1. Lorenz Curve and Gini Coefficient

a) See Figure at the end.

b) The results using the linear interpolation are:

<table>
<thead>
<tr>
<th>Year</th>
<th>Top 10% Share</th>
<th>Top 1% Share</th>
<th>90/50 Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>44%</td>
<td>18%</td>
<td>3.3</td>
</tr>
<tr>
<td>2017</td>
<td>45%</td>
<td>21%</td>
<td>4.2</td>
</tr>
</tbody>
</table>

c) The results using the Pareto interpolation are:

<table>
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Elaborate explanation on Pareto interpolations

We will use a Pareto interpolation following the method discussed in section 3 in Atkinson et al. (2011) and section 5.1 in Atkinson (2005). We will begin by describing the Pareto distribution and some of its useful properties, and then discuss the Pareto interpolation procedure.

The Pareto cumulative distribution function $F(y)$ for $y$ is,

$$F(y) = 1 - \left( \frac{k}{y} \right)^\alpha, \quad \alpha > 1, \quad k > 0$$

$$\Rightarrow f(y) = \frac{\alpha k^\alpha}{y^{1+\alpha}}$$

where $k$ and $\alpha$ are given parameters, $\alpha$ is referred to as the Pareto parameter. The $\alpha$ parameter determines the thickness of the tail of the distribution. The key property of the Pareto distribution is that the ratio of average income $y^*(y)$ of individ-
uals above \( y \) to \( y \) does not depend on the income threshold of \( y \) 

\[
y^*(y) = \frac{\int_{z>y} zf(z)dz}{\int_{z>y} f(z)dz} = \frac{\int_{z>y} z\alpha k^\alpha / z^{1+\alpha}dz}{\int_{z>y} \alpha k^\alpha / z^{1+\alpha}dz} = \frac{\int_{z>y} 1/z^\alpha dz}{\int_{z>y} 1/z^{1+\alpha}dz} \\
= \frac{-1}{alpha(z\alpha-1)}|_y^\infty = \frac{0 - \frac{1}{(\alpha-1)y^\alpha-1}}{0 - \frac{1}{\alpha y^\alpha}} = y \cdot \alpha/(\alpha - 1)
\]

\( \Rightarrow \frac{y^*(y)}{y} = \beta, \ \text{with} \ \beta = \alpha/(\alpha - 1) \)

The parameter \( \beta \) has a more intuitive interpretation than \( \alpha \), if \( \beta = 2 \) the average income of individuals with income above $100,000 is $200,000. A higher \( \beta \) means a flatter upper tail of the income distribution. We will use Atkinson (2005) notation. Denote by \( H(y) \) the share of individual with income higher than \( y \),

\[
H(y) = 1 - F(y) = \left(\frac{k}{y}\right)^\alpha = Ay^{-\alpha}, \ \text{with} \ A = k^\alpha
\]

Denote by \( G(y) \) the total income of received by individuals with income \( y \) or higher. According to the properties of the Pareto distribution,

\[
G(y) = \frac{\alpha}{\alpha - 1} Ay^{-(\alpha - 1)}
\]

The share in total income of people with incomes \( y \) or higher can be obtained by dividing \( G(y) \) by \( \mu \), where \( \mu \) is the mean income, denote by \( \Omega(y) = \frac{G(y)}{\mu} \).

Let \( y_p \) the desired income percentile. There are to kinds of interpolations:

1. When we observe tabulations of incomes shares lower and higher then \( y_p \), in this case we can use interpolation between the two adjacent tabulations that \( y_p \) is in the middle of them. For example we observe the total income of the top 3.677% and the top 0.7688%, however we do not observe the total income of the top 1%.

2. The second scenario is when we want to estimate the share in total income of a percentile we observe only percentiles that are higher than it. For example the smallest percentile we observe is 0.0123%, hence if we want to estimate the share in total income of the top 0.001% we will have only tabulations from one side. This is referred as open range tabulation. In this exercise all the tabulations we are asked to perform are from the first type.

\footnote{Using the notation from Atkinson et al. (2011)}
Pareto interpolations

I will begin by interpolating the share in total income of the top 1%. Denote by $G_1, G_2,$ and $G^*$ the share in total of the top $0.768\%, 3.677\%$ and $1\%$ respectively. Denote by $H_1$ and $H_2$ the $H(y)$ of the two tabulations to the left and right of $H(y) = 1$ (e.g., $H_1 = 0.00768$ and $H_2 = 0.03677$).

It follows from the Pareto distribution that

$$\frac{\alpha}{\alpha - 1} = \frac{\log (H_1/H_2)}{\log (G_1/G_2)}$$

(2)

Therefore using equation (2) we can estimate $\frac{\alpha}{\alpha - 1}$ for the top 1 percent. Note, that this is making a Pareto interpolation between the two tabulations closest to the desired quantity.

$$\beta = \frac{\log (H_1/H_2)}{\log (G_1/G_2)}$$

$$\Rightarrow \alpha = \frac{\beta}{\beta - 1}$$

Next we use $G_1$ and $H_1$ to approximate the total earnings of the top 1% (it does not matter if we choose $H_2$ and $G_2$, the result would have been the same).

$$\beta \approx \frac{\log (H_1/0.01)}{\log (G_2/G^*)} \Rightarrow G^* = \log \left( \frac{G_2}{G^*} \right) = \frac{1}{\beta} \cdot \log (H_1/0.01)$$

$$\Rightarrow G_2/G^* = \exp \left( \frac{1}{\beta} \cdot \log (H_1/0.01) \right) = (H_1/0.01)^{\frac{1}{\beta}}$$

$$G^* = G_2 \cdot (H_1/0.01)^{\frac{1}{\beta}}$$

Next we use the same procedure as before to estimate $\alpha$ for each percentile, and then use the CDF of the Pareto distribution. Denote by $p$ the percentile of interest and by $y_p$ income of the individual at the $p$ percentile. Given a tabulation we can calculate $\alpha$. Next we extrapolate the parameter $k$ (or $A$ in Atkinson’s notation) for each of the tabulations above and below the desired $p$,

$$k = (1 - F(y))^{1/\alpha} \cdot y$$

Proof: Let $y_1$ and $y_2$ be two income levels such that $y_1 > y_2$ and therefore $H(y_1) < H(y_2)$.

$$\frac{\log (H_1/H_2)}{\log (G_1/G_2)} = \frac{\log \left( \frac{A y_1^{-\alpha} / A y_2^{-\alpha} }{y_1 / y_2} \right) }{\log \left( \frac{A y_1^{-\alpha} / A y_2^{-\alpha} }{y_1 / y_2} \right) } = \frac{\log \left( \frac{(y_1/y_2)^{-\alpha} }{(y_1/y_2)^{-\alpha-1} } \right) }{\log \left( \frac{(y_1/y_2)^{-\alpha-1} }{(y_1/y_2)^{-\alpha} } \right) }$$

$$= -\alpha \log \left( \frac{(y_1/y_2)}{(y_1/y_2)} \right) - (\alpha - 1) \log \left( \frac{(y_1/y_2)}{(y_1/y_2)} \right) = \frac{\alpha}{\alpha - 1} \square$$
After we have $k$ and $\alpha$ the desired income threshold, $y_p$ is,

$$y_p = \frac{k}{(1 - F(y_p))^{1/\alpha}}$$  \hspace{1cm} (3)

We need to decide which of the tabulations to use in order to calculate $k$. Piketty 2001, Appendix B, (Page 598 in the book) suggest to choose the tabulation that is closest to the desired percentile. It is clear that the choice of which tabulation to use has a significant impact on the results.


d) See second figure at the end of the solutions for the Lorenz curve.

<table>
<thead>
<tr>
<th></th>
<th>2004</th>
<th>2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top 10 percent share</td>
<td>41%</td>
<td>42%</td>
</tr>
<tr>
<td>Top 1 percent share</td>
<td>16%</td>
<td>18%</td>
</tr>
<tr>
<td>90/50 ratio</td>
<td>3.3</td>
<td>4.2</td>
</tr>
</tbody>
</table>

e) Mis-ranking always under-estimates the Gini coefficient (as the Lorenz curve is furthest away from the diagonal precisely when individuals are perfectly ranked by income).

2. Optimal Commodity Taxation with Linear Income Tax

This is the Atkinson-Stiglitz theorem in the case of linear income taxation.

a) 

$$V(y, q) = \max_c v(c) \text{ st } q \cdot c \leq y$$

$$V(\lambda y, q) = \max_c v(c) \text{ st } q \cdot c \leq \lambda y$$

Denoting $c' = c/\lambda$,

$$V(\lambda y, q) = \max_{c'} v(\lambda c') \text{ st } q \cdot c' \leq y,$$

We have $v(c) = \lambda v(c')$, hence

$$V(\lambda y, q) = \lambda \max_{c'} v(c') \text{ st } q \cdot c' \leq y,$$

Hence $V(\lambda y, q) = \lambda V(y, q)$ is linear in $y$ and hence takes the form $\phi(q) \cdot y$ with $\phi(q) = V(1, q)$. Similarly, the solution is scaled by a factor $\lambda$ so that $c_k(y, q)$ is also linear in $y$ for each $k$.

b) This follows exactly the proof of the Atkinson-Stiglitz theorem seen in class. The same proof carries over if any tax system $(\tau, R, t)$ can be replaced by a pure income tax $(\bar{\tau}, \bar{R}, t = 0)$ such that $V((1-\tau)z+\bar{R}, p+t) = V((1-\bar{\tau})z+\bar{R}, p)$ for all $z$. This is possible if $V(y, q)$ takes the linear form $\phi(q) \cdot y$. You need to pick $\bar{\tau}$ and $\bar{R}$ so that $\phi(p+t) \cdot [(1-\tau)z+\bar{R}] = \phi(p) \cdot [(1-\bar{\tau})z+\bar{R}]$. I.e., $1 - \bar{\tau} = (1-\tau)\phi(p+t)/\phi(p)$ and $\bar{R} = R\phi(p+t)/\phi(p)$. 

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In that case each individual has the same utility under the two systems and pick the same $z$ under the two systems. Exactly as the Atkinson-Stiglitz theorem proof seen in class, you can show that the new system raises as much revenue as the old one.

c) Intuitively, in the linear tax case, unless Engel curves are linear, commodity taxation can be useful to “non-linearize” the tax system. Adding nonlinearity through commodity taxes is then in general desirable (note that this is less efficient that doing it directly through the nonlinear income per the standard Atkinson-Stiglitz theorem).
Figures

2017 Data

Gini coefficient: 0.5957

Cumulative share of returns from lowest to highest AGI
(Only positive AGI included)

2004 Data

Gini coefficient: 0.5711

Cumulative share of returns from lowest to highest AGI
(Only positive AGI included)

2017 Data – After tax

Gini coefficient: 0.562

Cumulative share of returns from lowest to highest AGI
(Only positive AGI included)

2004 Data – After tax

Gini coefficient: 0.5414

Cumulative share of returns from lowest to highest AGI
(Only positive AGI included)
References
