Problem Set 1

1. Optimal Commodity Taxation with Linear Income Tax

This is the Atkinson-Stiglitz theorem in the case of linear income taxation.

a) 

\[ V(y, q) = \max_c v(c) \text{ st } q \cdot c \leq y \]

\[ V(\lambda y, q) = \max_c v(c) \text{ st } q \cdot c \leq \lambda y \]

Denoting \( c' = c/\lambda \),

\[ V(\lambda y, q) = \max_{c'} v(\lambda c') \text{ st } q \cdot c' \leq y, \]

We have \( v(c) = \lambda v(c') \), hence

\[ V(\lambda y, q) = \lambda \max_{c'} v(c') \text{ st } q \cdot c' \leq y, \]

Hence \( V(\lambda y, q) = \lambda V(y, q) \) is linear in \( y \) and hence takes the form \( \phi(q) \cdot y \) with \( \phi(q) = V(1, q) \).

Similarly, the solution is scaled by a factor \( \lambda \) so that \( c_k(y, q) \) is also linear in \( y \) for each \( k \).

b) This follows exactly the proof of the Atkinson-Stiglitz theorem seen in class. The same proof carries over if any tax system \((\tau, R, t)\) can be replaced by a pure income tax \((\bar{\tau}, \bar{R}, t = 0)\) such that \( V((1-\tau)z + \bar{R}, p + t) = V((1-\bar{\tau})z + \bar{R}, p) \) for all \( z \). This is possible if \( V(y, q) \) takes the linear form \( \phi(q) \cdot y \). You need to pick \( \bar{\tau} \) and \( \bar{R} \) so that \( \phi(p + t) \cdot [(1-\tau)z + \bar{R}] = \phi(p) \cdot [(1-\bar{\tau})z + \bar{R}] \).

I.e., \( 1 - \bar{\tau} = (1 - \tau)\phi(p + t)/\phi(p) \) and \( \bar{R} = R\phi(p + t)/\phi(p) \).

In that case each individual has the same utility under the two systems and pick the same \( z \) under the two systems. Exactly as the Atkinson-Stiglitz theorem proof seen in class, you can show that the new system raises as much revenue as the old one.

c) Intuitively, in the linear tax case, unless Engel curves are linear, commodity taxation can be useful to “non-linearize” the tax system. Adding nonlinearity through commodity taxes is then in general desirable (note that this is less efficient that doing it directly through the nonlinear income per the standard Atkinson-Stiglitz theorem).
3. Optimal Top Income Tax Rate

This exercise follows Saez Restud’01 section 3 (see the paper for more details).

a) In the tax reform graph drawn in class, increasing $\tau$ by $d\tau$ creates a negative substitution effect (less slope) leading to less work, and a negative income effect leading to more work. Hence, $e$ is a mix of substitution and income effects.

b) Let $z^i(1 - \tau, R)$ be the earnings supply function obtained from solving the individual utility maximization problem under a linear tax:

$$\max_{c,z} u^i(c,z) \text{ st } c = (1 - \tau)z + R$$

We denote by $e^i_u$ the uncompensated elasticity of $z^i$ with respect to $1 - \tau$ and by $\eta^i = (1 - \tau)\partial z^i / \partial R$ the income effect parameter. As a simple graph shows (see Saez Restud’01 Section 3), the reform changes $1 - \tau$ by $-d\tau$ and changes $R$ by $dR = z^*d\tau$. Note, that the virtual income is defined as $R \equiv z^*\tau$ (in Saez 2001 it is written as $R \equiv \bar{z}\tau$), and as we assume that $z^*$ is fixed it follows immediately that $dR = z^*d\tau$. Hence, we have:

$$dz^i = -d\tau \frac{\partial z^i}{\partial (1 - \tau)} + z^*d\tau \frac{\partial z^i}{\partial R} = -\frac{d\tau}{1 - \tau} z^i \frac{1 - \tau}{z^i} \frac{\partial z^i}{\partial (1 - \tau)} + \frac{d\tau}{1 - \tau} z^*(1 - \tau) \frac{\partial z^i}{\partial R} = -\frac{d\tau}{1 - \tau} z^i e^i_u + \frac{d\tau}{1 - \tau} z^* \eta^i$$

c) Recall that the elasticity $e$ is defined as

$$e = \frac{1 - \tau}{\sum z^i d(1 - \tau)} \sum dz^i$$

Hence we have

$$e = \frac{1 - \tau}{\sum z^i} \left[ \frac{1}{1 - \tau} z^i e^i_u - \frac{1}{1 - \tau} z^* \eta^i \right] = \frac{\sum z^i e^i_u}{\sum z^i} - \frac{z^*}{z^m} \frac{\sum \eta^i}{N}$$

with $N$ number of top bracket taxpayers. Hence $e = \bar{e}_u - \frac{a - 1}{a} \bar{\eta}$ with $\bar{e}_u$ the income weighted average of $e^i_u$ and $\bar{\eta}$ the straight average of $\eta^i$ among top bracket taxpayers.

The uncompensated elasticity is income weighted because those with higher income should count more in the response. In contrast, the income effect parameter is not an elasticity and hence should not be income weighted.

d) $$\max_{c,z} c - a^i \cdot z^{1 + 1/\varepsilon} \text{ st } c = (1 - \tau)z + R$$

$$\Rightarrow \max_{z} (1 - \tau)z + R - a^i \cdot z^{1 + 1/\varepsilon}$$

$$\Rightarrow 1 - \tau = a^i (1 + 1/\varepsilon) \cdot z^{1/\varepsilon}$$

$$\Rightarrow z^i(1 - \tau, R) = \text{constant} \cdot (1 - \tau)^{\varepsilon}$$

$z^i(1 - \tau, R)$ depends only on $1 - \tau$ and not $R$. This is the classic case with no income effects studied in Diamond AER1998.
\[ \Rightarrow e^i_u = \varepsilon, \eta^i = 0, e^i_c = \varepsilon \]

c) implies that \( e = \varepsilon \) and hence the optimal top tax rate is \( \tau = 1/(1 + a \cdot \varepsilon) \).

e)-f) Suppose, each individual utility takes the form \( u^i(c, z) = \log c - a^i \cdot z^{1+1/\varepsilon} \) with \( a^i \) is a parameter (that can vary across individuals). The individual solves \( \max_z u^i((1 - \tau)z + R, z) = \log((1 - \tau)z + R) - a^i \cdot z^{1+1/\varepsilon} \)

The individual FOC in \( z \) is:

\[ \frac{1 - \tau}{(1 - \tau)z + R} = a^i(1 + 1/\varepsilon)z^{\frac{1}{\varepsilon}} \]

or

\[ \log(1 - \tau) - \log((1 - \tau)z + R) = \frac{1}{\varepsilon} \log(z) + \log[a^i(1 + 1/\varepsilon)] \]

This defines implicitly \( z^i(1 - \tau, R) \) with the following comparative statics. A small change \( dR \) leads to \( dz^i \) such that:

\[ dz^i \left[ \frac{1}{\varepsilon z} + \frac{1 - \tau}{(1 - \tau)z + R} \right] = -\frac{dR}{(1 - \tau)z + R} \]

Hence

\[ \eta^i = (1 - \tau) \frac{dz^i}{dR} = -\frac{1}{1 + \frac{z + R/(1 - \tau)}{\varepsilon z}} \rightarrow_{z \to \infty} -\frac{\varepsilon}{\varepsilon + 1} \]

A small change \( d(1 - \tau) \) leads to \( dz^i \) such that:

\[ dz^i \left[ \frac{1}{\varepsilon z} + \frac{1 - \tau}{(1 - \tau)z + R} \right] = \frac{d(1 - \tau)}{1 - \tau} - \frac{zd(1 - \tau)}{(1 - \tau)z + R} \]

Hence

\[ (1 - \tau) \frac{dz^i}{d(1 - \tau)} \left[ \frac{1}{\varepsilon z} + \frac{1}{z + R/(1 - \tau)} \right] = 1 - \frac{z}{z + R/(1 - \tau)} \]

\[ e^i_u = \frac{1 - \tau}{z} \frac{dz^i}{d(1 - \tau)} = \frac{1 - \frac{z + R/(1 - \tau)}{\varepsilon z}}{z + R/(1 - \tau)} \rightarrow_{z \to \infty} 0 \]

f) Use c) to obtain an optimal top tax rate formula as a function of \( \varepsilon \) and the Pareto parameter \( a \) in that case. We have \( e = \bar{e}_a = \frac{a - 1}{a} \bar{\eta} \approx \frac{a - 1}{a} \frac{\varepsilon}{\varepsilon + 1} \) and hence:

\[ \tau = \frac{1}{1 + ae} = \frac{1}{1 + (a - 1)\frac{\varepsilon}{\varepsilon + 1}} = \frac{\varepsilon + 1}{1 + a\varepsilon} \]

If \( a = 1.5 \) and \( \varepsilon = 1 \), we have \( \tau = 1/(1 + .5 \cdot .5) = 80\% \).

With \( \varepsilon = \infty \), we get \( \tau = 1/(1 + .5) = 66.6\% \).

This utility specification has zero uncompensated elasticity and hence large income effects when the compensated elasticity is large. As a result, the tax rate is substantial, even with a very large Frisch elasticity.