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ABSTRACT

This paper extends the New Keynesian model by introducing wealth, in the form of government bonds, into the utility function. The extension modifies the Euler equation: in steady state the real interest rate is negatively related to consumption instead of being constant, equal to the time discount rate. Thus, when the marginal utility of wealth is large enough, the dynamical system representing the equilibrium is a source not only in normal times but also at the zero lower bound. This property eliminates the zero-lower-bound anomalies of the New Keynesian model, such as explosive output and inflation, and forward-guidance puzzle.

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1. Introduction

A current issue in monetary economics is that the textbook New Keynesian model does not behave well at the zero lower bound on nominal interest rates (ZLB). While the topic of the ZLB was once exotic, discussed only in the context of the Great Depression and Japan, it has become prominent in the aftermath of the Great Recession.

In the textbook model, the equilibrium is represented by a dynamical system composed of two equations: the Phillips curve, describing producers' optimal pricing, and the Euler equation, describing consumers' optimal saving. The dynamical system is a saddle at the ZLB, which generates several anomalies: output and inflation follow explosive paths when the ZLB is long-lasting (Cochrane 2017); forward-guidance policies have implausible effects (Del Negro, Giannoni, and Patterson 2012); and so on.

This paper proposes to remedy this issue by introducing wealth, in the form of government bonds, into the utility function. This extension modifies the Euler equation: in steady state the real interest rate is negatively related to consumption instead of being constant, equal to the time discount rate. Intuitively, since people save not only for future consumption but also because they enjoy holding wealth, in steady state they accept to save at a real rate below the discount rate; and they accommodate a lower real rate by increasing consumption relative to wealth. Then, with sufficient marginal utility of wealth, the Euler equation is sufficiently altered that the dynamical system representing the ZLB equilibrium becomes a source instead of a saddle. At that point all the New Keynesian anomalies at the ZLB disappear. This is because when the system is a source, the response to any temporary shock is a muted version of the response to a permanent shock, which is itself finite.

Although the assumption of wealth in the utility function is nonstandard in modern economics, Adam Smith, Ricardo, J.S. Mill, Marshall, Veblen, Keynes, and Irving Fisher all recognized that people accumulate wealth for many reasons beside future consumption: because it brings social status; because it brings political influence; and because people derive satisfaction from the process of wealth accumulation (Steedman 1981). Their views are also aligned with Weber's thesis in *The Protestant Ethic and the Spirit of Capitalism* and with modern neuroscientific evidence, which shows that people derive a different pleasure from wealth than from present or future consumption (Camerer, Loewenstein, and Prelec 2005, pp. 35–37). In fact an emerging literature in economics has been using the assumption to revisit various topics: long-run growth (Kurz 1968), long-run stagnation (Ono 2015; Michau 2018), unemployment fluctuations (Michaillat and Saez 2014), financial crises (Kumhof, Ranciere, and Winant 2015), life-cycle saving (Carroll 2000), asset pricing (Bakshi and Chen 1996), and capital taxation (Saez and Stantcheva 2018).

In our model wealth takes the form of government bonds, so preferences for government bonds could also justify our assumption. These preferences capture in reduced form the special features of bonds relative to other assets, such as safety and liquidity (Poterba and Rotemberg 1987; Krishnamurthy and Vissing-Jorgensen 2012). Moreover, macroeconomic models with bonds in the utility function describe the data well: they resolve asset-pricing puzzles (Krishnamurthy and Vissing-Jorgensen 2012), explain short-run fluctuations (Fisher 2015), and generate consumption responses to income shocks that are nearly as accurate as rich heterogeneous-agent models (Auclert, Rognlie, and Straub 2018, p. 19).

Before us, Eggertsson and Mehrotra (2014), Gabaix (2016), Diba and Loisel (2017), and Cochrane (2018) have developed variants of the New Keynesian model that cure all its ZLB anomalies. Eggertsson and Mehrotra modify the Euler equation with overlapping generations and the Phillips curve with downward nominal wage rigidity. Gabaix modifies the Euler equation and Phillips curve with boundedly rational households and firms. Diba and Loisel modify the Euler equation with a spread between interest rates on bonds and on bank reserves, and the Phillips curve with a banking cost. Finally, Cochrane maintains Euler equation and Phillips curve but inserts a third equation: a fiscal theory of the price level. These strategies and ours have a commonality: they result in determinacy of the ZLB equilibrium. Our finding is that it is sufficient to modify the Euler equation to address all ZLB anomalies, and that this can be done by deviating only minimally from the textbook model.

Finally, our paper goes in the same direction as McKay, Nakamura, and Steinsson (2016) and Bilbiie (2017). They develop discounted Euler equations by introducing heterogeneous agents facing borrowing constraints in the New Keynesian model, and they show that discounting mitigates the forward-guidance puzzle. By introducing wealth in the utility function we also generate a discounted Euler equation; but going one step further, we find that sufficient discounting transforms the dynamical system representing the ZLB equilibrium from a saddle to a source, resolving at once all anomalies at the ZLB.

2. The Model

We start from the textbook New Keynesian model of Gali (2008, chap. 3), slightly modified as in Benhabib, Schmitt-Grohe, and Uribe (2001) to improve tractability. The three modifications are: continuous time instead of discrete time; self-employed households instead of separate firms and households; and Rotemberg (1982) pricing instead of Calvo (1983) pricing. We extend this model by assuming that households also derive utility from holding real wealth.

2.1. Assumptions

The economy is composed of a measure 1 of self-employed households. Each household $j \in [0, 1]$ produces $y_j(t)$ units of a differentiated good j, sold to other households at a price $p_j(t)$. Household j's production function is

$$y_j(t) = ah_j(t), \tag{1}$$

where the parameter $a > 0$ represents the level of technology, and the variable $h_j(t)$ is hours of work. Household j incurs a disutility $h_j(t)$ from working.

The goods produced by households are imperfect substitutes for one another, so each household exercises some monopoly power. Moreover, households face a quadratic cost when they change their price: if household j changes its price at a rate $\pi_j(t) = \dot{p}_j(t)/p_j(t)$, it incurs a disutility

$$\frac{\gamma}{2}\pi_j(t)^2.$$

The parameter $\gamma > 0$ governs the cost to change prices and thus price rigidity.

Each household consumes goods produced by other households. Household j buys amounts $c_{jk}(t)$ of the different goods $k \in [0, 1]$, which are aggregated into a consumption index

$$c_j(t) = \left[\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} dk\right]^{\epsilon/(\epsilon-1)}, \tag{2}$$

where $\epsilon > 1$ is the elasticity of substitution between goods. The consumption index yields utility $\ln(c_j(t))$ to the household. The price index is

$$p(t) = \left[\int_0^1 p_j(t)^{1-\epsilon} di\right]^{1/(1-\epsilon)}; \tag{3}$$

when households optimally allocate their consumption expenditure across goods, $p(t)$ is the price of one unit of consumption index. The inflation rate is $\pi(t) = \dot{p}(t)/p(t)$.

Each household j also holds a nominal quantity of bonds $b_j(t)$. Bonds are used to smooth consumption; in addition, they provide utility

$$u\left(\frac{b_j(t)}{p(t)}\right),$$

where the function u is increasing and concave. The law of motion of bond holdings is

$$\dot{b}_j(t) = i(t)b_j(t) + p_j(t)y_j(t) - \int_0^1 p_k(t)c_{jk}(t)dk, \tag{4}$$

where $i(t)$ is the nominal interest rate, set by the central bank. The term $i(t)b_j(t)$ is interest income, $p_j(t)y_j(t)$ is labor income, and $\int_0^1 p_k(t)c_{jk}(t)dk$ is consumption expenditure.

Then, taking as given aggregate variables, initial wealth $b_j(0)$, and initial price $p_j(0)$, each household j chooses time paths for $y_j(t)$, $p_j(t)$, $h_j(t)$, $\pi_j(t)$, $c_{jk}(t)$ for all $k \in [0,1]$, and $b_j(t)$ to maximize the discounted sum of instantaneous utilities

$$\int_0^\infty e^{-\delta t}\left[\ln(c_j(t)) + u\left(\frac{b_j(t)}{p(t)}\right) - h_j(t) - \frac{\gamma}{2}\pi_j(t)^2\right]dt, \tag{5}$$

where $\delta > 0$ is the time discount rate. The household faces four constraints: the production function (1); the budget constraint (4); the law of motion $\dot{p}_j(t) = \pi_j(t)p_j(t)$; and the demand for good j coming from other households' maximization:

$$y_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t), \tag{6}$$

where $c(t) = \int_0^1 c_k(t)dk$ is aggregate consumption. The household also faces a borrowing constraint preventing Ponzi schemes. Finally, we assume that all households face the same initial conditions, so they all behave the same.

2.2. Equilibrium

The equilibrium is described by two differential equations: a Phillips curve and an Euler equation. These two equations determine two variables: inflation π and aggregate output y. (The derivation of the equations is standard and relegated to appendix A.)

The Phillips curve arises from households' optimal pricing decisions:

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon}{\gamma a}\left[y(t) - y^n\right], \tag{7}$$

where

$$y^n = a\,\frac{\epsilon - 1}{\epsilon} \tag{8}$$

is the natural level of output, prevailing when prices are flexible ($\gamma = 0$). The natural level of output is the level of output maximizing profits in the absence of price-adjustment cost: at that level the relative price charged by any firm (1 in equilibrium) features the desired markup ($\epsilon/(\epsilon-1)$) over the real marginal cost—which is the marginal rate of substitution between labor and consumption ($1/(1/y) = y$ in equilibrium) divided by the marginal product of labor (a). The steady-state Phillips curve, obtained by setting $\dot{\pi} = 0$ in (7), describes inflation as a linearly increasing function of output:

$$\pi = \frac{\epsilon}{\delta\gamma a}\left(y - y^n\right). \tag{9}$$

Inflation is zero in (9) when output is at its natural level.

The Euler equation arises from households' optimal saving decisions:

$$\frac{\dot{y}(t)}{y(t)} = r(\pi(t)) + u'(0)\, y(t) - \delta, \tag{10}$$

where $r(\pi)$ is a linear function of inflation describing the real interest rate set by the central bank, and the marginal utility of wealth, u', is evaluated at zero because government bonds are in zero net supply. The steady-state Euler equation, obtained by setting $\dot{y} = 0$ in (10), describes inflation as a linear function of output:

$$r(\pi) = \delta - u'(0)\, y. \tag{11}$$

Through this Euler equation, output is a decreasing function of the real interest rate—as in the old-fashioned Keynesian IS curve. Output is at its natural level in (11) when the real interest rate equals the natural rate of interest

$$r^n = \delta - u'(0) y^n. \tag{12}$$

All the properties of the model arise from the Phillips curve and Euler equation, so we review their economic interpretations here. The Phillips curve is conventional. Since households discount the future, they tend to start with low inflation or deflation and raise it over time in order to postpone the utility cost of changing prices. This explains the term $\delta\pi$ in (7). But households may start with high inflation early and raise it less over time if a high price early is more profitable. This happens when the household produces too much: $y > y^n$. This explains the term $-\epsilon(y - y^n)/(\gamma a)$ in (7). When these two forces balance out, inflation is constant, given by the steady-state Phillips curve: then inflation is positive whenever output is above its natural level. Intuitively, when inflation is positive, reducing inflation lowers the price-adjustment cost; since pricing is optimal, there must also be a cost to reducing inflation and thus increasing production; therefore, output must be above its natural level, indicating already excessive production.

Next, the Euler equation is unconventional because wealth enters the utility function. In the textbook model, consumption is governed by the cost of delaying consumption, given by the time discount rate δ, and the return on saving, given by the real interest rate r. With wealth in the utility function, the financial return on saving is supplemented by an hedonic return on saving, given by the marginal rate of substitution between real wealth and consumption $u'(0)y$; thus the total return on saving is $r + u'(0)y$. Following the usual logic, people front-load saving and increase consumption over time when the return on saving exceeds time discounting. When these two forces balance out, consumption is constant, given by the steady-state Euler equation: then consumption is a decreasing function of the real interest rate. Intuitively, when the real rate is higher, people have an incentive to save now and postpone consumption; people keep consumption constant only if the hedonic return on saving $u'(0)y$ falls. Since the marginal utility of wealth is fixed at $u'(0)$, the marginal utility of consumption $1/y$ must increase: consumption must decline.

Because the optimal consumption path depends not only on interest rates but also on the marginal rate of substitution between wealth and consumption, future interest rates have less impact on today's consumption. In fact, in appendix B, we derive the discrete-time version of the model and show that the Euler equation is discounted exactly as in McKay, Nakamura, and Steinsson (2017), and that discounting is stronger with higher marginal utility of wealth.

We now compare two submodels:

DEFINITION 1: *The New Keynesian (NK) model has zero marginal utility of wealth: $u'(0) = 0$. The wealth-in-the-utility New Keynesian (WUNK) model has sufficient marginal utility of wealth:*

$$u'(0) > \frac{\epsilon}{\delta \gamma a}. \tag{13}$$

The NK model is the textbook model; the WUNK model is the alternative proposed in this paper. When prices are completely fixed ($\gamma \to \infty$), condition (13) simply becomes $u'(0) > 0$; when prices are perfectly flexible ($\gamma = 0$), condition (13) becomes $u'(0) > \infty$, which cannot be satisfied; hence, the WUNK model is well defined at the fixed-price limit but not at the flexible-price limit. In the WUNK model, we also impose $\delta > \sqrt{(\epsilon - 1)/\gamma}$ so that the model accommodates positive natural rates of interest while respecting (13).

In appendix C, we review empirical evidence and find that (13) holds most of the time. First, using (8) and (12), we rewrite (13) as

$$(14) \qquad (\delta - r^n)\,\delta > \frac{\epsilon - 1}{\gamma}.$$

The statistic $(\epsilon - 1)/\gamma$ is the coefficient on the output gap in the Phillips curve; inflation does not respond much to the output gap, so this coefficient is estimated to be quite low: $(\epsilon - 1)/\gamma \approx 3\%$. Direct measures of the price-adjustment cost (γ) and market power (ϵ) lead to a similar estimate of $(\epsilon - 1)/\gamma$. Next the natural rate of interest is estimated to be $r^n \approx 2\%$. Finally, most studies find high subjective time discount rates, with a median estimate of $\delta \approx 40\%$. The time discount rate being significantly above the natural rate of interest is consistent with positive marginal utility of wealth, since $u'(0) = (\delta - r^n)/y^n$. Overall, these estimates suggest that (14) holds: $0.4 \times (0.4 - 0.02) = 0.15 > 0.03$. In fact, (14) holds for most estimates: it would only fail if the price-adjustment cost and time discount rate were simultaneously at the low end of available estimates.

2.3. Dynamical Properties

We now describe the dynamical properties of the NK and WUNK models. This analysis is abstract but is a required preliminary to the study of the ZLB.

In normal times, the natural rate of interest is positive, and the central bank follows a simple interest-rate rule:

$$(15) \qquad i(\pi) = r^n + \phi\pi,$$

which implies a real interest rate

$$(16) \qquad r(\pi) = r^n + (\phi - 1)\,\pi.$$

The parameter $\phi \geq 0$ governs the response of monetary policy to inflation: monetary policy is active when $\phi > 1$ and passive when $\phi < 1$.

In such normal times, the models have the following dynamical properties:

PROPOSITION 1: *Assume that the natural rate of interest is positive ($r^n > 0$) and monetary policy follows* (15). *Then there is a unique steady-state equilibrium, with zero inflation ($\pi = 0$) and output at its natural level ($y = y^n$). At this natural steady state the ZLB is not binding*

($i = r^n > 0$). Around the natural steady state, equilibrium dynamics are governed by

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)y^n & (\phi - 1)y^n \\ -\epsilon/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y - y^n \\ \pi \end{bmatrix}. \tag{17}$$

- *NK model: The linear system* (17) *is a source when monetary policy is active ($\phi > 1$) and a saddle when monetary policy is passive ($\phi < 1$).*

- *WUNK model: The linear system* (17) *is a source whether monetary policy is active or passive.*

When the natural rate of interest is negative, however, the natural steady state cannot be achieved because it would imply a negative nominal interest rate, violating the ZLB constraint. In the WUNK model, the natural rate becomes negative because the marginal utility of wealth increases sufficiently: $u'(0) > \delta/y^n$. In the NK model, we follow the literature and assume that the time discount rate in the Euler equation (now denoted $r^n < 0$) becomes negative, while the time discount rate in the Phillips curve (still denoted $\delta > 0$) does not change. This assumption can been justified in various ways: firm managers have constant discount rate while households' discount rate fluctuates (McKay, Nakamura, and Steinsson 2017, pp. 826–827); or financial intermediation creates a fluctuating spread between the central bank's interest rate and the interest rate used for intertemporal allocation of consumption expenditure (Woodford 2011, p. 16).[1]

Once the natural rate of interest is negative, the central bank moves away from (15) and sets the nominal interest rate to zero in order to stimulate the economy. The dynamical properties of the models are modified as follows:

PROPOSITION 2: *Assume that the natural rate of interest is negative ($r^n < 0$) and monetary policy is at the ZLB: $i(\pi) = 0$ and $r(\pi) = -\pi$. Then there is a unique steady-state equilibrium, denoted $[y^z, \pi^z]$. Around this ZLB steady state, equilibrium dynamics are governed by*

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)y^z & -y^z \\ -\epsilon/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y - y^z \\ \pi - \pi^z \end{bmatrix}, \tag{18}$$

- *NK model: The ZLB steady state has positive inflation ($\pi^z = -r^n > 0$) and above-natural output ($y^z > y^n$). The linear system* (18) *is a saddle.*

- *WUNK model: The ZLB steady state has deflation ($\pi^z < 0$) and below-natural output ($y^z < y^n$). The linear system* (18) *is a source.*

[1]Changing the discount rate only in the Euler equation does not affect the results: the properties of the NK model at the ZLB would remain the same if the discount rates in both Phillips curve and Euler equation became negative.

The proofs of the two propositions are simple and relegated to appendix D: the steady states are determined by the steady-state Phillips curve and Euler equation; the systems (17) and (18) are obtained by linearizing the system generated by the Phillips curve and Euler equation around the natural and ZLB steady states; and the properties of (17) and (18) are obtained using standard results on planar linear systems.[2]

The results in the propositions are illustrated by the phase diagrams in figure 1. The phase diagrams represent the dynamical systems (17) and (18) in the NK and WUNK models; they are constructed using standard methods for planar linear systems. Five points are worth noting.

First, because the model is nearly linear, the π-nullcline is the same as the steady-state Phillips curve, and the y-nullcline the same as the steady-state Euler equation.

Second, the sole difference between NK and WUNK models is the Euler line (y-nullcline). In the NK model the Euler line is always horizontal because the steady-state Euler equation imposes that the real rate $r(\pi)$ equals the time discount rate, which determines inflation independently of output. In the WUNK model the Euler line is not horizontal because the steady-state Euler equation makes the real rate $r(\pi)$ a decreasing function of output. When monetary policy is active, $r(\pi)$ is increasing in π, so the Euler line slopes downward; when monetary policy is passive, including at the ZLB, $r(\pi)$ is decreasing in π, so the Euler line slopes upward.

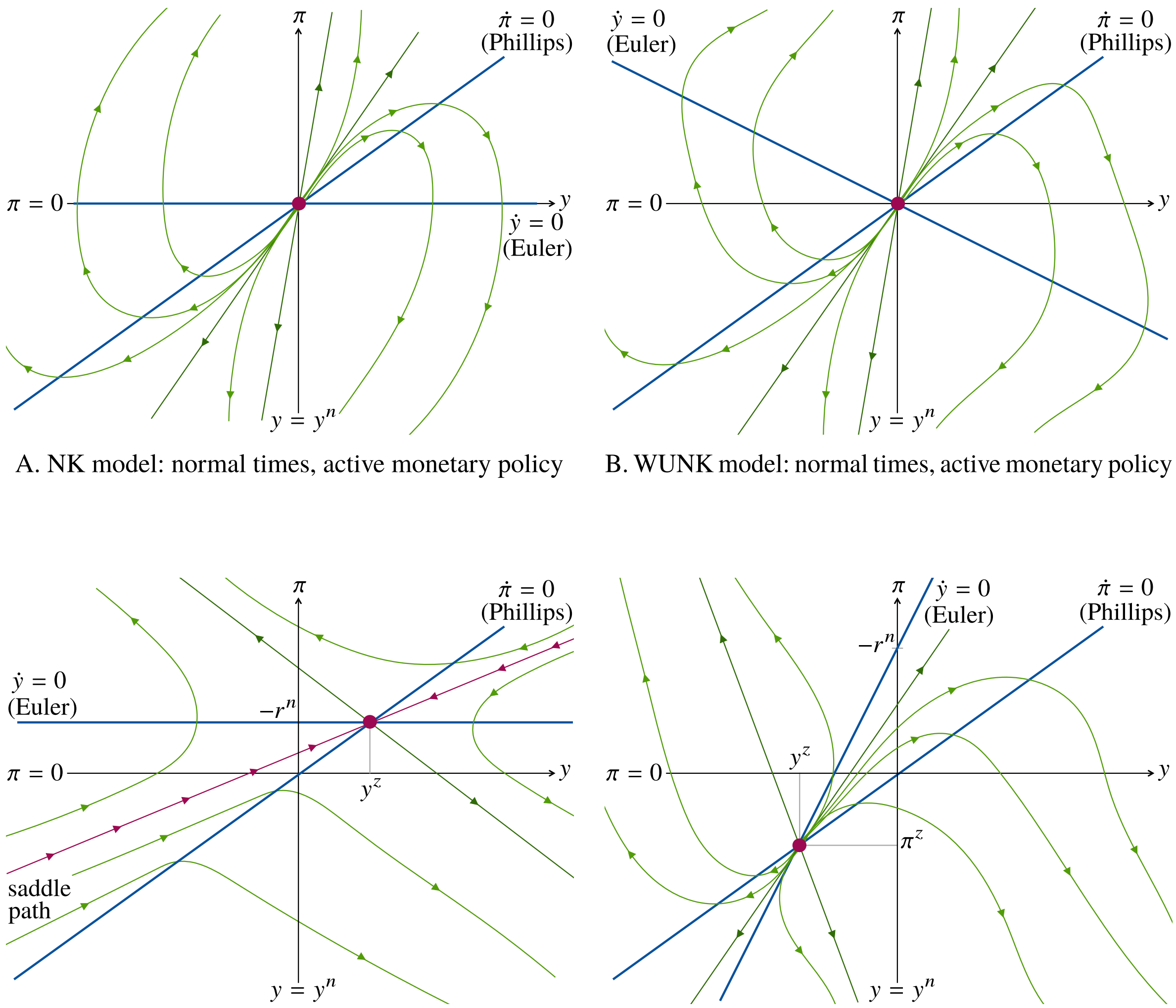
Third, only the Euler line changes between normal times and ZLB, because the natural rate of interest and monetary policy change. In the NK model, the Euler line shifts up from $\pi = 0$ to $\pi = -r^n > 0$. In the WUNK model, the Euler line is upward sloping (as monetary policy is passive) and shifts inward to cross the point $[y^n, -r^n]$ instead of the point $[y^n, 0]$.

Fourth, we omit the phase diagrams when monetary policy is passive in normal times, but they can easily be visualized. These phase diagrams are like the ZLB phase diagrams, except that the Euler line shifts to go through the point $[y^n, 0]$.

Fifth, panel D illustrate the origin of condition (13). The dynamical system representing the WUNK equilibrium remains a source at the ZLB as long as the Euler line is steeper than the Phillips line (π-nullcline). The slope of the Euler line at the ZLB is the marginal utility of wealth, $u'(0)$, so a certain marginal utility of wealth is required—this amount is given by (13).

The propositions have several implications. First, they have implications for equilibrium

[2] Following most of the literature, we separately consider the monetary policy (15) away from the ZLB and the monetary policy $i = 0$ at the ZLB. Then, we check that the ZLB is indeed not binding at the equilibrium obtained under (15). A more rigorous but more cumbersome approach would consider a monetary-policy rule that directly incorporates the ZLB: $i(\pi) = \max\{0, r^n + \phi\pi\}$. In the WUNK model, the two approaches are equivalent, so this would not make a difference. Indeed, appendix E shows that when $i(\pi) = \max\{0, r^n + \phi\pi\}$, there are two cases: if $r^n > 0$, the natural steady state is the unique steady state, and local dynamics follow (17); if $r^n < 0$, the ZLB steady state is the unique steady state, and local dynamics follow (18). In the NK model, however, things are more complicated: when $r^n > 0$, the natural steady state may coexist with another, ZLB steady state (Benhabib, Schmitt-Grohe, and Uribe 2001).



A. NK model: normal times, active monetary policy

B. WUNK model: normal times, active monetary policy

C. NK model: ZLB

D. WUNK model: ZLB

Figure 1. Equilibrium Phase Diagrams in Normal Times and at the ZLB

Notes: π is inflation, y is output, y^n is the natural level of output, r^n is the natural rate of interest, and $[y^z, \pi^z]$ is the ZLB steady state. The NK model has no marginal utility of wealth: $u'(0) = 0$. The WUNK model has sufficient marginal utility of wealth: $u'(0) > \epsilon/(\delta\gamma a)$. In normal times, $r^n > 0$ and the monetary policy is given by $i = r^n + \phi\pi$. Monetary policy is active when $\phi > 1$ and passive when $\phi < 1$. At the ZLB, $r^n < 0$ and the monetary policy is given by $i = 0$. The phase diagrams in panels A and B are constructed from (17) using standard methods from planar linear systems. The phase diagrams in panels C and D are constructed similarly from (18). The systems (17) and (18) are obtained by linearizing the Phillips curve and Euler equation around the natural and ZLB steady states; they describe local equilibrium dynamics. Panel A shows that in the NK model the equilibrium system is a source in normal times when monetary policy is active. Panels B and D show that in the WUNK model the equilibrium system is a source both in normal times and at the ZLB. In panels A and B, we have plotted a nodal source, but the system could also be a spiral source, depending on value of ϕ; in panel D the system is always a nodal source. In these three panels, the equilibrium is determinate: the only bounded equilibrium trajectory is to jump to the steady state and stay there. Finally, panel C shows that in the NK model the equilibrium system is a saddle at the ZLB. Hence the equilibrium is indeterminate: any trajectory jumping on the saddle path and converging to the steady state is an equilibrium.

determinacy. When the system is a source, the equilibrium is determinate: the only bounded equilibrium trajectory is to jump to the steady state and stay there; if the economy jumped somewhere else, output or inflation would become infinite following a trajectory similar to those plotted in panels A, B, or D. When the system is a saddle, in contrast, the equilibrium is indeterminate: any trajectory jumping somewhere on the saddle path and converging to the steady state is an equilibrium, as illustrated in panel C. Hence, in the NK model, the ZLB equilibrium is indeterminate, which has forced researchers to focus on temporary ZLB episodes, following Krugman (1998) and Eggertsson and Woodford (2003). In the WUNK model, on the other hand, the ZLB equilibrium is determinate: the economy jumps to the ZLB steady state and remains there. The ZLB could last forever: absent fundamental shocks, no market forces or change in sentiment could pull the economy away from it.

Second, the propositions imply that away from the ZLB, monetary policy plays quite a different role in the NK and WUNK models. In the NK model, the equilibrium is determinate only when monetary policy is active—the Taylor principle (Woodford 2001). Hence, the central bank must always adhere to an active policy to avoid indeterminacy. The Taylor principle does not hold in the WUNK model, however: the equilibrium is determinate whether monetary policy is active or passive. Hence, the central bank does not need to worry about how strongly it responds to inflation; it only needs to worry about the rate of interest.

Third, the propositions are the basis for the analysis in the rest of the paper. In particular, as we will see in the next section, it is because the dynamical system representing the ZLB equilibrium is a source in the WUNK model instead of a saddle that the anomalies of the NK model at the ZLB disappear in the WUNK model.

3. New Keynesian Anomalies at the ZLB

We illustrate the anomalies of the NK model at the ZLB and show that the anomalies do not exist in the WUNK model. We focus on the two most prominent anomalies: the explosive paths of output and inflation, and the forward-guidance puzzle.

3.1. Explosive Paths of Output and Inflation

We study the paths of output and inflation during a temporary ZLB episode. We consider a typical, two-stage scenario. First, between times 0 and $T > 0$, the natural rate of interest becomes negative: $r^n = r^z < 0$. In response, the central bank maintains the nominal interest rate at the ZLB: $i = 0$. Second, after time T, the natural rate becomes positive again, and the central bank returns to the normal monetary-policy rule, given by (15). (In the NK model, monetary policy must be active to ensure equilibrium determinacy.)

We analyze the ZLB episode by going backward in time. In the second stage, $t > T$, monetary policy maintains the economy at the natural steady state. We then move back to the ZLB episode, $t \in (0, T)$. The corresponding phase diagrams are shown in figure 2. The phase diagrams for the NK model are the same as in panel C of figure 1. At the end of the ZLB, at time T, the economy must be at the natural steady state. Because the equilibrium system is a saddle, at time 0 inflation and output jump down to $\pi(0) < 0$ and $y(0) < y^n$. After that initial contraction, they recover following the unique trajectory leading to $y(T) = y^n$ and $\pi(T) = 0$. Hence the ZLB creates a slump, with below-natural output and deflation. Furthermore output and inflation become unboundedly low as the ZLB lasts longer (panel C). Such pattern is difficult to reconcile with real-world observations: the Japanese ZLB episode that started in 1995 lasted more than twenty years without sustained deflation. Such explosive paths of output and inflation at the ZLB is a well-known anomaly of the NK model (Eggertsson and Woodford 2004, fig. 1; Cochrane 2017).

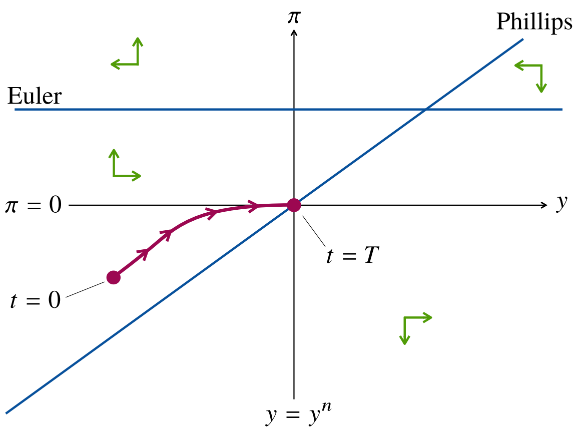
In the WUNK model, output and inflation do not take explosive paths during the ZLB. The phase diagrams for the ZLB episode (panels B and D) are the same as in panel D of figure 1. Because the equilibrium system remains a source at the ZLB, at time 0 inflation and output jump down toward the ZLB steady state, but not all the way: $\pi^z < \pi(0) < 0$ and $y^z < y(0) < y^n$. Then they recover over time, following the unique trajectory leading to $\pi(T) = 0$ and $y(T) = y^n$. Thus, as in the NK model, the ZLB episode creates a slump. Furthermore, the initial contraction is deeper when the ZLB lasts longer (panel D). But unlike in the NK model, the paths of output and inflation are bounded below: irrespective of the length of the ZLB, inflation and output always fall less than if the ZLB were permanent. In fact, if the natural rate of interest is negative but close to 0 at the ZLB, such that π^z is close to 0 and y^z to y^n, output and inflation barely deviate from the natural steady state during the ZLB, even if the ZLB lasts an arbitrarily long time.

The following proposition summarizes these results:[3]

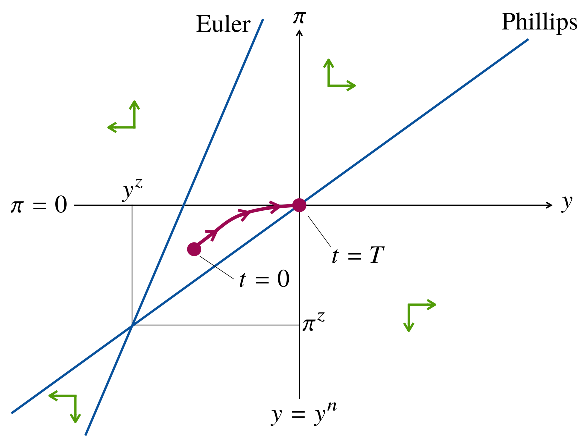
PROPOSITION 3: *A ZLB episode between times* 0 *and* T *has the following properties:*

- *NK model: The economy enters a slump:* $y(t) < y^n$ *and* $\pi(t) < 0$ *for all* $t \in (0, T)$*. The slump becomes infinitely severe as the ZLB becomes infinitely long:* $\lim_{T\to\infty} y(0) = -\infty$ *and* $\lim_{T\to\infty} \pi(0) = -\infty$.

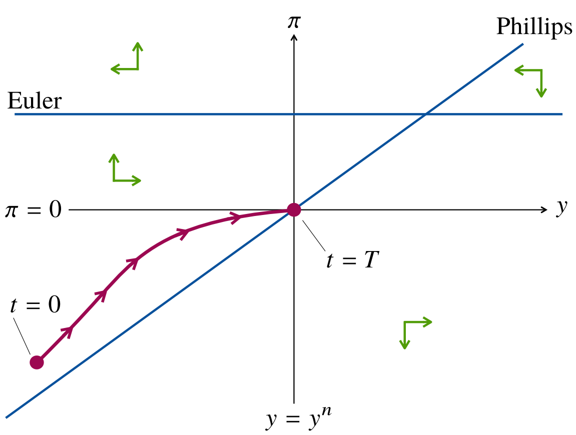
[3]The result in proposition 3 that in the NK model output becomes infinitely negative when the ZLB becomes infinitely long should not be interpreted literally—because the result is obtained using a local approximation of the dynamical system (7)-(10). The interpretation is that output falls much below its natural level (so much so that the local approximation stops being valid). The nonapproximated dynamical system guarantees that output remains positive.



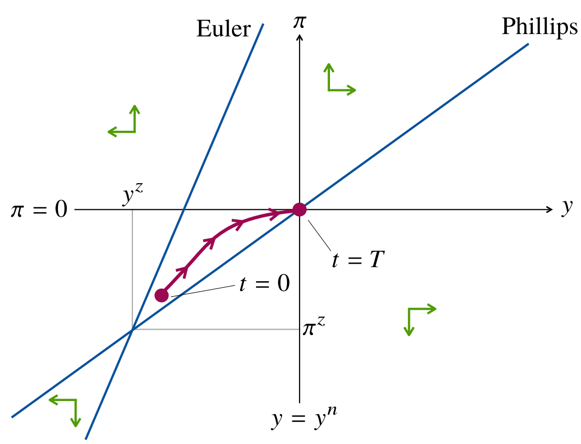
A. NK model: short ZLB



B. WUNK model: short ZLB



C. NK model: long ZLB



D. WUNK model: long ZLB

Figure 2. Output and Inflation During a ZLB Episode

Notes: The figure displays the equilibrium trajectory in the NK and WUNK models during a ZLB episode. A ZLB episode lasts between times 0 and T. During the episode, the natural rate of interest is negative and the central bank responds by setting the nominal interest rate to zero. At the end of the ZLB, at time T, the central bank brings the economy to the natural steady state, where $y = y^n$ and $\pi = 0$. The phase diagrams in panels A and C come from figure 1, panel C. The phase diagrams in panels B and D come from figure 1, panel D. (The Phillips line is the π-nullcline; the Euler line is the y-nullcline.) The figure contrasts a short ZLB episode (small T) in panels A and B with a long ZLB episode (large T) in panels C and D. In both models, at time 0, inflation is negative and output is below its natural level: the ZLB creates a slump. Then the economy recovers along the unique trajectory reaching the natural steady state at time T. Critically, the WUNK model eliminates the explosive paths of output and inflation during long ZLB episodes: in the NK model, the initial slump becomes unboundedly large as the ZLB becomes longer; in the WUNK model, in contrast, the initial slump is bounded below by the ZLB steady state, where $y = y^z$ and $\pi = \pi^z$.

– *WUNK model: The economy enters a slump:* $y(t) < y^n$ *and* $\pi(t) < 0$ *for all* $t \in (0, T)$*. The slump is bounded below by the ZLB steady state:* $y^z < y(t)$ *and* $\pi^z < \pi(t)$ *for all* $t \in (0, T)$*. In fact, the slump approaches the ZLB steady state as the ZLB becomes infinitely long:* $\lim_{T\to\infty} y(0) = y^z$ *and* $\lim_{T\to\infty} \pi(0) = \pi^z$*.*

3.2. Forward-Guidance Puzzle

Here we study the paths of output and inflation when the central bank conducts forward guidance during the ZLB. We consider a typical, three-stage scenario. First, between time 0 and time $T > 0$, there is a ZLB episode, as in the previous section. To alleviate the ZLB, the central bank makes a forward-guidance promise at time 0: that it will maintain the nominal interest rate at zero for a duration Δ once the ZLB is over. Second, between time T and time $T + \Delta$, the natural rate of interest becomes positive: $r^n = r^f > 0$. The central bank abides by its forward-guidance promise and keeps the nominal interest rate at zero. Third, after time $T + \Delta$, the natural rate of interest is still positive, and the central bank returns to normal monetary policy, given by (15).

Once again, we construct the paths of output and inflation by going backward in time. In the third stage, $t > T + \Delta$, monetary policy maintains the economy at the natural steady state. We then move to the forward-guidance episode, $t \in (T, T + \Delta)$. The corresponding phase diagrams are depicted in panels A and B of figure 3. The phase diagrams are similar to those in panels C and D of figure 1, except that $r^n > 0$ instead of $r^n < 0$. Since the Euler line goes through the point $[y^n, -r^n]$, we infer that the Euler line is lower in the NK model and shifted outward in the WUNK model, and that the steady state, denoted $[y^f, \pi^f]$, shifts accordingly. Nevertheless, the equilibrium system retains the same property: saddle in the NK model, source in the WUNK model.[4] Following the logic of figure 2, we find that at time T, inflation and output must be at $\pi(T) > 0$ and $y(T) > y^n$; then they decrease over time, following the unique trajectory leading to $\pi(T + \Delta) = 0$ and $y(T + \Delta) = y^n$. Accordingly, the economy booms during forward guidance: there is positive inflation and above-natural output. Furthermore, as the duration of forward guidance increases in the NK model, output and inflation become unboundedly high. This is a first aspect of the forward-guidance puzzle, emphasized by Del Negro, Giannoni, and Patterson (2012): the stimulative effect of forward guidance is implausibly large its duration increases. This anomaly is resolved in the WUNK model: the paths of output and inflation are bounded above by the forward-guidance steady state, $[y^f, \pi^f]$: irrespective of the duration of forward guidance,

[4]Proposition 2 was derived for a ZLB situation: $i = 0$ and $r^n < 0$. But almost all the results of the proposition also hold for a forward-guidance situation: $i = 0$ and $r^n > 0$. The only different results concern the steady state: in the NK model, the steady state has deflation and below-natural output (in addition the steady state only exists if $r^n < (\epsilon - 1)/(\delta\gamma)$); in the WUNK model the steady state has positive inflation and above-natural output.

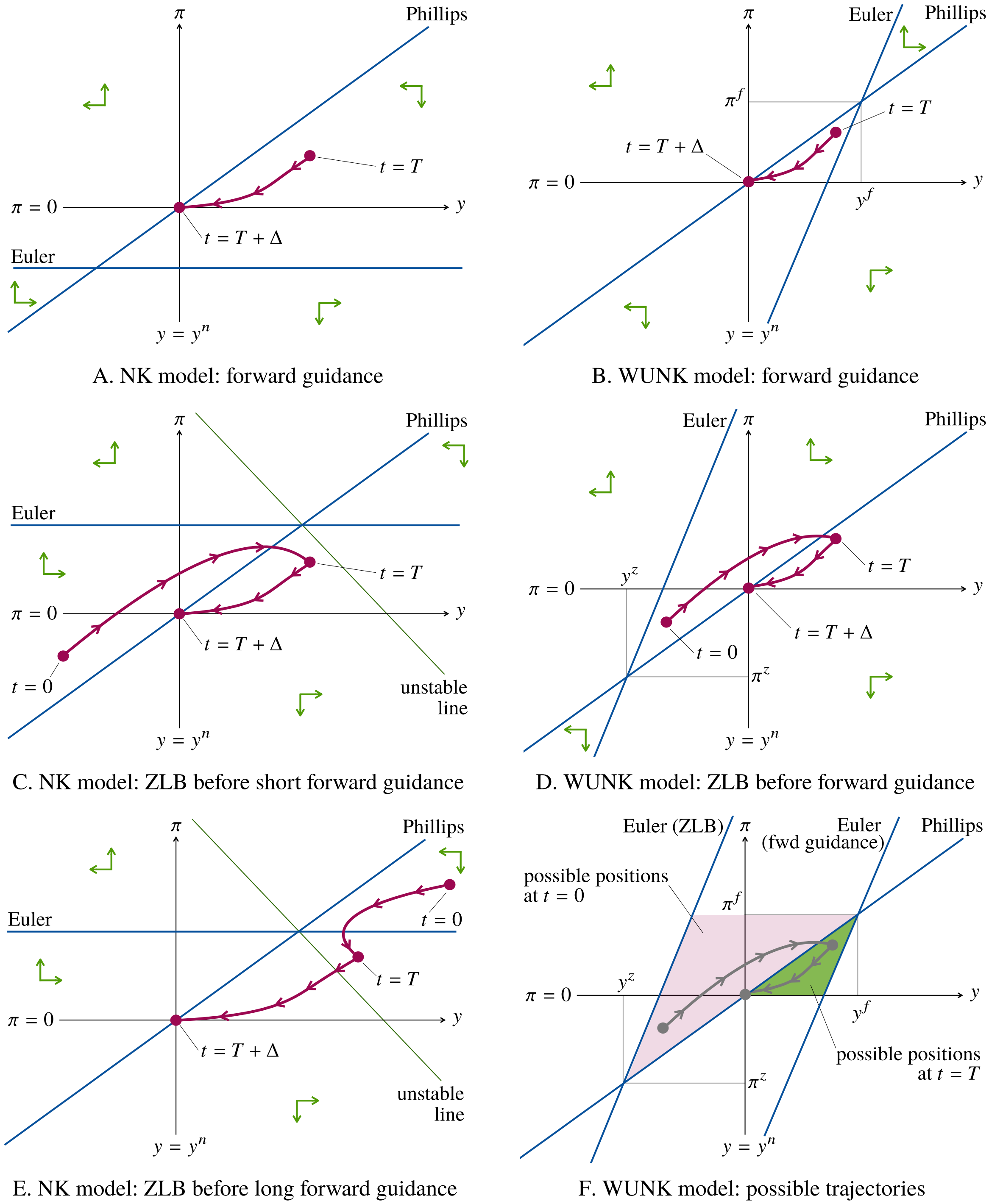


Figure 3. Output and Inflation when a Forward-Guidance Episode Follows a ZLB Episode

Notes: The ZLB episode between times 0 and T is as in figure 2; it is followed by a forward-guidance episode between times T and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. At the end of forward guidance, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagrams in panels A and B (similar to panels C and D in figure 1 but with $r^n > 0$) depict the forward-guidance episode and its equilibrium trajectory. The phase diagrams in panels C and D (same as panels C and D in figure 1) depict the ZLB episode and its equilibrium trajectory; they also show the equilibrium trajectory during forward guidance. Panel E is like panel C, but with longer forward guidance (longer Δ). In panels C and E, the unstable line represents the eigenvector associated with the positive eigenvalue of (18). Panel F is a generic version of panel D, describing any duration of ZLB and forward guidance. The NK model suffers from two problems, which are two facets of the forward-guidance puzzle: the forward-guidance boom becomes arbitrarily large as forward guidance becomes longer (panel A); and there is a reversal as the duration of forward guidance crosses a threshold, whereby the ZLB slump becomes a boom (panel C versus panel E). The forward-guidance puzzle disappears in the WUNK model: equilibrium trajectories are bounded in the green and red areas and there is no reversal (panel F).

inflation and output always rise less than if forward guidance were permanent.

Finally, we turn to the ZLB episode, $t \in (0, T)$. The relevant phase diagrams are depicted in panels C, D, and E of figure 3; the phase diagrams are the same as those in panels C and D of figure 1. Because of the boom engineered through forward guidance, the situation is improved at the ZLB: instead of reaching the natural steady state at time T, the economy must reach $\pi(T) > 0$ and $y(T) > y^n$, so at any time before T, the economy tends to have higher inflation and output than without forward guidance (panels C and D). In the NK model, however, forward guidance has a highly discontinuous effect. For small durations of forward guidance, the initial position of the economy, $[y(0), \pi(0)]$, varies continuously with the duration; but as the duration of forward guidance increases, the position at the beginning of forward guidance, $[y(T), \pi(T)]$, crosses the unstable line of the dynamical system representing the ZLB equilibrium. Suddenly, $[y(T), \pi(T)]$ is connected to a new set of trajectories: these trajectories come from the northeast quadrant of the ZLB phase diagram, not from the southwest quadrant. As a result $[y(0), \pi(0)]$ has much higher output and inflation than when $[y(T), \pi(T)]$ was on the other side of the unstable line. This is a second aspect of the forward-guidance puzzle in the NK model: an infinitesimal increase in the duration of forward guidance when $[y(T), \pi(T)]$ is at the unstable line of the ZLB system generates a discrete increase in output and inflation at the beginning of the ZLB. In fact if the ZLB is very long-lasting, the infinitesimal duration increase transforms a major slump into a major boom (compare panel C to panel E). Once forward guidance creates a ZLB boom (panel E), the boom becomes larger when the ZLB lasts longer.

This second anomaly is also resolved in the WUNK model: $[y(T), \pi(T)]$ can only be connected to trajectories coming from the northeast quadrant of the ZLB phase diagram; therefore, the duration of forward guidance always has a continuous effect on the equilibrium trajectory. In panel F: $[y(T), \pi(T)]$ necessarily falls in the green area so $[y(0), \pi(0)]$ is necessarily in the red area (or green area if T is tiny). One notable implication is that the effect of forward guidance on inflation and output becomes vanishingly small as the ZLB lasts an arbitrarily long time: then the economy simply jumps near the ZLB steady state at time 0, irrespective of forward guidance.

The following proposition summarizes these results:

PROPOSITION 4: *A ZLB episode between times* 0 *and* T *followed by a forward-guidance episode between times* T *and* $T + \Delta$ *have the following properties:*

- *NK model: The economy booms during forward guidance:* $y(t) > y^n$ *and* $\pi(t) > 0$ *for all* $t \in (T, T + \Delta)$*. The boom becomes infinitely large as forward guidance becomes infinitely long:* $\lim_{\Delta \to \infty} y(T) = +\infty$ *and* $\lim_{\Delta \to \infty} \pi(T) = +\infty$*. Forward guidance has a highly discontinuous effect: there exists a forward-guidance duration* Δ^* *such that a long-enough ZLB is an arbitrarily severe slump for any shorter forward guidance and an arbitrarily large boom* *for any longer forward guidance. Formally: for any* $\Delta < \Delta^*$, $\lim_{T\to\infty} y(0) = -\infty$ *and* $\lim_{T\to\infty} \pi(0) = -\infty$*; and for any* $\Delta > \Delta^*$, $\lim_{T\to\infty} y(0) = +\infty$ *and* $\lim_{T\to\infty} \pi(0) = +\infty$.

– *WUNK model: The economy booms during forward guidance:* $y(t) > y^n$ *and* $\pi(t) > 0$ *for all* $t \in (T, T + \Delta)$*. The boom is bounded above by the forward-guidance steady state:* $y(t) < y^f$ *and* $\pi(t) < \pi^f$ *for all* $t \in (T, T + \Delta)$*. Finally, the effect of forward-guidance vanishes as the ZLB becomes infinitely long: for any* Δ, $\lim_{T\to\infty} y(0) = y^z$ *and* $\lim_{T\to\infty} \pi(0) = \pi^z$.

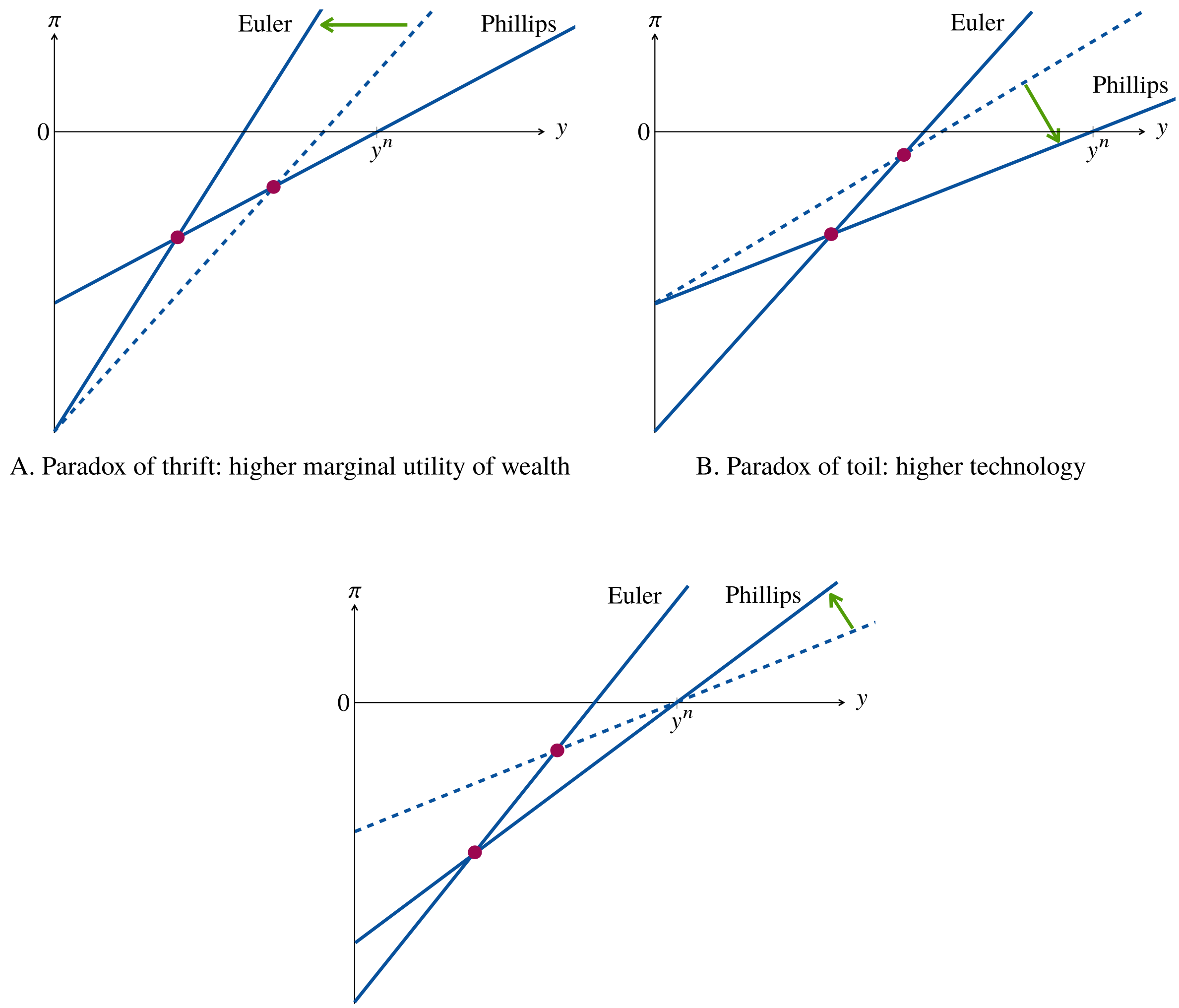
4. New Keynesian Paradoxes at the ZLB

Beside the anomalous properties described in the previous section, the New Keynesian model has several other, intriguing properties at the ZLB. These properties are labeled "paradoxes": not because they are anomalous but because they defy usual economic logic. Here we show that the WUNK model shares these properties: the paradox of thrift, paradox of toil, and paradox of flexibility.

Since the ZLB equilibrium is determinate in the WUNK model, we are not constrained to introduce temporary ZLB episodes: we simply analyze these paradoxes at a permanent ZLB. We assume that the natural rate of interest is negative and the central bank maintains the nominal interest rate at zero. The only equilibrium is to jump to and stay at the ZLB steady state, where the economy is in a slump: inflation is negative and output is below its natural level. Then an unexpected permanent shock occurs, immediately bringing the economy to a new ZLB steady state. We aim to compare the new equilibrium to the old one. Since each equilibrium is the intersection of the steady-state Phillips curve and steady-state Euler equation (11), we determine the effects of the shocks graphically, as shown in figure 4.

We first study the effect of an increase in the marginal utility of wealth ($u'(0)$). At the ZLB the steady-state Euler equation is $\pi = -\delta + u'(0)y$. Hence the increase in $u'(0)$ makes the Euler line steeper and moves the economy inward along the Phillips line: both output and inflation decrease (panel A). The increase in the marginal utility of wealth operates exactly like the Keynesian paradox of thrift, which also appears in the New Keynesian model (Eggertsson 2010, p. 16): after the increase, households want to save more and accumulate more wealth; but aggregate wealth is fixed since government bonds are in fixed supply; hence, in equilibrium, the only way to hold more wealth relative to consumption is to reduce consumption. In normal times, the central bank would offset this reduction in aggregate demand by reducing the nominal interest rate; but this is not an option at the ZLB, so output falls.

Next we examine the effect of an increase in technology (a). Using (8), the steady-state Phillips curve (9) can be rewritten as $\pi = \epsilon y/(\delta\gamma a) - (\epsilon - 1)/(\delta\gamma)$. Hence the increase in a



A. Paradox of thrift: higher marginal utility of wealth

B. Paradox of toil: higher technology

C. Paradox of flexibility: lower price-adjustment cost

Figure 4. New Keynesian Paradoxes at the ZLB in the WUNK Model

Notes: This figure displays the equilibrium during a permanent ZLB in the WUNK model, and its response to three unexpected permanent shocks. π is inflation, y is output, and y^n is the natural output. The Phillips line represents the steady-state Phillips curve, given by (9). The Euler line represents the steady-state Euler equation at the ZLB, given by (11) with $r(\pi) = -\pi$ and $r^n < 0$. The ZLB equilibrium is given by the intersection of the two lines: output is below its natural level and inflation is negative. Panel A illustrates the paradox of thrift: an increase in the marginal utility for wealth moves the Euler line inward, which depresses output and inflation. Panel B illustrates the paradox of toil: an increase in technology moves the Phillips line outward, which depresses output and inflation. Panel C illustrates the paradox of flexibility: a decrease in the price-adjustment cost rotates the Phillips line counterclockwise around the point $[y^n, 0]$, which depresses output and inflation.

makes the Phillips line flatter and moves the economy inward along the Euler line: both output and inflation decrease (panel B). This reduction in output is the paradox of toil, discovered by Eggertsson (2010). With higher technology, real marginal costs are lower, and the natural level of output is higher: firms would like to produce and sell more. Hence, firms tend to reduce their prices, reducing inflation. Away from the ZLB, the central bank would offset this reduction by lowering the nominal interest rate; but this cannot happen at the ZLB. Then the reduction in inflation raises the real interest rate—as the nominal interest rate is at zero—which makes households more prone to save and in equilibrium leads to lower output.

We finally turn to the effect of a decrease in the price-adjustment cost (γ). The steady-state Phillips curve (9) shows that the increase in γ leads to a counterclockwise rotation of the Phillips line around the point $[y^n, 0]$, which moves the economy inward along the Euler line: both output and inflation decrease (panel C). This reduction in output is the paradox of flexibility, introduced in the New Keynesian model by Eggertsson and Krugman (2012, pp. 1487–1488). With a lower price-adjustment cost, firms are keener to adjust their prices to bring production closer to the natural level of output, which accentuates the existing deflation. Hence, as in the paradox of toil, the real interest rate rises, which results in lower output.

The following proposition summarizes the results:

PROPOSITION 5: *At the ZLB in the WUNK model, unexpected permanent shocks have the following effects:*

- *Paradox of thrift: an increase in marginal utility of wealth reduces output and inflation.*
- *Paradox of toil: an increase in technology reduces output and inflation.*
- *Paradox of flexibility: a decrease in price-adjustment cost reduces output and inflation.*

5. Conclusion

This paper extends the textbook New Keynesian model by introducing wealth, in the form of government bonds, into the utility function. The marginal utility of wealth is assumed to be above a threshold that depends on price rigidity and the time discount rate—an assumption that seems to hold in the data. Although our extended model deviates only minimally from the textbook model, it resolves all its anomalies at the ZLB: there is no explosive slump if the ZLB is long-lasting, no explosive boom if forward guidance is long-lasting, and no reversal from slump to boom when the duration of forward guidance crosses some threshold. Hence in our model, the ZLB is not necessarily short-lived—it might last forever as a steady state—and forward-guidance policies have plausible, limited effects. At the same time, our model retains other ZLB properties: paradox of thrift, paradox of toil, and paradox of flexibility.

Beyond the New Keynesian model, the wealth-in-the-utility-function assumption might be a simple way to model people's saving behavior better. First, it reconciles the single-digit interest rates observed on many markets with the double-digit time discount rates measured in most studies. Second, it leads to a negative relationship between consumption and real interest rate, as in the old-fashioned IS curve. Third and related, it accommodates a broad range of steady-state real interest rates—including negative ones. Fourth, it reduces the effect of future interest rates on today's consumption. These properties contrast with the standard model, in which the steady-state real interest rate equals the time discount rate, and future interest rates have an implausibly large effect on today's consumption.

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Appendix A. Derivation of Phillips Curve and Euler Equation

We derive the two differential equations that govern the equilibrium of our New Keynesian model: the Phillips curve, given by (7), and the Euler equation, given by (10).

Saving and Pricing by Households. We begin by characterizing households' optimal saving and pricing. To solve household j's optimization problem, we write the current-value Hamiltonian:

$$\begin{aligned}\mathcal{H}_j = \frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}dk\right) + u\left(\frac{b_j(t)}{p(t)}\right) - \frac{y_j^d(p_j(t),t)}{a} - \frac{\gamma}{2}\pi_j(t)^2 \\ + \mathcal{A}_j(t)\left[i(t)b_j(t) + p_j(t)y_j^d(p_j(t),t) - \int_0^1 p_k(t)c_{jk}(t)dk\right] + \mathcal{B}_j(t)\pi_j(t)p_j(t)\end{aligned}$$

with control variables $c_{jk}(t)$ for all $k \in [0,1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian. To ease notation, we now drop the time index t.

The first optimality condition is $\partial\mathcal{H}_j/\partial c_{jk} = 0$, which yields

$$\text{(A1)} \qquad \frac{1}{c_j}\left(\frac{c_{jk}}{c_j}\right)^{-1/\epsilon} = \mathcal{A}_j p_k.$$

Appropriately integrating this expression over all $k \in [0,1]$, and using (2) and (3), we find

$$\text{(A2)} \qquad \mathcal{A}_j = \frac{1}{pc_j}.$$

Using (A1) and (A2), we obtain

$$c_{jk} = \left(\frac{p_k}{p}\right)^{-\epsilon} c_j.$$

We then get the usual demand for good k:

$$\text{(A3)} \qquad y_k^d(p_k) = \int_0^1 c_{jk}dj = \left(\frac{p_k}{p}\right)^{-\epsilon} c,$$

where $c = \int_0^1 c_j dj$ measures aggregate consumption. We use this expression for $y_k^d(p_k)$ in household k's Hamiltonian. We also obtain $\int_0^1 p_k c_{jk} dk = pc_j$: the price of one unit of consumption index is indeed p.

The second optimality condition is $\partial\mathcal{H}_j/\partial b_j = \delta\mathcal{A}_j - \dot{\mathcal{A}}_j$, which implies

$$-\frac{\dot{\mathcal{A}}_j}{\mathcal{A}_j} = i + \frac{u'\left(b_j/p\right)}{p\mathcal{A}_j} - \delta.$$

Using (A2), we obtain the household's Euler equation:

$$\frac{\dot{c}_j}{c_j} = i - \pi + u'\left(b_j/p\right) c_j - \delta. \tag{A4}$$

This Euler equation describes the optimal path of household j's consumption.

The third optimality condition is $\partial\mathcal{H}_j/\partial\pi_j = 0$, which implies

$$\mathcal{B}_j p_j = \gamma\pi_j. \tag{A5}$$

Differentiating (A5) with respect to time, we obtain

$$\frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j. \tag{A6}$$

The fourth optimality condition is $\partial\mathcal{H}_j/\partial p_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j$, which implies

$$\pi_j - \frac{(\epsilon-1)y_j\mathcal{A}_j}{\mathcal{B}_j p_j}\left(p_j - \frac{\epsilon}{\epsilon-1}\cdot\frac{1}{a\mathcal{A}_j}\right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Then, using (A2), (A5), and (A6), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon-1)y_j}{\gamma c_j\pi_j}\left(\frac{p_j}{p} - \frac{\epsilon}{\epsilon-1}\cdot\frac{c_j}{a}\right). \tag{A7}$$

This equation describes the optimal path of the price set by household j.

The previous four conditions, together with the transversality conditions $\lim_{t\to+\infty} e^{-\delta t}\mathcal{B}_j(t)p_j(t) = 0$ and $\lim_{t\to+\infty} e^{-\delta t}\mathcal{A}_j(t)b_j(t) = 0$, are necessary and sufficient for a maximum to the household's problem (Benhabib, Schmitt-Grohe, and Uribe 2001, p. 48).

Saving and Pricing in Equilibrium. We now turn to saving and pricing in equilibrium. All households have the same initial wealth and initial price, so they all behave the same. We therefore omit the subscripts j and k.

Then, we simplify the household's Euler equation, given by (A4), and the household's Phillips curve, given by (A7), using two equilibrium conditions: government bonds are in zero net supply, so $b = 0$; and production and consumption are equal, so $y = c$. Accordingly, we simplify the household's Euler equation to

$$\frac{\dot{y}}{y} = r - \delta + u'(0)\, y, \tag{A8}$$

where $r = i - \pi$. This is just the Euler equation given by (10). Since $y^n = (\epsilon - 1)a/\epsilon$, we also simplify the household's Phillips curve to

$$\dot{\pi} = \delta\pi - \frac{\epsilon}{\gamma a}\left(y - y^n\right). \tag{A9}$$

This is the Phillips curve given by (7).

Saving and Pricing in Steady State. Finally, we describe saving and pricing in steady state. Plugging $\dot{y} = 0$ into (A8), we obtain

$$r = \delta - u'(0)\, y.$$

This is just the steady-state Euler equation given by (11).

Next, plugging $\dot{\pi} = 0$ into (A9), we find

$$\pi = \frac{\epsilon}{\delta\gamma a}\left(y - y^n\right).$$

This is the steady-state Phillips curve given by (9).

Appendix B. Phillips Curve and Euler Equation in Discrete Time

We recast the Phillips curve (7) and Euler equation (10) in discrete time. These discrete-time equations should be helpful to compare our model with the textbook New Keynesian model, which is usually presented in discrete time. The discrete-time formulation also shows that introducing wealth in the utility function yields a discounted Euler equation.

In discrete time, households trade one-period government bonds. Bonds purchased in period t have a price $q(t)$, mature in period $t+1$, and pay one unit of money at maturity. The nominal interest rate between t and $t+1$ is defined as $i(t) = -\ln(q(t))$.

Saving and Pricing by Households. We begin by characterizing households' optimal saving and pricing. Household j chooses sequences $\left\{y_j(t), p_j(t), h_j(t), \left[c_{jk}(t)\right]_{k=0}^{1}, b_j(t)\right\}_{t=0}^{\infty}$ to maximize the discounted sum of instantaneous utilities

$$\sum_{t=0}^{\infty}\beta^t\left\{\frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}dk\right)+u\left(\frac{b_j(t)}{p(t)}\right)-h_j(t)-\frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)}-1\right]^2\right\}dt,$$

where $\beta<1$ is the time discount factor. The maximization is subject to three constraints. First, there is the production function (1). Second, there is the demand for good j, given by (6). The demand for good j remains the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time. And third, there is a budget constraint:

$$\int_0^1 p_k(t)c_{jk}(t)dk+q(t)b_j(t)=p_j(t)y_j(t)+b_j(t-1).$$

Household j is also subject to a solvency constraint preventing Ponzi schemes: $\lim_{T\to\infty}b_j(T)\geq 0$. Finally, household j takes as given the initial conditions $b_j(-1)$ and $p_j(-1)$, as well as the sequences of aggregate variables $\{p(t),q(t),c(t)\}_{t=0}^{\infty}$.

To solve household j's problem, we set up the Lagrangian:

$$\begin{aligned}\mathcal{L}_j=\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}dk\right)+u\left(\frac{b_j(t)}{p(t)}\right)-\frac{y_j^d(p_j(t),t)}{a}-\frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)}-1\right]^2\\&+\mathcal{A}_j(t)\left[p_j(t)y_j^d(p_j(t),t)+b_j(t-1)-\int_0^1 p_k(t)c_{jk}(t)dk-q(t)b_j(t)\right]\Bigg\}\end{aligned}$$

where $\mathcal{A}_j(t)$ is the Lagrange multiplier on the budget constraint in period t, and

$$y_j^d(p_j(t),t)=\left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon}c(t) \tag{A10}$$

is the demand for good j in period t. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Lagrangian.

We begin by computing the first-order conditions with respect to $c_{jk}(t)$. As in the continuous-time case, we obtain

$$\mathcal{A}_j(t)=\frac{1}{p(t)c_j(t)}. \tag{A11}$$

We then turn to the first-order condition with respect to $b_j(t)$. The condition is

$$q(t)\mathcal{A}_j(t) = \frac{1}{p(t)}u'\left(\frac{b_j(t)}{p(t)}\right) + \beta\mathcal{A}_j(t+1).$$

Using (A11), we obtain a consumption Euler equation:

$$q(t) = c_j(t)u'\left(\frac{b_j(t)}{p(t)}\right) + \beta\frac{p(t)c_j(t)}{p(t+1)c_j(t+1)}. \tag{A12}$$

Finally, the first-order condition with respect to $p_j(t)$ is

$$\epsilon\frac{y_j(t)}{a} + \beta\gamma\frac{p_j(t+1)}{p_j(t)}\left[\frac{p_j(t+1)}{p_j(t)} - 1\right] = \gamma\frac{p_j(t)}{p_j(t-1)}\left[\frac{p_j(t)}{p_j(t-1)} - 1\right] + (\epsilon-1)\mathcal{A}_j(t)p_j(t)y_j(t).$$

Using (A11) again, we obtain the household's Phillips curve:

$$\epsilon\frac{y_j(t)}{a} + \beta\gamma\frac{p_j(t+1)}{p_j(t)}\left[\frac{p_j(t+1)}{p_j(t)} - 1\right] = \gamma\frac{p_j(t)}{p_j(t-1)}\left[\frac{p_j(t)}{p_j(t-1)} - 1\right] + (\epsilon-1)\frac{p_j(t)y_j(t)}{p(t)c_j(t)}. \tag{A13}$$

Saving and Pricing in Equilibrium. We now turn to saving and pricing in a symmetric equilibrium. First, since all households behave the same in equilibrium, we drop the subscripts j and k. Second, since government bonds are in zero net supply, we set $b(t) = 0$. Third, since production and consumption are equal in equilibrium, we set $y(t) = c(t)$. Then, from (A12), we obtain the Euler equation:

$$q(t) = u'(0)y(t) + \beta\,\frac{p(t)y(t)}{p(t+1)y(t+1)}.$$

Similarly, from (A13), we obtain the Phillips curve:

$$\frac{p(t)}{p(t-1)}\left[\frac{p(t)}{p(t-1)} - 1\right] = \beta\,\frac{p(t+1)}{p(t)}\left[\frac{p(t+1)}{p(t)} - 1\right] - \frac{\epsilon}{\gamma a}\left[y(t) - y^n\right].$$

The last step is to log-linearize the Euler equation and Phillips curve around the natural steady state, where $y = y^n$, $\pi = 0$, and $i = r^n$. We introduce the log-deviation of output from its steady-state level: $\hat{y}(t) = \ln(y(t)) - \ln(y^n)$. We also introduce the inflation rate between periods t and $t+1$: $\pi(t+1) = \ln(p(t+1)) - \ln(p(t))$.

We start by log-linearizing the Euler equation. We obtain

$$-i(t) = -r^n + (1-\alpha)\hat{y}(t) + \alpha\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right]$$

where

$$\alpha = \frac{\beta}{\beta + u'(0)y^n}.$$

Because the marginal utility of wealth is positive, $\alpha < 1$, which then yields a discounted Euler equation:

$$\hat{y}(t) = \alpha\hat{y}(t+1) - [i(t) - r^n - \alpha\pi(t+1)] \tag{A14}$$

The Euler equation is discounted because future consumption ($\hat{y}(t+1)$) appears discounted by a coefficient $\alpha < 1$. In McKay, Nakamura, and Steinsson (2017), such discounting occurs because of income risk and borrowing constraints. Diba and Loisel (2017, sec. 6) show that such discounting occurs in a variety of models. To make discounting even more apparent, we solve the Euler equation forward. This yields

$$\hat{y}(t) = -\sum_{k=0}^{+\infty} \alpha^k \left[i(t+k) - r^n - \alpha\pi(t+k+1)\right].$$

As noted by McKay, Nakamura, and Steinsson (2017, p. 821), in this type of Euler equations, the effect on current consumption of interest rates k periods in the future is discounted by $\alpha^k < 1$. Discounting is stronger for interest rates further away in the future.

Next, we log-linearize the Phillips curve. We obtain

$$\pi(t) = \beta\pi(t+1) + \kappa\hat{y}(t), \tag{A15}$$

where

$$\kappa = \frac{\epsilon}{\gamma} \cdot \frac{y^n}{a} = \frac{\epsilon - 1}{\gamma}.$$

Despite wealth in the utility function, the Phillips curve remains exactly the same as in typical New Keynesian models: for instance, (A15) is the same as equation (21) in Gali (2008, p. 49).

Appendix C. Empirical Support for the WUNK Assumption

In the WUNK model the marginal utility of wealth is above the threshold specified in (13). We now survey estimates of the time discount rate and price rigidity to assess this assumption. We find that (13) usually holds.

A vast literature has attempted to estimate time discount rates. Frederick, Loewenstein, and O'Donoghue (2002) survey the estimates obtained from real-world behavior and elicitation in field or laboratory experiments. There is a lot of variation in the estimates, but the majority of them points to high time discounting, much higher than prevailing market interest rates (table 1). We take the average estimate in each of the 43 studies covered and then take the median of these averages. We obtain an annual discount rate of $\delta = 0.35$.

Most studies discussed by Frederick, Loewenstein, and O'Donoghue assume that people use a single rate to exponentially discount future utility. This exponential-discounting model is subject to many anomalies, such as hyperbolic discounting. Recent papers allow for more general discounting; they also separate between time discounting and risk aversion, which were sometimes mingled in previous studies. Andersen et al. (2014, table 3) survey 16 such studies. Taking again the average estimate in each study, and then the median of these averages, we obtain $\delta = 0.43$.

Overall, a time discount rate of $\delta = 0.4$ seems to be a reasonable midpoint estimate. Additionally, the Euler equation (11) implies that in the natural steady state,

$$u'(0)\, y^n = \delta - r^n. \tag{A16}$$

Del Negro et al. (2017, fig. 1) and Holston, Laubach, and Williams (2017, fig. 1) estimate the US natural rate of interest around $r^n = 2\%$, which yields the natural marginal rate of substitution between real wealth and consumption: $u'(0)y^n = 0.4 - 0.02 = 0.38$. Although the financial return on wealth is much lower than the time discount rate, people are willing to hold wealth because they derive direct utility from it.

A large literature also estimates price rigidity. Price rigidity is estimated in macro data from the slope of the Phillips Curve. The Phillips curve (7) is a differential equation. As shown in appendix B, the equivalent difference equation is

$$\pi(t) = (1 - \delta)\, \pi(t + 1) + \kappa\, \frac{y(t) - y^n}{y^n}, \tag{A17}$$

where $[y(t) - y^n]\,/y^n$ is the output gap and

$$\kappa = \frac{\epsilon - 1}{\gamma}. \tag{A18}$$

Mavroeidis, Plagborg-Moller, and Stock (2014) review the literature that estimates such Phillips curve and propose their own estimates. For a specification similar to (A17), estimates vary between $\kappa = 0.005$ and $\kappa = 0.08$, with a median estimate across 16 studies of $\kappa = 0.03$ (fig. 3). Their own estimate is slightly lower: $\kappa = 0.018$ (table 3). We use $\kappa = 0.03$ as a midpoint estimate.

We can also measure $\kappa = (\epsilon - 1)/\gamma$ from microestimates of ϵ and γ. Using firm-level data, De Loecker and Eeckhout (2017) measure the goods-market markup in the United States. They find that the average markup $\epsilon/(\epsilon - 1)$ hovers between 1.2 and 1.3 in the 1950–1980 period before continuously rising to 1.7 in the 1980–2014 period (fig. 1). Since 1990 the average markup is around 1.5, implying an average elasticity $\epsilon = 3$, which we use as our midrange estimate. Following Michaillat (2014, p. 206), we calibrate the price-adjustment cost to $\gamma = 61$. This estimate is obtained from microevidence provided by Zbaracki et al. (2004), who analyze the pricing process of a large industrial firm and measure the physical, managerial, and customer costs of changing prices. Combining these estimates for ϵ and γ yields $\kappa = (3 - 1)/61 = 0.033$. This number is close to the estimate from the Phillips-curve literature, so we set $\kappa = 0.03$.

We use our estimates of δ, $u'(0)y^n$, and κ to assess (13). Using (8) and (A18), we rewrite (13) as $u'(0)y^n\delta > \kappa$. Our estimates imply that the condition normally holds: $u'(0)y^n\delta = 0.38\times 0.4 = 0.15$, far above estimates of κ.

While there remains uncertainty about the estimates of κ, δ, and $u'(0)y^n$, (13) seems to hold in most situations. It holds for the highest value of κ, around $\kappa = 0.08$, well below 0.15. The robustness analysis conducted by Mavroeidis, Plagborg-Moller, and Stock (2014) shows that it is not impossible to obtain estimates of κ above 0.15, but it is quite unlikely. Next, the bottom third of estimates of δ in Frederick, Loewenstein, and O'Donoghue (2002) is below $\delta = 0.2$; in Andersen et al. (2014), only the bottom 10% of estimates is below $\delta = 0.2$. This lower estimate of δ implies $u'(0)y^n = \delta - r^n = 0.2 - 0.02 = 0.18$ and $u'(0)y^n\delta = 0.18 \times 0.2 = 0.036$. This value remains above the median estimate of κ. To conclude, (13) would only fail if κ was at the high end of available estimates and simultaneously δ was at the low end of available estimates.

Appendix D. Proofs

Proof of Proposition 1

A steady state must satisfy the steady-state Phillips curve (9), and the steady-state Euler equation (11) combined with the monetary-policy rule (16). These equations form the following linear system:

$$\pi = \frac{\epsilon}{\delta\gamma a}\left(y - y^n\right) \tag{A19}$$

$$(\phi - 1)\,\pi = -u'(0)\left(y - y^n\right). \tag{A20}$$

As $[y, \pi] = [y^n, 0]$ satisfies both equations, it is a steady state. Furthermore the two equations are non-parallel. In the NK model this is obvious since $u'(0) = 0$. In the WUNK model the slope of the second equation is $-u'(0)/(\phi - 1)$: if $\phi > 1$, the slope is negative; if $\phi \in [0, 1)$, the slope is positive and greater than $u'(0)$ and thus than $\epsilon/(\delta\gamma a)$ as (13) holds; in both cases the slope of the second equation is different from that of the first equation. We conclude that the two equations cannot admit more than one intersection, so $[y^n, 0]$ is the unique steady state. The nominal interest rate at $[y^n, 0]$ is given by (15): $i = r^n + \phi \times 0 = r^n > 0$.

Equilibrium dynamics are governed by the nonlinear dynamical system generated by the Phillips curve (7) and Euler equation (10). Hence, around the natural steady state, equilibrium dynamics are approximately given by the system resulting from the linearization of that nonlinear system, which is (17).

Next, we classify the linear system (17) following the methodology in Hirsch, Smale, and Devaney (2013, pp. 61–64). We denote by $\mathbf{M}$ the matrix in (17). To classify the linear system we compute the trace and determinant of $\mathbf{M}$:

$$\begin{aligned} \operatorname{tr}(\mathbf{M}) &= \delta + u'(0)y^n \\ \det(\mathbf{M}) &= \delta u'(0)y^n + \frac{(\phi - 1)y^n\epsilon}{\gamma a}. \end{aligned}$$

Since $\delta > 0$ and $u'(0) \geq 0$, then $\operatorname{tr}(\mathbf{M}) > 0$.

In the NK model, $u'(0) = 0$ so $\det(\mathbf{M}) = (\phi - 1)y^n\epsilon/(\gamma a)$. Accordingly, when $\phi > 1$, $\det(\mathbf{M}) > 0$; since $\operatorname{tr}(\mathbf{M}) > 0$, (17) is a source. Conversely, when $\phi < 1$, $\det(\mathbf{M}) < 0$, indicating that (17) is a saddle.

In the WUNK model, using (13) and $\phi - 1 \geq -1$, we have

$$\det(\mathbf{M}) \geq \delta u'(0)y^n - \frac{y^n\epsilon}{\gamma a} = \delta y^n \left[u'(0) - \frac{\epsilon}{\delta\gamma a}\right] > 0.$$

Hence, $\det(\mathbf{M}) > 0$ and $\operatorname{tr}(\mathbf{M}) > 0$, indicating that (17) is a source.

Proof of Proposition 2

The steady-state Phillips curve (9) and the steady-state Euler equation (11) combined with the monetary-policy rule $r(\pi) = -\pi$ form the following linear system:

$$\pi = \frac{\epsilon}{\delta\gamma a}(y - y^n) \tag{A21}$$

$$\pi = -r^n + u'(0)(y - y^n). \tag{A22}$$

A steady state is a solution to this system with positive output.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

$$\pi = -r^n \tag{A23}$$

$$y = y^n - \frac{\delta\gamma a r^n}{\epsilon}. \tag{A24}$$

Since $r^n < 0$, $y > y^n$ and $y > 0$: the solution to the system is a steady state. Hence the NK model admits a unique steady state at the ZLB, where inflation is positive (since $r^n < 0$) and output is above its natural level (since $\delta\gamma a r^n/\epsilon < 0$).

When $r^n > 0$ (for instance, during forward guidance) the steady state may not exist, however: $y > 0$ only when $(\delta\gamma a r^n)/\epsilon < y^n$ or $r^n < (\epsilon - 1)/(\delta\gamma)$. When $r^n > (\epsilon - 1)/(\delta\gamma)$, then $y < 0$ and no steady state exists.

In the WUNK model, since (13) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Inflation in that unique solution is given by

$$\pi = \frac{r^n}{u'(0)\delta\gamma a/\epsilon - 1}. \tag{A25}$$

Condition (13) implies that $u'(0)\delta\gamma a/\epsilon > 1$, so π has the sign of r^n; since $r^n < 0$, we infer that $\pi < 0$. Since y and π in the solution are related by (A21) and $\pi < 0$, we infer that $y < y^n$. In addition, as $\epsilon y^n/(\delta\gamma a) = (\epsilon - 1)/(\delta\gamma)$ and $r^n + u'(0)y^n = \delta$, output in the unique solution can be expressed as

$$y = \frac{\delta - (\epsilon - 1)/(\delta\gamma)}{u'(0) - \epsilon/(\delta\gamma a)}. \tag{A26}$$

In the WUNK model, we have assumed $u'(0) > \epsilon/(\delta\gamma a)$ and $\delta > \sqrt{(\epsilon - 1)/\gamma}$, which implies $\delta - (\epsilon - 1)/(\delta\gamma) > 0$. We infer that $y > 0$. Hence, the solution to the system is indeed a steady state. In sum, the WUNK model admits a unique steady state at the ZLB, where inflation is negative and output is below its natural level.

Next, as in the proof of proposition 1, around the ZLB steady state, equilibrium dynamics are approximately given by the system resulting from the linearization of the dynamical system generated by the Phillips curve (7) and Euler equation (10). This linear system is (18).

Once again, we classify the linear system (18) following the methodology in Hirsch, Smale, and Devaney (2013, pp. 61–64). We denote by $\mathbf{M}$ the matrix in (18). As in the proof of proposition 1, we have $\mathrm{tr}(\mathbf{M}) > 0$ and

$$\det(\mathbf{M}) = \delta y^n \left[u'(0) - \frac{\epsilon}{\delta\gamma a}\right].$$

In the NK model $u'(0) = 0$ so $\det(\mathbf{M}) < 0$, implying that (18) is a saddle. In the WUNK model, (13) implies that $\det(\mathbf{M}) > 0$; since $\mathrm{tr}(\mathbf{M}) > 0$, (18) is a source. In fact, the discriminant of the characteristic equation of the matrix $\mathbf{M}$ is strictly positive:

$$\mathrm{tr}(\mathbf{M})^2 - 4\det(\mathbf{M}) = \delta^2 + (u'(0)y^n)^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{y^n\epsilon}{\gamma a} = (\delta - u'(0)y^n)^2 + 4\frac{y^n\epsilon}{\gamma a} > 0.$$

Hence (18) is a nodal source, not a spiral source (the eigenvalues of $\mathbf{M}$ are real, not complex).

Appendix E. Unique Steady State in the WUNK Model

We establish the result, discussed at the end of section 2, that the WUNK model admits a unique steady state under the global monetary-policy rule

$$i(\pi) = \max\{0, r^n + \phi\pi\}. \tag{A27}$$

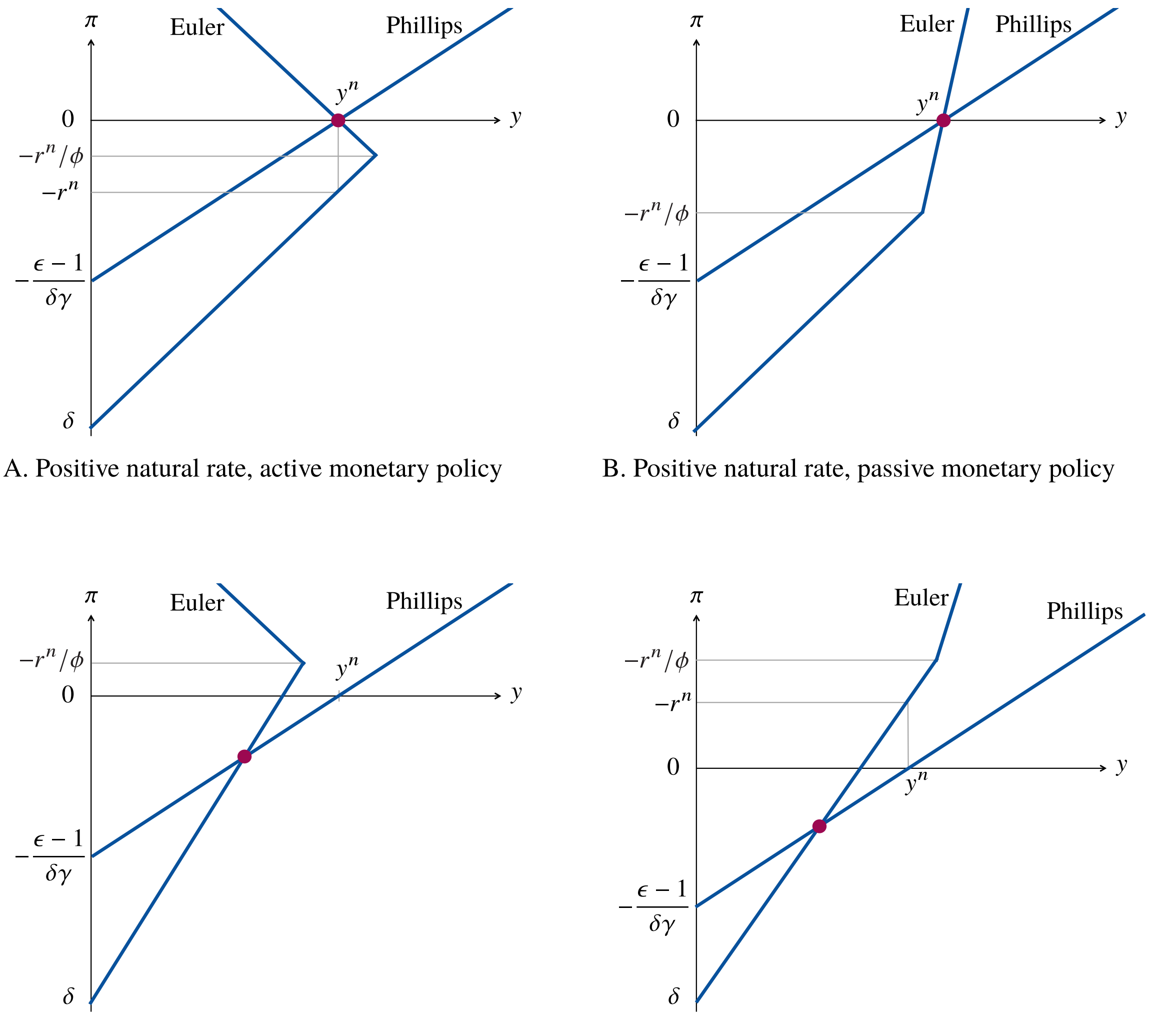
The monetary-policy rule implies a real interest rate

$$r(\pi) = \max\{-\pi, r^n + (\phi - 1)\pi\}. \tag{A28}$$

A steady-state equilibrium must satisfy the steady-state Phillips curve, given by (9), and the steady-state Euler equation, given by (11), in which the real interest rate comes from (A28). The monetary-policy rule (A28) implies that $r(\pi) = -\pi$ if $\pi < -r^n/\phi$ and $r(\pi) = r^n + (\phi - 1)\pi$ if $\pi \geq -r^n/\phi$. Hence, we first search for potential steady states with $\pi \geq -r^n/\phi$, before turning to potential steady states with $\pi < -r^n/\phi$.

We start with the range $\pi \geq -r^n/\phi$. In that range, $r(\pi) = r^n + (\phi - 1)\pi$, so the steady-state Phillips curve and Euler equation form the linear system (A19)–(A20). As argued in the proof of proposition 1, $[y, \pi] = [y^n, 0]$ is the unique solution of the system. But that solution is a steady state only when $\pi \geq -r^n/\phi$. Since $\pi = 0$ and $\phi > 0$, that solution is a steady state only when $r^n \geq 0$. When $r^n < 0$, there is no steady state in the range $\pi \geq -r^n/\phi$.

We then turn to the range $\pi < -r^n/\phi$. In that range, $r(\pi) = -\pi$, so the steady-state Phillips curve and Euler equation form the linear system (A21)–(A22). As argued in the proof of proposition 2, the system admits a unique solution $[y, \pi] \in \mathbb{R}^2$. Inflation in that unique solution is given by (A25). Condition (13) implies that $u'(0)\delta\gamma a/\epsilon > 1$, so π has the sign of r^n. For that solution to be a steady state, however, we need $\pi < -r^n/\phi$. Hence, as $\phi > 0$, the inequality $\pi < -r^n/\phi$ holds only when $r^n < 0$. This means that there is no steady state on the range $\pi < -r^n/\phi$ when $r^n \geq 0$, but there may be one when $r^n < 0$: the requirement is that $y \geq 0$. We



A. Positive natural rate, active monetary policy

B. Positive natural rate, passive monetary policy

C. Negative natural rate, active monetary policy

D. Negative natural rate, passive monetary policy

Figure A1. Steady State in the WUNK Model

Notes: π is inflation, y is output, y^n is the natural level of output, r^n is the natural rate of interest, and ϕ indicates the response of monetary policy to inflation ($\phi > 1$: active monetary policy; $\phi < 1$: passive monetary policy). The Phillips line represents the steady-state Phillips curve, given by (9). The Euler curve represents the steady-state Euler equation for the WUNK model, given by (11) and (A28) with the marginal utility of wealth satisfying (13). The intersection of the Phillips line and Euler curve gives the steady state of the WUNK model. The four panels describe the steady state in four different situations: positive versus negative natural rate of interest, and active versus passive monetary policy. The main result illustrated in the figure is that in the WUNK model the steady state is unique. If the natural rate of interest is positive, the ZLB is not binding and the steady state has zero inflation and output at its natural level. If the natural rate of interest is negative, the ZLB is binding and the steady state has deflation and output below its natural level.

have seen in the proof of proposition 2 that this requirement is always satisfied; therefore, the solution to the system (A21)–(A22) is a steady state when $r^n < 0$. This steady state has $\pi < 0$ (from (A25) and $r^n < 0$) and $y < y^n$ (from (A21) and $\pi < 0$).

These results are summarized in the following proposition:

PROPOSITION A1: *Under the monetary-policy rule* (A27)*, the WUNK model admits a unique steady-state equilibrium. If the natural rate of interest is positive, the unique steady state is the natural steady state, with zero inflation, output at its natural level, and nominal interest rate at the natural rate of interest. If the natural rate of interest is negative, the unique steady state is a ZLB steady state, with deflation, output below its natural level, and zero nominal interest rate.*

These results are illustrated in figure A1. The Phillips line represents the steady-state Phillips curve (9): it is linear, upward sloping, and goes through the point $[y, \pi] = [y^n, 0]$.

The Euler curve represents the steady-state Euler equation (11) combined with the interest-rate rule (A28). The Euler curve is composed of two linear branches because (A28) defines the real interest rate as a piecewise linear function of inflation. The non-ZLB branch is defined for $\pi > -r^n/\phi$ by (11) and $r = r^n + (\phi - 1)\pi$ (corresponding to $i = r^n + \phi\pi$). The ZLB branch is defined for $\pi < -r^n/\phi$ by (11) and $r = -\pi$ (corresponding to $i = 0$). The non-ZLB branch is downward sloping when monetary policy is active, and upward sloping when monetary is passive. Because of (13), when it is upward sloping, the non-ZLB branch is always steeper than the Phillips line. The non-ZLB branch (or its continuation) goes through the point $[y, \pi] = [y^n, 0]$. The ZLB branch is upward sloping and steeper than the Phillips line due to (13). The condition $\delta > \sqrt{(\epsilon - 1)/\gamma}$ ensures that the intercept of the ZLB branch with the y-axis is below the intercept of the Phillips line. Finally, the ZLB branch (or its continuation) goes through the point $[y, \pi] = [y^n, -r^n]$.

Consequently, if $r^n > 0$, then $[y^n, 0]$ is the unique steady state, whether monetary policy is active (panel A) or passive (panel B). This is the natural steady state, at the intersection of the Phillips line and non-ZLB Euler branch. If $r^n < 0$, on the other hand, $[y^n, 0]$ is not a steady state. Instead, the unique steady state is a ZLB steady state, at the intersection of the Phillips line and the ZLB Euler branch. This steady state features deflation and below-natural output. Such a ZLB steady state may prevails with active monetary policy away from the ZLB (panel C) or passive monetary policy away from the ZLB (panel D).

To conclude, we note that the results in this appendix are less robust than the results in the main text. The uniqueness of the steady state proven here relies on the functional-form assumptions of log utility over consumption and linear disutility of labor. These assumptions lead to the linearity of the steady-state Phillips curve and Euler equation, which ensures the uniqueness of the steady state. With general isoelastic utility functions instead, the steady-state

Phillips curve and Euler equation would be nonlinear, and several new steady states may appear. The results in the text, on the other hand, would continue to hold with general isoelastic utility functions, as long as the analysis concentrates on the neighborhood of the natural steady state (when the natural rate of interest is positive) or the closest ZLB steady state (when the natural rate of interest is negative).

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