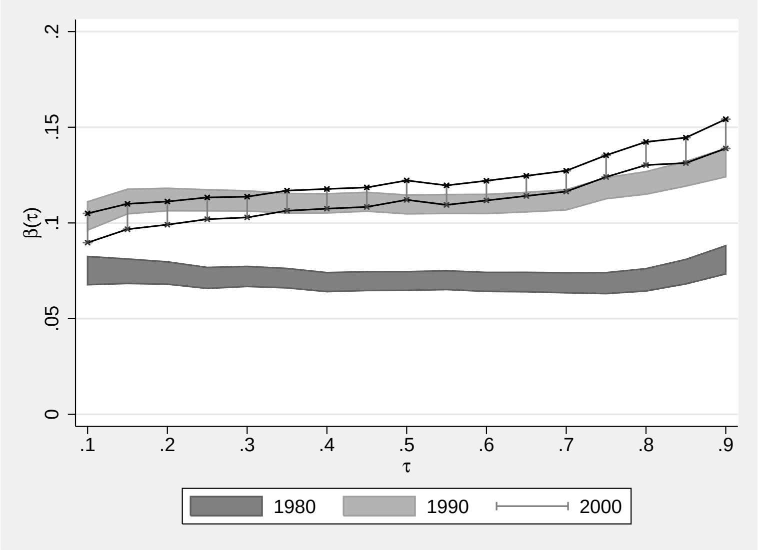


Figure: Uniform Confidence Regions for Schooling Coefficients by Quantile and Year Under $\mathcal{S}(F) \leq 0.05$.

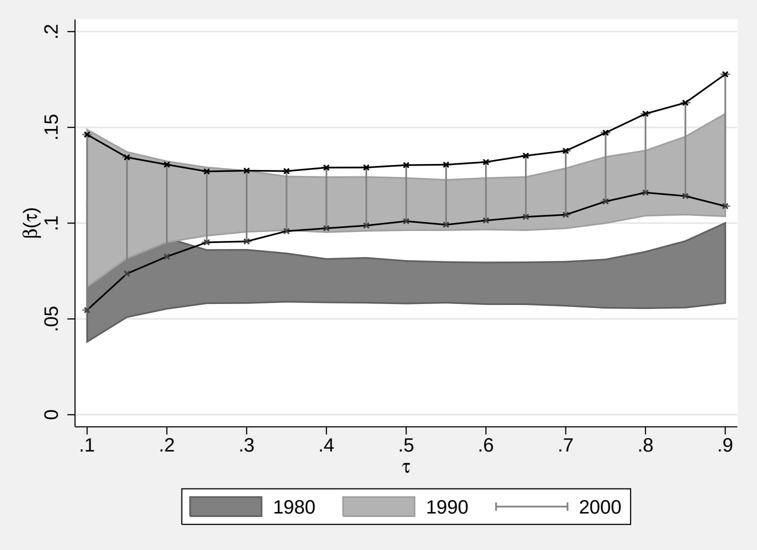


Figure: Uniform Confidence Regions for Schooling Coefficients by Quantile and Year Under $\mathcal{S}(F) \leq 0.175$ (1980 vs. 1990).

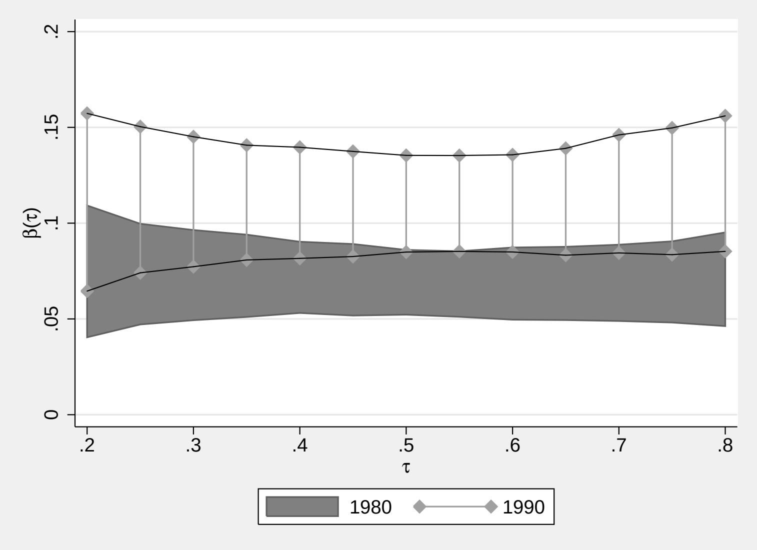
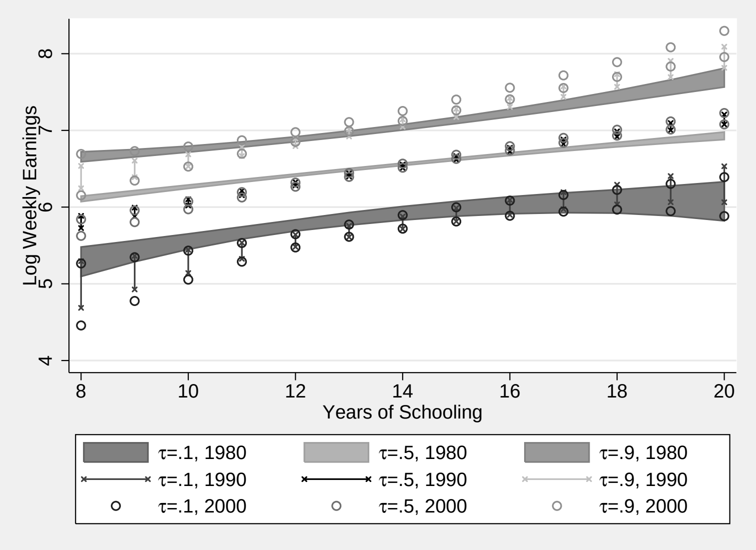


Figure: Confidence Intervals for Fitted Values Under $\mathcal{S}(F) \leq 0.05$.



Distributional Sensitivity

Found critical k at which $\pi_U^{80}(\tau, k) \geq \pi_L^{90}(\tau, k)$ for all τ.

... more informative find a τ specific critical k for each τ.

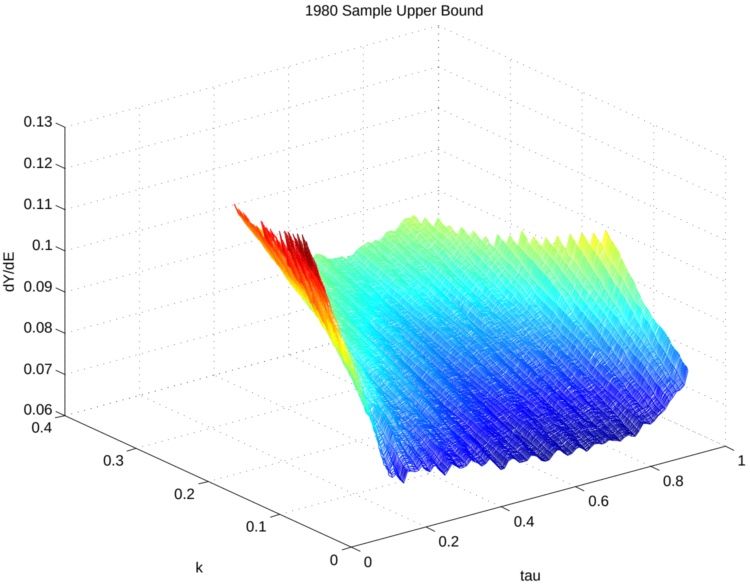
Define τ-"breakdown" point $\kappa_0(\tau)$ as the smallest $k \in [0, 1]$ for which

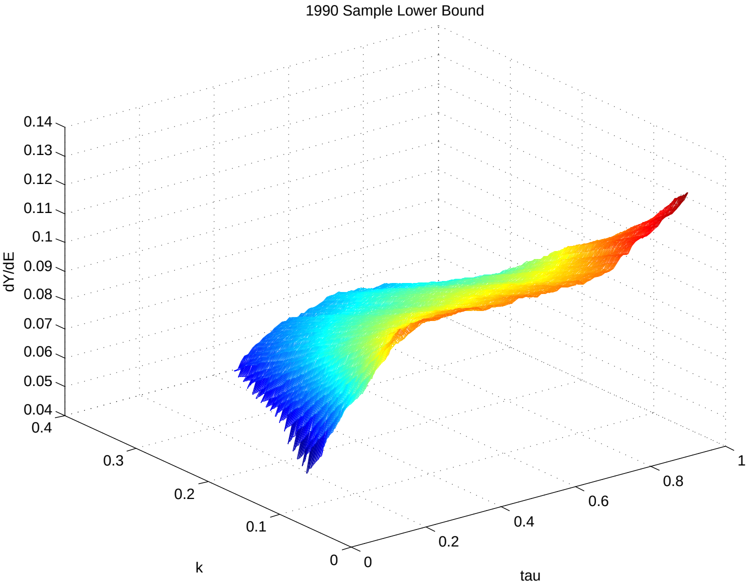
$$\pi_U^{80}(\tau, k) - \pi_L^{90}(\tau, k) \geq 0$$

$\Rightarrow$ pointwise defines a function κ_0 which at each τ gives critical k.

Comments

- κ_0 function summarizes distributional sensitivity to MAR assumption.
- Use (τ, k) uniformity to build confidence interval for $\kappa_0(\tau)$ uniform in τ.


1980 Sample Upper Bound
dY/dE
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.4
0.3
0.2
0.1
0
k
0
0.2
0.4
0.6
0.8
1
tau


1990 Sample Lower Bound
dY/dE
0.14
0.13
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
k
0.4
0.3
0.2
0.1
0
tau
0
0.2
0.4
0.6
0.8
1

Intersection of Sample Bounds

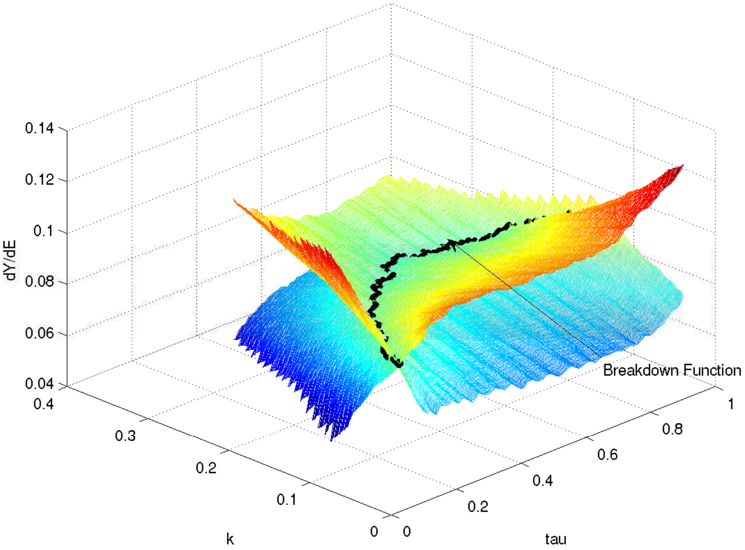
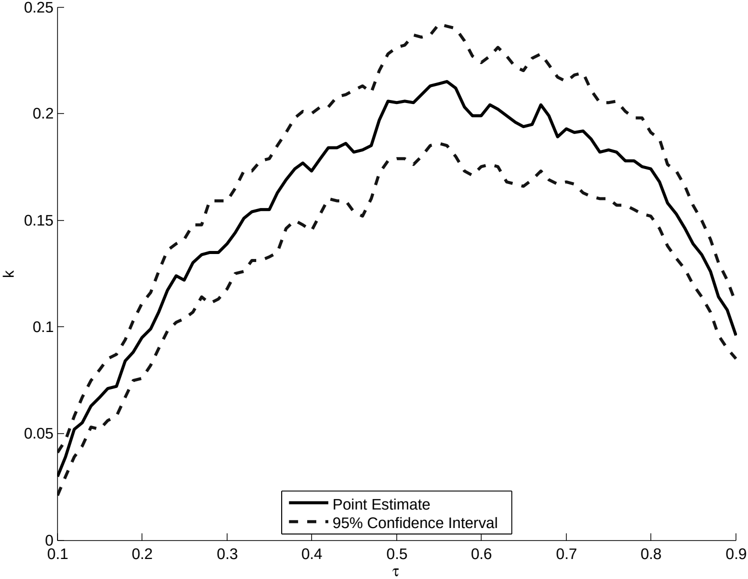


Figure: Breakdown Curve (1980 vs 1990).



How worried should we be?

Goal: Employ 1973 CPS-SSA File to assess $\mathcal{S}(F)$.

Data on SSA and IRS earnings for respondents to March CPS

Sample Restrictions

- Black and white men between ages of 25 and 55.
- More than 6 years of schooling.
- Must have reported working at least one week in past year.
- Drop self-employed and occupations likely to receive tips.
- Drop observations with IRS earnings $\leq \$1000$ or $\geq \$50000$.

Comments

- Roughly 7.2$\%$ of observations have unreported CPS earnings.
- Use IRS rather than SSA earnings due to topcoding.

Assessing $\mathcal{S}(F)$

Define,

$$p_L(x,\tau) \equiv P(D=1|X=x, F_{y|x}(Y) \leq \tau) \qquad p_U(x,\tau) \equiv P(D=1|X=x, F_{y|x}(Y) > \tau)$$

Leads to alternative expression for distance between $F_{y|1,x}$ and $F_{y|0,x}$

$$|F_{y|1,x}(q(\tau|x)) - F_{y|0,x}(q(\tau|x))| = \frac{\sqrt{(p_L(x,\tau) - p(x))(p_U(x,\tau) - p(x))\tau(1-\tau)}}{p(x)(1-p(x))}$$

Comments

- Emphasizes the effect of selection.
- Only need estimate of $P(D=1|X=x, F_{y|x}(Y)=\tau)$.
- Use earnings information on nonrespondents to estimate selection.

Three Logit Models

$$P(D=0|X=x, F_{y|x}(Y)=\tau) = \Lambda(\beta_1\tau + \beta_2\tau^2 + \delta_x) \quad (1)$$

$$P(D=0|X=x, F_{y|x}(Y)=\tau) = \Lambda(\beta_1\tau + \beta_2\tau^2 + \gamma_1\delta_x\tau + \gamma_2\delta_x\tau^2 + \delta_x) \quad (2)$$

$$P(D=0|X=x, F_{y|x}(Y)=\tau) = \Lambda(\beta_{1,x}\tau + \beta_{2,x}\tau^2 + \delta_x) \quad (3)$$

Comments

- Five year age categories, Four schooling $(<12, 12, 13-15, 16)$.
- Drop small cells (<50 obs).
- Only need estimate of $P(D=1|X=x, F_{y|x}(Y)=\tau)$.
- Model (2) substantially increases Likelihood over model (1).
- LR test cannot reject model (2) for model (3).

Table: Logit Estimates of $P(D=0|X=x, F_{y|x}(Y)=\tau)$ in 1973 CPS-IRS Sample

	Model 1	Model 2	Model 3
b_1	-1.06	0.05	
	(0.43)	(5.44)	
b_2	1.09	3.75	
	(0.41)	(4.08)	
γ_1		0.45	
		(2.30)	
γ_2		1.15	
		(1.73)	
Log-Likelihood	-3,802.91	-3798.48	-3759.97
Parameters	37	39	105
Number of observations	15,027	15,027	15,027
Demographic Cells	35	35	35
Ages 25-55			
Min KS Distance	0.02	0.02	0.01
Median KS Distance	0.02	0.05	0.12
Max KS Distance ($\mathcal{S}(F)$)	0.02	0.17	0.67
Ages 40-49			
Min KS Distance	0.02	0.02	0.01
Median KS Distance	0.02	0.05	0.08
Max KS Distance ($\mathcal{S}(F)$)	0.02	0.09	0.39

Note: Asymptotic standard errors in parentheses.

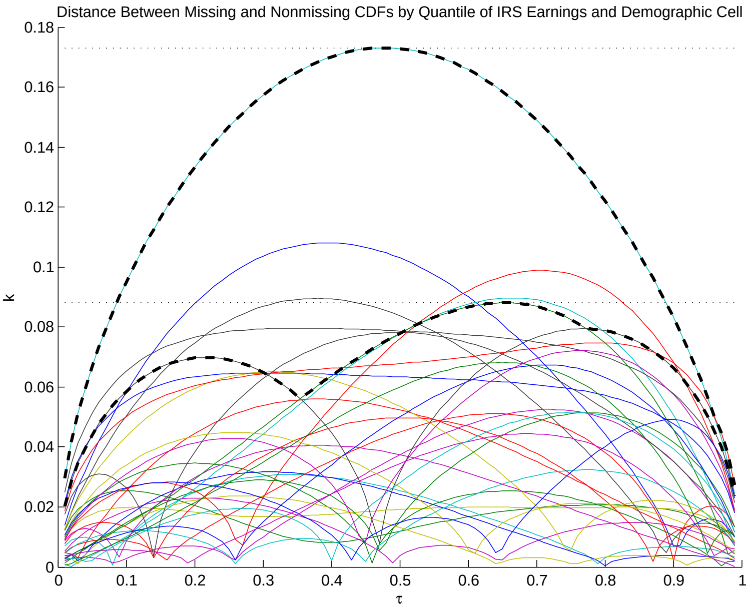
Comments

MAR clearly violated

- Very high and very low earning individuals mostly likely to have missing earnings on average.
- But missingness pattern appears to be heterogenous across demographic cells.
- Difficult to have guessed pattern a priori.

Degree of Heterogeneity Affects Bottom Line

- Model 1: $\mathcal{S}(F) = 0.02$
- Model 2: $\mathcal{S}(F) = 0.09$
- Model 3: $\mathcal{S}(F) = 0.39$


Distance Between Missing and Nonmissing CDFs by Quantile of IRS Earnings and Demographic Cell
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
k
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
τ

Conclusion

Theory: When data are poor, useful to check sensitivity to violations of MAR.

- KS provides natural metric for assessing violations of MAR.
- Methods developed here enable study of parametric approximating models.
- And allow for assessment of distributional sensitivity to MAR assumption.

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Empirics: Reexamine the quantile specific returns to education.

- Measured changes in wage structure between 1980-1990 fairly robust (except at low end of distribution).
- But changes over 1990-2000 easily confounded by a bit of selection and deterioration in quality of Census data.
- 1973 CPS-SSA file provides evidence of selection and heterogeneity.