At the zero lower bound, the New Keynesian model predicts that output and inflation collapse to implausibly low levels, and that government spending and forward guidance have implausibly large effects. To resolve these anomalies, we introduce wealth into the utility function; the justification is that wealth is a marker of social status, and people value social status. Since people save not only for future consumption but also to accrue social status, the Euler equation is modified. As a result, when the marginal utility of wealth is sufficiently large, the dynamical system representing the equilibrium at the zero lower bound transforms from a saddle to a source—which resolves all the anomalies.

1. Introduction

A current issue in monetary economics is that the New Keynesian model makes several anomalous predictions when the zero lower bound on nominal interest rates (ZLB) is binding: implausibly large collapse of output and inflation (Eggertsson and Woodford 2004; Eggertsson 2011; Werning 2011);
implausibly large effect of forward guidance (Del Negro, Giannoni, and Patterson 2015; Carlstrom, Fuerst, and Paustian 2015; Cochrane 2017); and implausibly large effect of government spending (Christiano, Eichenbaum, and Rebelo 2011; Woodford 2011; Cochrane 2017).

Several papers have developed variants of the New Keynesian model that behave well at the ZLB (Gabaix 2016; Diba and Loisel 2019; Cochrane 2018; Bilbiie 2019; Acharya and Dogra 2019). But these variants are more complex than the standard model. In some cases the derivations are complicated by bounded rationality or heterogeneity. In other cases the dynamical system representing the equilibrium—normally composed of an Euler equation and a Phillips curve—must include additional differential equations to describe bank-reserve dynamics, price-level dynamics, or the evolution of the wealth distribution. Moreover, a good chunk of the description and resolution of the anomalies is conducted by numerical simulations. Hence, it is sometimes difficult to grasp the nature of the anomalies and their resolutions.

It may therefore be valuable to strip the logic to the bone. We do so using a New Keynesian model in which relative wealth enters the utility function. The justification for the assumption is that relative wealth is a marker of social status, and people value high social status. We deviate from the standard model only minimally: the derivations are the same; the equilibrium is described by a dynamical system composed of an Euler equation and a Phillips curve; the only difference is an extra term in the Euler equation. We also veer away from numerical simulations and establish our results with phase diagrams describing the dynamics of output and inflation given by the Euler-Phillips system. The model’s simplicity and the phase diagrams will allow us to gain new insights into the anomalies and their resolutions.¹

Using the phase diagrams, we begin by depicting the anomalies. First, we find that output

¹Our approach relates to the work of Michaillat and Saez (2014), Ono and Yamada (2018), and Michau (2018). By assuming wealth in the utility function, they obtain non-New-Keynesian models that behave well at the ZLB. But their results are not portable to the New Keynesian framework because they require strong forms of wage or price rigidity (exogenous wages, fixed inflation, or downward nominal wage rigidity). Our approach also relates to the work of Fisher (2015) and Campbell et al. (2017), who build New Keynesian models with government bonds in the utility function. The bonds-in-the-utility assumption captures special features of government bonds relative to other assets, such as safety and liquidity (for example, Krishnamurthy and Vissing-Jorgensen 2012). While their assumption and ours are conceptually different, they affect equilibrium conditions in a similar way. These papers use their assumption to generate risk-premium shocks (Fisher) and to alleviate the forward-guidance puzzle (Campbell et al.).
and inflation become unboundedly low when the ZLB episode is arbitrarily long-lasting. Second, we find that there is a duration of forward guidance above which any ZLB slump, irrespective of its duration, is transformed into a boom. Such boom is unbounded when the ZLB episode is arbitrarily long-lasting. Third, we find that there is an amount of government spending at which the government-spending multiplier becomes infinite when the ZLB episode lasts an arbitrarily long time. Furthermore, if government spending exceeds this amount, the economy experiences an unbounded boom when the ZLB episode is arbitrarily long-lasting.

The phase diagrams also pinpoint the origin of the anomalies: they arise because the Euler-Phillips system is a saddle at the ZLB. In normal times, in contrast, the Euler-Phillips system is source, so anomalies are absent. In economic terms, the anomalies arise because household consumption (given by the Euler equation) responds too strongly to the real interest rate. Indeed, since the only motive for saving is future consumption, households are very forward-looking, and their response to interest rates is strong.

Once wealth enters the utility function, however, people save not only for future consumption but also because they enjoy holding wealth in itself. They are less forward-looking, so their consumption responds less to future interest rates. With enough marginal utility of wealth, all the anomalies disappear. Mathematically, with enough marginal utility of wealth, the Euler equation is sufficiently “discounted”—in the sense of McKay, Nakamura, and Steinsson (2017)—that at the ZLB the Euler-Phillips system transforms from a saddle to a source. At that point the response to a temporary shock, however long-lasting, is a muted version of the response to a permanent shock, which is finite, so the anomalies disappear.

Indeed, our model makes reasonable predictions at the ZLB. First, output and inflation never collapse: they are bounded below by the ZLB steady state. Second, irrespective of the duration of forward guidance, when the ZLB episode is long enough, the economy necessarily experiences a slump. Third, government-spending multipliers are always finite, irrespective of the duration of the ZLB episode.

Beside its anomalies, the New Keynesian model has several other intriguing properties at the
ZLB—some labeled “paradoxes” because they defy usual economic logic (Eggertsson 2010; Werning 2011; Eggertsson and Krugman 2012). Our model shares these properties. First, the paradox of thrift holds: when households desire to save more than their neighbors, the economy contracts and they end up saving the same amount as the neighbors. The paradox of toil also holds: when households desire to work more, the economy contracts and they end up working less. The paradox of flexibility is present too: the economy contracts when prices become more flexible. Last, the government-spending multiplier is above one, so government spending stimulates private consumption.

2. Justification for wealth in the utility function

Before delving into the model, we justify our assumption of wealth in the utility function.

The standard model assumes that people save to smooth consumption over time, but it has long been recognized that people seem to enjoy accumulating wealth irrespective of future consumption. Describing the European upper class of the early 20th century, Keynes (1919, chap. 2) noted that “The duty of saving became nine-tenths of virtue and the growth of the cake the object of true religion. . . . Saving was for old age or for your children; but this was only in theory—the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you.” Irving Fisher added that “A man may include in the benefits of his wealth . . . the social standing he thinks it gives him, or political power and influence, or the mere miserly sense of possession, or the satisfaction in the mere process of further accumulation” (Fisher 1930, p. 17). Fisher’s perspective is interesting since he developed the theory of saving based on consumption smoothing.

Neuroscientific evidence confirms that wealth itself provides utility, independently of the consumption it can buy. Camerer, Loewenstein, and Prelec (2005, p. 32) note that “brain-scans conducted while people win or lose money suggest that money activates similar reward areas as do other ‘primary reinforcers’ like food and drugs, which implies that money confers direct utility, rather than simply being valued only for what it can buy.”

Among all the reasons why people may value wealth, we focus on social status: we postulate that people enjoy wealth because it provides social status; we therefore introduce relative (not absolute)
Wealth into the utility function. The assumption is convenient: in equilibrium everybody is the same, so relative wealth is zero. And the assumption seems plausible. Adam Smith, Ricardo, John Rae, J.S. Mill, Marshall, Veblen, and Frank Knight all believed that people accumulate wealth to attain high social status (Steedman 1981). More recently, a broad literature has documented that people seek to achieve high social status, and that accumulating wealth is a prevalent pathway to do so (Weiss and Fershtman 1998; Heffetz and Frank 2011; Fiske 2010; Anderson, Hildreth, and Howland 2015; Cheng and Tracy 2013; Ridgeway 2014; Mattan, Kubota, and Cloutier 2017).

3. New Keynesian model with wealth in the utility function

We extend the New Keynesian model by assuming that households derive utility not only from consumption and leisure but also from relative wealth. To simplify derivations and be able to represent the equilibrium with phase diagrams, we use an alternative formulation of the New Keynesian model, inspired by Benhabib, Schmitt-Grohe, and Uribe (2001) and Werning (2011). Our formulation features continuous time instead of discrete time; self-employed households instead of firms and households; and Rotemberg (1982) pricing instead of Calvo (1983) pricing.

3.1. Assumptions

The economy is composed of a measure 1 of self-employed households. Each household \( j \in [0, 1] \) produces \( y_j(t) \) units of a good \( j \) at time \( t \), sold to other households at a price \( p_j(t) \). The household’s production function is \( y_j(t) = ah_j(t) \), where \( a > 0 \) represents the level of technology, and \( h_j(t) \) is hours of work. Working causes a disutility \( \kappa h_j(t) \), where \( \kappa > 0 \) is the marginal disutility of labor.

The goods produced by households are imperfect substitutes for one another, so each household

---

2 Cole, Mailath, and Postlewaite (1992, 1995) develop models in which relative wealth does not directly confer utility but has other attributes such that people behave as if wealth entered their utility function. In one such model, wealthier individuals have higher social rankings, which allows them to marry wealthier partners and enjoy higher utility.

3 The wealth-in-the-utility assumption has been found useful in models of long-run growth (Kurz 1968; Konrad 1992; Zou 1994; Corneo and Jeanne 1997; Futagami and Shibata 1998), risk attitudes (Robson 1992; Clemens 2004), asset pricing (Bakshi and Chen 1996; Gong and Zou 2002; Kamihigashi 2008; Michau, Ono, and Schlegl 2018), life-cycle consumption (Zou 1995; Carroll 2000; Francis 2009; Straub 2019), social stratification (Long and Shimomura 2004), international macroeconomics (Fisher 2005; Fisher and Hof 2005), financial crises (Kumhof, Ranciere, and Winant 2015), and optimal taxation (Saez and Stantcheva 2018). Such usefulness lends further support to the assumption.
exercises some monopoly power. Moreover, households face a quadratic cost when they change their price: changing a price at a rate \( \pi_j(t) = \dot{p}_j(t)/p_j(t) \) causes a disutility \( \gamma \pi_j(t)^2/2 \). The parameter \( \gamma > 0 \) governs the cost to change prices and thus price rigidity.

Each household consumes goods produced by other households. Household \( j \) buys quantities \( c_{jk}(t) \) of the goods \( k \in [0, 1] \). These quantities are aggregated into a consumption index \( c_j(t) = [\int_0^1 c_{jk}(t)^{\epsilon-1}/\epsilon \, dk]^{\epsilon/(\epsilon-1)} \), where \( \epsilon > 1 \) is the elasticity of substitution between goods. The consumption index yields utility \( \ln(c_j(t)) \). Given the consumption index, the relevant price index is \( p(t) = [\int_0^1 p_j(t)^{1-\epsilon} \, dt]^{1/(1-\epsilon)} \). When households optimally allocate their consumption expenditure across goods, \( p(t) \) is the price of one unit of consumption index. The inflation rate is defined as \( \pi(t) = \dot{p}(t)/p(t) \).

Households save using government bonds. Since we postulate that people derive utility from their relative real wealth, and since bonds are the only store of wealth, holding bonds provides direct utility. Formally, holding a nominal quantity of bonds \( b_j(t) \) yields utility \( u((b_j(t) - b(t))/p(t)) \). The function \( u : \mathbb{R} \to \mathbb{R} \) is increasing and concave, \( b(t) = \int_0^1 b_k(t) \, dk \) is average nominal wealth, and \( (b_j(t) - b(t))/p(t) \) is the household \( j \)'s relative real wealth.

Bonds earn a nominal interest rate \( i^h(t) = i(t) + \sigma(t) \), where \( i(t) \geq 0 \) is the nominal interest rate set by the central bank, and \( \sigma(t) \geq 0 \) is a spread between the monetary-policy rate \( (i(t)) \) and the rate used by households for saving decisions \( (i^h(t)) \). The spread \( \sigma(t) \) captures the efficiency of financial intermediation (Woodford 2011); the spread is large when financial intermediation is severely disrupted, as during the Great Depression or Great Recession. The law of motion of bond holdings is
\[
\dot{b}_j(t) = i^h(t)b_j(t) + p_j(t)y_j(t) - \int_0^1 p_k(t)c_{jk}(t) \, dk - \tau(t).
\]
The term \( i^h(t)b_j(t) \) is interest income; \( p_j(t)y_j(t) \) is labor income; \( \int_0^1 p_k(t)c_{jk}(t) \, dk \) is consumption expenditure; and \( \tau(t) \) is a lump-sum tax (used among other things to service government debt).

Lastly, the problem of household \( j \) is to choose time paths for \( y_j(t), p_j(t), h_j(t), \pi_j(t), c_{jk}(t) \) for
all \( k \in [0, 1] \), and \( b_j(t) \) to maximize the discounted sum of instantaneous utilities

\[
\int_0^{\infty} e^{-\delta t} \left[ \ln(c_j(t)) + u\left( \frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)} \right) - \kappa h_j(t) - \frac{y}{2} \pi_j(t)^2 \right] dt,
\]

where \( \delta > 0 \) is the time discount rate. The household faces four constraints: the production function; the budget constraint; the law of motion \( \dot{p}_j(t) = \pi_j(t)p_j(t) \); and the demand for good \( j \) coming from other households’ maximization: \( y_j(t) = \left[ p_j(t)/p(t) \right]^{-\epsilon} c(t) \), where \( c(t) = \int_0^1 c_k(t) \, dk \) is aggregate consumption. The household also faces a borrowing constraint preventing Ponzi schemes. The household takes as given aggregate variables, initial wealth \( b_j(0) \), and initial price \( p_j(0) \). All households are assumed to face the same initial conditions, so they will all behave the same.

### 3.2. Euler equation and Phillips curve

The equilibrium is described by a system of two differential equations: an Euler equation and a Phillips curve. The Euler-Phillips system governs the dynamics of output \( y(t) \) and inflation \( \pi(t) \). Here we present the system; formal and heuristic derivations are in appendices A and B; a discrete-time version is in appendix C.

The Phillips curve arises from households’ optimal pricing decisions:

\[
\hat{\pi}(t) = \delta \pi(t) - \frac{\epsilon \kappa}{\gamma a} \left[ y(t) - y^n \right],
\]

where

\[
y^n = \frac{\epsilon - 1}{\epsilon} \cdot \frac{a}{\kappa}.
\]

The Phillips curve is not modified by wealth in the utility function.

The steady-state Phillips curve, obtained by setting \( \hat{\pi} = 0 \) in (1), describes inflation as a linearly
increasing function of output:

\[ \pi = \frac{\epsilon \kappa}{\delta y} (y - y^n). \]

We see that \( y^n \) is the natural level of output: the level at which producers keep their prices constant.

The Euler equation arises from households’ optimal saving decisions:

\[ \frac{\dot{y}(t)}{y(t)} = r(t) - r^n + u'(0) [y(t) - y^n], \]

where \( r(t) = i(t) - \pi(t) \) is the real monetary-policy rate and

\[ r^n = \delta - \sigma - u'(0)y^n. \]

The real rate \( r(t) \) governs the financial return on saving. It therefore is a key determinant of optimal saving and of the growth rate of consumption, \( \dot{y}(t)/y(t) \). Unlike in the standard model, people also enjoy holding wealth, so a new term appears in the equation to capture the hedonic return on saving: the marginal rate of substitution between wealth and consumption, \( u'(0)y(t) \). In the marginal rate of substitution, the marginal utility of wealth is \( u'(0) \) because in equilibrium all households hold the same wealth so relative wealth is zero; the marginal utility of consumption is \( 1/y(t) \) because consumption utility is log. Because consumption depends not only on interest rates but also on the marginal rate of substitution between wealth and consumption, future interest rates have less impact on today’s consumption than in the standard model: the Euler equation is discounted. In fact, the discrete-time version of our Euler equation features discounting exactly as in McKay, Nakamura, and Steinsson (2017) (see appendix C).

The steady-state Euler equation is obtained by setting \( \dot{y} = 0 \) in (4):

\[ u'(0)(y - y^n) = r^n - r. \]
When $u'(0) > 0$, the equation describes output as a linearly decreasing function of the real monetary-policy rate—as in the old-fashioned, Keynesian IS curve. We also see that $r^n$ is the natural rate of interest: the real policy rate at which households consume a quantity $y^n$.

The wealth-in-the-utility assumption only adds one parameter to the equilibrium conditions: $u'(0)$. We compare two submodels:

**Definition 1.** The New Keynesian (NK) model has zero marginal utility of wealth: $u'(0) = 0$. The wealth-in-the-utility New Keynesian (WUNK) model has sufficient marginal utility of wealth:

$$\frac{\epsilon K}{\delta y a}$$

The NK model is the standard model; the WUNK model is the extension proposed in this paper. When prices are fixed ($\gamma \to \infty$), condition (7) becomes $u'(0) > 0$; when prices are perfectly flexible ($\gamma = 0$), condition (7) becomes $u'(0) > \infty$. Hence, at the fixed-price limit, the WUNK model only requires an infinitesimal marginal utility of wealth; at the flexible-price limit, the WUNK model is not well-defined. In the WUNK model we also impose $\delta > \sqrt{\epsilon (\epsilon - 1) / \gamma}$ in order to accommodate positive natural rates of interest.\(^4\)

### 3.3. Natural rate of interest and monetary policy

The central bank aims to maintain the economy at the natural steady state, where inflation is at zero and output at its natural level.

In normal times, the natural rate of interest is positive, and the central bank is able to maintain

\[^4\text{Indeed, using (2) and (7), we see that in the WUNK model} \]

$$\frac{u'(0)y^n}{\delta} > \frac{1}{\delta^2} \cdot \frac{\epsilon - 1}{\gamma}.$$  

This implies that the natural rate of interest, $r^n = \delta \left[1 - u'(0)y^n / \delta\right]$, is bounded above:

$$r^n < \delta \left[1 - \frac{1}{\delta^2} \cdot \frac{\epsilon - 1}{\gamma}\right].$$

For the WUNK model to accommodate positive natural rates of interest, the upper bound on the natural rate must be positive, which requires $\delta > \sqrt{\epsilon (\epsilon - 1) / \gamma}$. 

---

9
the economy at the natural steady state using the simple policy rule \(i(\pi(t)) = r^n + \phi\pi(t)\). The corresponding real policy rate is \(r(\pi(t)) = r^n + (\phi - 1)\pi(t)\). The parameter \(\phi \geq 0\) governs the response of interest rates to inflation: monetary policy is active when \(\phi > 1\) and passive when \(\phi < 1\).

When the natural rate of interest is negative, however, the natural steady state cannot be achieved—because this would require the central bank to set a negative nominal policy rate, which would violate the ZLB. In that case, the central bank moves to the ZLB: \(i(t) = 0\), so \(r(t) = -\pi(t)\).

What could cause the natural rate of interest to be negative? A first possibility is a banking crisis, which disrupts financial intermediation and raises the interest-rate spread (Woodford 2011; Eggertsson 2011). When the spread is large enough \((\sigma > \delta - u'(0)y^n)\), the natural rate of interest turns negative. Another possibility in the WUNK model is drop in consumer sentiment, which leads households to favor saving over consumption, and can be parameterized by an increase in the marginal utility of wealth. When the marginal utility is large enough \((u'(0) > (\delta - \sigma)/y^n)\), the natural rate of interest turns negative.

### 3.4. Properties of the Euler-Phillips system

We now study the dynamical system generated by the Euler equation (4) and Phillips curve (1). The Euler-Phillips system governs the dynamics of output and inflation. We contrast normal times and the ZLB, as well as the NK and WUNK models.

To establish the properties of the Euler-Phillips system, we construct its phase diagrams in various situations. The diagrams are displayed in figure 1. We omit phase diagrams for passive monetary policy because they would be similar to the ZLB diagrams. (Appendix D establishes the properties using an algebraic approach.)

The first step in constructing the phase diagrams is to plot the locus \(\dot{y} = 0\) and the locus \(\dot{\pi} = 0\). The locus \(\dot{y} = 0\) is given by the steady-state Euler equation (6), and is represented by the line labeled “Euler.” The locus \(\dot{\pi} = 0\) is given by the steady-state Phillips curve (3), and is represented by the line labeled “Phillips.”

The Phillips line is the same in the NK and WUNK models, and in normal times and at the ZLB:
it is upward-sloping and goes through the point \([y = y^n, \pi = 0]\). The Euler line, on the other hand, is different in each case.

In the NK model, the Euler line is horizontal because when \(u'(0) = 0\), (6) imposes that \(r(\pi) = r^n\), which determines inflation independently of output (panels A and C). In the WUNK model, the Euler line is not horizontal because when \(u'(0) > 0\), (6) makes output a decreasing function of the real policy rate \(r(\pi)\). When monetary policy is active (\(\phi > 1\)), \(r(\pi) = r^n + (\phi - 1)\pi\) is increasing in \(\pi\), so the Euler line slopes downward (panel B); at the ZLB, \(r(\pi) = -\pi\) is decreasing in \(\pi\), so the Euler line slopes upward (panel D).

The Euler line also changes between normal times and the ZLB, because the natural rate of interest and monetary policy change. In the NK model the Euler line shifts up from \(\pi = 0\) in normal times to \(\pi = -r^n > 0\) at the ZLB (panels A and C). In the WUNK model, in normal times with active monetary policy, the Euler line is

\[
\pi = -\frac{u'(0)}{\phi - 1}(y - y^n),
\]

so it is downward sloping and goes through the point \([y = y^n, \pi = 0]\) (panel B). At the ZLB the Euler line is

(8) \[
\pi = -r^n + u'(0)(y - y^n),
\]

so it is upward sloping and goes through the point \([y = y^n + r^n/u'(0), \pi = 0]\) (panel D). Since \(r^n \leq 0\) at the ZLB, the ZLB Euler line is always inward of the point \([y = y^n, \pi = 0]\), explaining why the central bank is unable to achieve the natural steady state. Further, since the slope of the Phillips line is \(\epsilon\kappa/(\delta\gamma a)\) and the slope of the ZLB Euler line is \(u'(0)\), the WUNK condition (7) ensures that the ZLB Euler line is steeper than the Phillips line.

In normal times, in both models, the Phillips and Euler lines intersect at the point \([y = y^n, \pi = 0]\); therefore, the Euler-Phillips system admits a unique steady state with zero inflation and natural output (panels A and B). At the ZLB, in both models, the Phillips and Euler lines have a unique
intersection, so the Euler-Phillips system admits a unique steady state. In the NK model, the steady state has positive inflation and above-natural output (panel C); in the WUNK model, the steady state has negative inflation and below-natural output (panel D).

The second step in constructing the phase diagrams is to plot the arrows giving the directions of the trajectories satisfying the Euler-Phillips system. We first determine the sign of $\dot{\pi}$, which is given by the Phillips curve (1). Any point above the Phillips line (where $\dot{\pi} = 0$) has $\dot{\pi} > 0$, and any point below the line has $\dot{\pi} < 0$. So in all the panels inflation is rising above the Phillips line and falling below it.

Next we examine the sign of $\dot{y}$, which is given by the Euler equation (4). In the NK model in normal times with active monetary policy (panel A), the Euler equation is

$$\frac{\dot{y}}{y} = (\phi - 1)\pi$$

with $\phi > 1$. Hence any point above the Euler line (where $\dot{y} = 0$) has $\dot{y} > 0$, and any point below the line has $\dot{y} < 0$. Accordingly, in the four quadrants delimited by the Phillips and Euler lines, the trajectories move away from the steady state: the Euler-Phillips system a source.

In the NK model at the ZLB (panel C), the Euler equation is

$$\frac{\dot{y}}{y} = -\pi - r^n.$$  

Hence any point above the Euler line has $\dot{y} < 0$, and any point below the line has $\dot{y} > 0$. We conclude that the Euler-Phillips system is a saddle, because in the southwest and northeast quadrants the trajectories move toward the steady state, whereas in the southeast and northwest quadrants the trajectories move away from it.

In the WUNK model in normal times with active monetary policy (panel B), the Euler equation is

$$\frac{\dot{y}}{y} = (\phi - 1)\pi + u'(0)(y - y^n)$$

In the WUNK model we also check that the intersection has positive output (appendix D).
A. NK model: normal times, active monetary policy

B. WUNK model: normal times, active monetary policy

C. NK model: ZLB

D. WUNK model: ZLB

**Figure 1. Phase diagrams of the Euler-Phillips system**

Notation: $\pi$ is inflation; $y$ is output; $y^n$ is the natural level of output. The phase diagrams represent the dynamical system generated by the Euler equation (4) and Phillips curve (1) in various cases. The Euler line is the locus $\dot{y} = 0$; the Phillips line is the locus $\dot{\pi} = 0$. The trajectories are solutions to the Euler-Phillips system linearized around its steady state. The NK model is the standard New Keynesian model. The WUNK model is the same as the NK model, except that wealth enters the utility function, and the marginal utility of wealth is sufficiently large to satisfy (7). In normal times, the natural rate of interest ($r^n$) is positive, and the nominal interest rate is given by $i = r^n + \phi \pi$; when monetary policy is active, $\phi > 1$. At the ZLB, the natural rate of interest is negative, and the monetary-policy rate is zero. Panel A shows that in the NK model, in normal times, the Euler-Phillips system is a source when monetary policy is active. Panel C shows that in the NK model the Euler-Phillips system is a saddle at the ZLB. Panels B and D show that in the WUNK model the Euler-Phillips system is a source both in normal times and at the ZLB. (In panels A and B, we have plotted a nodal source, but the system could also be a spiral source, depending on the value of $\phi$; in panel D the system is always a nodal source.)
with $\phi > 1$. Accordingly, any point above the Euler line has $\dot{y} > 0$, and any point below the line has $\dot{y} < 0$. In all four quadrants the trajectories move away from the steady state, so the Euler-Phillips system is a source.

Last, in the WUNK model at the ZLB (panel D), the Euler equation is

$$\frac{\dot{y}}{y} = -\pi - r^n + u'(0)(y - y^n).$$

Any point above the Euler line has $\dot{y} < 0$, and any point below it has $\dot{y} > 0$. Thus in all four quadrants the trajectories move away from the steady state: the Euler-Phillips system remains a source.

For completeness, we also plot sample solutions to the Euler-Phillips system. The trajectories are obtained by linearizing the Euler-Phillips system at its steady state, so technically the trajectories only approximate the exact solutions; but the approximation is very accurate in the neighborhood of the steady state. When the system is a source, there are two unstable lines—trajectories that move away from the steady state in a straight line. At $t \to -\infty$, all other trajectories are in the vicinity of the steady state and leave tangentially to one unstable line. At $t \to +\infty$, the trajectories move to infinity parallel to the second unstable line. When the system is a saddle, there is one stable line—a trajectory that goes to the steady state in a straight line—and one unstable line—a trajectory that escapes from the steady state in a straight line. All other trajectories come from infinity parallel to the stable line when $t \to -\infty$, and move to infinity parallel to the unstable line when $t \to +\infty$.

The following propositions summarize the results:

**Proposition 1.** Consider the Euler-Phillips system in normal times. The system admits a unique steady state, where output is at its natural level, inflation is zero, and the ZLB is not binding. In the NK model, the system is a source when monetary policy is active but a saddle when monetary policy is passive. In the WUNK model, the system is a source whether monetary policy is active or passive.

**Proposition 2.** Consider the Euler-Phillips system at the ZLB. In the NK model, the system admits a unique steady state, where output is above its natural level and inflation is positive; furthermore, the system is a saddle. In the WUNK model, the system admits a unique steady state, where output is
below its natural level and inflation is negative; furthermore, the system is a source.

The propositions give the key difference between the NK and WUNK models: at the ZLB, the Euler-Phillips system remains a source in the WUNK model, whereas it becomes a saddle in the NK model. This difference will explain why the WUNK model does not suffer from the anomalies plaguing the NK model at the ZLB. The phase diagrams also illustrate the origin of the WUNK condition (7). In the WUNK model, the Euler-Phillips system remains a source at the ZLB as long as the Euler line is steeper than the Phillips line (panel D of figure 1). The Euler line’s slope at the ZLB is the marginal utility of wealth, so that marginal utility is required to be above a certain level—which is given by (7).

The propositions have implications for equilibrium determinacy. When the Euler-Phillips system is a source, the equilibrium is determinate: the only equilibrium trajectory in the vicinity of the steady state is to jump to the steady state and stay there; if the economy jumped somewhere else, output or inflation would diverge following a trajectory similar to those plotted in panels A, B, and D of figure 1. When the system is a saddle, the equilibrium is indeterminate: any trajectory jumping somewhere on the saddle path and converging to the steady state is an equilibrium, as illustrated in panel C of figure 1. Hence, in the NK model, the equilibrium is determinate when monetary policy is active but indeterminate when monetary policy is passive and at the ZLB. In the WUNK model, the equilibrium is always determinate, even when monetary policy is passive and at the ZLB.

Accordingly, in the NK model, the Taylor principle holds: the central bank must adhere to an active monetary policy to avoid indeterminacy. From now on, we therefore assume that the NK central bank respects the Taylor principle whenever it can \((\phi > 1 \text{ whenever } r^n > 0)\). In the WUNK model, by contrast, indeterminacy is never a risk, so the central bank does not need to worry about how strongly its policy rate responds to inflation. The central bank could even follow an interest-rate peg without creating indeterminacy.

The NK results in the propositions are well-known (for example, Woodford 2001). The WUNK results are similar to those obtained by Gabaix (2016). He finds that when bounded rationality is strong enough, the Euler-Phillips system is a source even at the ZLB. He also finds that when prices
are more flexible, more bounded rationality is required to maintain the source property. The same is true here: when the price-adjustment cost $\gamma$ is lower, a higher marginal utility of wealth is required for (7) to hold. The logic is illustrated in panel D of figure 1. The system remains a source at the ZLB as long as the Euler line is steeper than the Phillips line. When prices are more flexible, the Phillips line becomes steeper, and the required steepness for the Euler line increases. As the slope of the Euler line is determined by bounded rationality in the Gabaix model and by marginal utility of wealth in our model, these need to be larger when prices are more flexible.

4. Description and resolution of the New Keynesian anomalies

We now describe the anomalous predictions of the NK model at the ZLB: implausibly large drop in output and inflation; and implausibly strong effects of forward guidance and government spending. We then show that these anomalies are absent from the WUNK model.

4.1. Drop in output and inflation

We consider a temporary ZLB episode, as in Werning (2011). Between times 0 and $T > 0$, the natural rate of interest is negative. In response, the central bank maintains its policy rate at zero. After time $T$, the natural rate is positive again, and monetary policy returns to normal. This scenario is summarized in panel A of table 1.

We start with the NK model. We analyze the ZLB episode by going backward in time, using the phase diagrams in panels A and C of figure 2. After time $T$, monetary policy maintains the economy at the natural steady state. Since equilibrium trajectories must be continuous, the economy must be at the natural steady state at the end of the ZLB, when $t = T$.

We then move back to the ZLB episode, when $t < T$. At time 0, the economy jumps to the unique position leading to $[y = y^n, \pi = 0]$ at time $T$. Hence, inflation and output initially jump down to $\pi(0) < 0$ and $y(0) < y^n$. After their initial collapse, inflation and output recover following the unique trajectory leading to $[y = y^n, \pi = 0]$. The ZLB therefore creates a slump, with below-natural output and deflation (panel A).
TABLE 1. ZLB scenarios

<table>
<thead>
<tr>
<th>A. ZLB episode</th>
<th>Timeline</th>
<th>Natural rate of interest</th>
<th>Monetary policy</th>
<th>Government spending</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZBL: ( t \in (0, T) )</td>
<td>( r^n &lt; 0 )</td>
<td>( i = 0 )</td>
<td>–</td>
<td></td>
</tr>
<tr>
<td>Normal times: ( t &gt; T )</td>
<td>( r^n &gt; 0 )</td>
<td>( i = r^n + \phi\pi )</td>
<td>–</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. ZLB episode with forward guidance</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZBL: ( t \in (0, T) )</td>
</tr>
<tr>
<td>Forward guidance: ( t \in (T, T + \Delta) )</td>
</tr>
<tr>
<td>Normal times: ( t &gt; T + \Delta )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C. ZLB episode with government spending</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZBL: ( t \in (0, T) )</td>
</tr>
<tr>
<td>Normal times: ( t &gt; T )</td>
</tr>
</tbody>
</table>

Critically, the economy is always on the same trajectory during the ZLB, irrespective of the ZLB duration \( T \). A longer ZLB only forces output and inflation to start from a lower position on the trajectory at time 0. Thus, as the ZLB lasts longer, initial output and inflation become unboundedly low (panel C).

In the WUNK model output and inflation never collapse during the ZLB, as shown by the phase diagrams in panels B and D of figure 2. Initially inflation and output jump down toward the ZLB steady state, but not all the way: \( \pi^z < \pi(0) < 0 \) and \( y^z < y(0) < y^n \). Then they recover, following the unique trajectory going through \([y = y^n, \pi = 0]\). Consequently the ZLB episode creates a slump (panel B), which is deeper when the ZLB lasts longer (panel D).

But unlike in the NK model, output and inflation are bounded below: irrespective of the length of the ZLB, they always fall less than if the ZLB were permanent. Moreover, if the natural rate of interest is negative but close to 0, such that \( \pi^z \) is close to 0 and \( y^z \) to \( y^n \), output and inflation barely deviate from the natural steady state during the ZLB, even if the ZLB lasts a very long time.

The following proposition records these results: 6

PROPOSITION 3. Consider a ZLB episode between times 0 and T. The economy enters a slump:

6The result that in the NK model output becomes infinitely negative when the ZLB becomes infinitely long should not be interpreted literally. It is obtained because we omitted the constraint that output must remain positive. The proper interpretation is that output falls much, much below its natural level—in fact it converges to zero.
The phase diagrams in panels A and C describe the NK Euler-Phillips system at the ZLB; they come from panel C in figure 1. The phase diagrams in panels B and D describe the WUNK Euler-Phillips system at the ZLB; they come from panel D in figure 1. The ZLB is binding between times 0 and T; then, at time T, the central bank brings the economy to the natural steady state, where inflation is zero and output is at its natural level. The equilibrium trajectories are the unique trajectories reaching the natural steady state at time T. The economy slumps during the ZLB: inflation is negative and output is below its natural level. In the NK model, the initial slump becomes unboundedly large as the ZLB becomes longer. In the WUNK model, there is no such collapse: output and inflation are bounded below by the ZLB steady state.
\( y(t) < y^\ast \) and \( \pi(t) < 0 \) for all \( t \in (0, T) \). In the NK model, the slump becomes infinitely severe as the ZLB becomes infinitely long: \( \lim_{T \to \infty} y(0) = \lim_{T \to \infty} \pi(0) = -\infty \). In the WUNK model, in contrast, the slump is bounded below by the ZLB steady state: \( y(t) > y^z \) and \( \pi(t) > \pi^z \) for all \( t \in (0, T) \). In fact, the slump approaches the ZLB steady state as the ZLB becomes infinitely long: \( \lim_{T \to \infty} y(0) = y^z \) and \( \lim_{T \to \infty} \pi(0) = \pi^z \).

In the NK model, output and inflation collapse when the ZLB is long-lasting—which is well-known (Eggertsson and Woodford 2004, fig. 1; Eggertsson 2011, fig. 1; Werning 2011, proposition 1). This collapse is difficult to reconcile with real-world observations. The ZLB episode that started in 1995 in Japan has been lasting for more than twenty years without sustained deflation. The ZLB episode that started in 2009 in the euro area has been lasting for more than 10 years; it did not yield sustained deflation either. The same is true of the ZLB episode that occurred in the United States between 2008 and 2015.

In the WUNK model, in contrast, the ZLB slump is bounded below by the ZLB steady state. So inflation and output never collapse at the ZLB, even if the ZLB lasts a very long time. Instead, as the duration of the ZLB increases, the economy converges to the ZLB steady state. And the ZLB steady state may not be far from the natural steady state: if the natural rate of interest is only slightly negative at the ZLB, steady-state inflation is only slightly below zero and steady-state output only slightly below its natural level. Gabaix (2016) obtains the results closest to ours: in his model output and inflation also converge to the ZLB steady state when the ZLB is arbitrarily long.

### 4.2. Forward guidance

We turn to the effects of forward guidance at the ZLB. We consider a three-stage scenario, as in Cochrane (2017). Between times 0 and \( T \), there is a ZLB episode, exactly as above. To alleviate the situation, the central bank makes a forward-guidance promise at time 0: that it will maintain the policy rate at zero for a duration \( \Delta \) once the ZLB is over. After time \( T \), the natural rate of interest is positive again. Between times \( T \) and \( T + \Delta \), the central bank fulfills its forward-guidance promise and keeps the policy rate at zero. After time \( T + \Delta \), monetary policy returns to normal. This scenario
is summarized in panel B of table 1.

Forward guidance in the NK model is analyzed in figure 3, by going backward in time. After time $T + \Delta$, monetary policy maintains the economy at the natural steady state. Between times $T$ and $T + \Delta$, the economy is in forward guidance, as depicted in panel A. The phase diagram is the same as in panel C of figure 1, except that $r^n > 0$ instead of $r^n < 0$: the Euler line, given by $\pi = -r^n$, is lower. Following the logic of figure 2, we find that at time $T$, inflation must be positive and output above its natural level. They subsequently decrease over time, following the unique trajectory leading to the natural steady state at time $T + \Delta$. Accordingly, the economy booms during forward guidance: inflation is positive and output above its natural level. Furthermore, as the duration of forward guidance increases, inflation and output at the beginning of forward guidance rise.

We look next at the ZLB episode, between times 0 and $T$. This episode is depicted in panels B, C, and D; the three panels differ by the duration of forward guidance after the ZLB episode. Since equilibrium trajectories are continuous, the economy must be at the same point at the end of the ZLB and at the beginning of forward guidance. Because of the boom engineered during forward guidance, then, the situation is improved at the ZLB. Instead of reaching the natural steady state at time $T$, the economy reaches a point with positive inflation and above-natural output, so at any time before $T$, inflation and output tend to be higher than without forward guidance.

Forward guidance can have tremendously strong effects. For small durations of forward guidance, the position at the beginning of forward guidance is below the ZLB unstable line. It is therefore connected to trajectories coming from the southwest quadrant of the phase diagram (panel B). As the ZLB lasts longer, initial output and inflation collapse. When the duration of forward guidance is such that the position at the beginning of forward guidance is exactly on the unstable line, the position at the beginning of the ZLB must be on the unstable line as well (panel C). As the ZLB lasts longer, the initial position inches closer to the ZLB steady state. For even longer forward guidance, the position at the beginning of forward guidance is above the unstable line, so it is connected to trajectories coming from the northeast quadrant of the phase diagram (panel D). From here, as the ZLB lasts longer, initial output and inflation become higher and higher. As a result, if the duration of
A. Forward guidance

B. ZLB before short forward guidance

C. ZLB before medium forward guidance

D. ZLB before long forward guidance

**Figure 3. NK model: ZLB followed by various durations of forward guidance**

The phase diagram in panel A describes the NK Euler-Phillips system during forward guidance; it is similar to the diagram in panel C of figure 1 but with $r^n > 0$. The phase diagrams in panels B, C, and D describe the NK Euler-Phillips system at the ZLB; they come from panel C of figure 1. The ZLB episode lasts between times $0$ and $T$. It is followed by forward guidance between times $T$ and $T + \Delta$: the natural rate of interest is positive but the central bank maintains the policy rate at zero. Then, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time $T + \Delta$. The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time $T$. The NK model suffers from a major anomaly: when forward guidance lasts sufficiently to bring $[y(T), \pi(T)]$ on the right-hand side of the unstable line, any ZLB episode—however long—will be a boom (panel B versus panel D).
A. Forward guidance

B. Short ZLB before forward guidance

C. Long ZLB before forward guidance

D. Possible trajectories

**Figure 4. WUNK model: ZLB followed by forward guidance**

The phase diagram in panel A describes the WUNK Euler-Phillips system during forward guidance; it is similar to the diagram in panel D of figure 1 but with \( r^n > 0 \). The phase diagrams in panels B and C describe the WUNK Euler-Phillips system at the ZLB; they come from panel D of figure 1. Panel D is a generic version of panels B and C, describing any duration of ZLB and forward guidance. The ZLB episode lasts between times 0 and \( T \). It is followed by forward guidance between times \( T \) and \( T + \Delta \); the natural rate of interest becomes positive but the central bank maintains the policy rate at zero. Then, at time \( T + \Delta \), the central bank brings the economy to the natural steady state. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time \( T + \Delta \). The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time \( T \). The anomaly of the NK model disappears in the WUNK model: a long-enough ZLB always leads to a slump (panels C and D versus panel B).
forward guidance is long enough, a deep slump can be transformed into a roaring boom; moreover, the forward-guidance duration threshold is independent of the ZLB duration.

The power of forward guidance is subdued in the WUNK model, as illustrated in figure 4. After time \( T + \Delta \), the economy is at the natural steady state. Between times \( T \) and \( T + \Delta \), forward guidance operates as shown in panel A. The phase diagram is the same as in panel D of figure 1, except that \( r^n > 0 \) instead of \( r^n < 0 \): the Euler line (8) is shifted outward. Inflation is positive and output is above its natural level at time \( T \); then they decrease over time, following the unique trajectory leading to the natural steady state at time \( T + \Delta \). The economy booms during forward guidance; but unlike in the NK model, output and inflation are bounded above by the forward-guidance steady state.

Before forward guidance comes the ZLB episode, depicted in panels B and C. Thanks to the boom engineered by forward guidance, the situation is improved at the ZLB: inflation and output tend to be higher than without forward guidance. Yet, unlike in the NK model, output during the ZLB episode is always below its level at the beginning of forward guidance: forward guidance cannot generate unbounded booms. The ZLB cannot generate unbounded slumps either, since output and inflation are bounded below by the ZLB steady state. These properties are summarized in panel F. Finally, for any forward-guidance duration, as the ZLB lasts longer, the economy converges to the ZLB steady state at time 0. The implication is that forward guidance can never prevent a slump when the ZLB lasts long enough.

Based on these dynamics, we identify two anomalies in the NK model, which are resolved in the WUNK model (appendix D fleshes out the proof):

**Proposition 4.** Consider a ZLB episode during \((0, T)\) followed by forward guidance during \((T, T + \Delta)\).

- *In the NK model, there exists a threshold \( \Delta^* \), such that any forward guidance longer than \( \Delta^* \) transforms a ZLB of any duration into a boom: let \( \Delta > \Delta^* \); then for any \( T \) and for all \( t \in (0, T + \Delta) \), \( y(t) > y^n \) and \( \pi(t) > 0 \). In addition, when the forward guidance is longer than \( \Delta^* \), a long-enough forward guidance or a long-enough ZLB generates an arbitrarily large boom: for any \( T \), \( \lim_{\Delta \to \infty} y(0) = \lim_{\Delta \to \infty} \pi(0) = +\infty \); and for any \( \Delta > \Delta^* \), \( \lim_{T \to \infty} y(0) = \lim_{T \to \infty} \pi(0) = +\infty \).*
• In the WUNK model, in contrast, there exists a threshold $T^*$, such that any ZLB longer than $T^*$ generates a slump, irrespective of the duration of forward guidance: let $T > T^*$; then for any $\Delta$, $y(0) < y^a$ and $\pi(0) < 0$. Furthermore, the slump approaches the ZLB steady state as the ZLB becomes infinitely long: for any $\Delta$, $\lim_{T \to \infty} y(0) = y^\pi$ and $\lim_{T \to \infty} \pi(0) = \pi^\pi$. In addition, the economy is bounded above by the forward-guidance steady state $[y^f, \pi^f]$: for any $T$ and $\Delta$, and for all $t \in (0, T + \Delta)$, $y(t) < y^f$ and $\pi(t) < \pi^f$.

The anomalies identified in the proposition correspond to the instances of the forward-guidance puzzle described by Carlstrom, Fuerst, and Paustian (2015, fig. 1) and Cochrane (2017, fig. 6). These papers also find that a long-enough forward guidance transforms a ZLB slump into a boom whose amplitude increases with the ZLB duration.

In the WUNK model, such anomalous patterns vanish. In the New Keynesian models by Gabaix (2016), Diba and Loisel (2019), Acharya and Dogra (2019), and Bilbiie (2019), forward guidance also has much more subdued effects than in the standard model. Besides, New Keynesian models have been developed with the sole goal of solving the forward-guidance puzzle. Among these, ours belongs to the group that uses discounted Euler equations. For example, Del Negro, Giannoni, and Patterson (2015) generate discounting from overlapping generations; McKay, Nakamura, and Steinsson (2016) from heterogeneous agents facing borrowing constraints and cyclical income risk; Angeletos and Lian (2018) from incomplete information; and Campbell et al. (2017) from government bonds in the utility function (which is closely related to our approach).

4.3. Government spending

Last we consider the effects of government spending at the ZLB. We first slightly extend the model by assuming that the government purchases goods from all households, which are aggregated into public consumption $g(t)$. To ensure that government spending affects inflation and private

---

7 In the literature the forward-guidance puzzle takes several forms. Their common element is that future monetary policy has an implausibly strong effect on output and inflation today.

8 Other approaches to solve the forward-guidance puzzle include modifying the Phillips curve (Carlstrom, Fuerst, and Paustian 2015), combining “reflective” expectations and temporary equilibrium (Garcia-Schmidt and Woodford 2019), combining bounded rationality and incomplete markets (Farhi and Werning 2019), or introducing an endogenous liquidity premium (Bredemeier, Kaufmann, and Schabert 2018).
consumption, we also assume that the disutility of labor is convex: household $j$ incurs disutility $\kappa^{1+\eta} h_j(t)^{1+\eta}/(1 + \eta)$ from working, where $\eta > 0$ is the inverse of the Frisch elasticity. Complete extended model, derivations, and additional results are presented in appendix E.

In this model, the Euler equation is unchanged, but the Phillips curve is modified because households must produce goods for the government, and because the marginal disutility of labor is no longer constant. The modification of the Phillips curve alters the analysis in two ways. First, we need to adjust the WUNK assumption: (7) is replaced by

$$u'(0) > (1 + \eta) \frac{e\kappa}{\delta y a} \left( \frac{e - 1}{\epsilon} \right)^{\eta/(1+\eta)}.$$

Naturally, for $\eta = 0$, this assumption reduces to (7). Second, the steady-state Phillips curve becomes nonlinear, which may introduce additional steady states. We handle this issue as in the literature: we linearize the Euler-Phillips system around the natural steady state without government spending, and then concentrate on the dynamics of the linearized system. These dynamics are described by the phase diagrams in figure 5 (for the NK model) and figure 6 (for the WUNK model). Despite the changes to the model, the phase diagrams are similar to those in figure 1; one small difference is that private consumption instead of output is on the horizontal axis.\(^9\)

We study a ZLB episode during which the government increases spending in an effort to stimulate the economy, as in Cochrane (2017). Between times 0 and $T$, there is a ZLB episode, exactly as above. To alleviate the situation, the government provides an amount $g > 0$ of public consumption. After time $T$, the natural rate of interest is positive again, government spending stops, and monetary policy returns to normal. This scenario is summarized in panel C of table 1.

We start with the NK model, illustrated in figure 5. We construct the equilibrium path by going backward in time, given that at time $T$, monetary policy brings the economy to the natural steady state. At the ZLB, government spending helps, but through a different mechanism than forward guidance.

\(^9\)The phase diagrams in the basic model have output $y$ on the horizontal axis, but output equals private consumption $c$ in that model (public consumption $g$ is zero), so phase diagrams with private consumption on the horizontal axis would be identical. In this model output and private consumption differ because public consumption is nonzero, so phase diagrams with output or private consumption on the horizontal axis would be different. We use private consumption here as it clarifies the analysis.
A. ZLB with no government spending  

B. ZLB with low government spending  

C. ZLB with medium government spending  

D. ZLB with high government spending  

**FIGURE 5. NK model: ZLB episodes with various levels of government spending**

Notation: $\pi$ is inflation; $c$ is private consumption; $c^n$ is the natural level of private consumption. The phase diagrams represent the linearized Euler-Phillips system for the NK model with government spending and convex disutility of labor. The Euler line is the locus $\dot{c} = 0$; the Phillips line is the locus $\dot{\pi} = 0$. The phase diagrams have the same properties as those in panel C of figure 1, except that the Phillips line shifts upward when government spending increases. The ZLB episode lasts between times $0$ and $T$. During the ZLB, government spending is positive. Then, at time $T$, government spending stops, and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time $T$. The NK model suffers from two anomalies: when government spending brings the unstable line below the natural steady state, any long-enough ZLB episode sees an arbitrarily large increase in output (panel B versus panel D); and then, any long-enough ZLB episode experiences an unboundedly large boom.
A. ZLB with no government spending

B. ZLB with low government spending

C. ZLB with medium government spending

D. ZLB with high government spending

**Figure 6. WUNK model: ZLB episodes with various levels of government spending**

Notation: \( \pi \) is inflation; \( c \) is private consumption; \( c^u \) is the natural level of private consumption. The phase diagrams represent the linearized Euler-Phillips system for the WUNK model with government spending and convex disutility of labor. The Euler line is the locus \( \dot{c} = 0 \); the Phillips line is the locus \( \dot{x} = 0 \). The phase diagrams have the same properties as those in panel D of figure 1, except that the Phillips line shifts upward when government spending increases. The ZLB episode lasts between times 0 and \( T \). During the ZLB, government spending is positive. Then, at time \( T \), government spending stops, and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time \( T \). The anomalies of the NK model disappear in the WUNK model: output multipliers are finite when the ZLB becomes arbitrarily long-lasting; and irrespective of the duration of the ZLB, the equilibrium trajectories are bounded.
Forward guidance improves the situation at the end of the ZLB, which pulls up the economy during the entire ZLB. With government spending the end of the ZLB is unchanged: the economy reaches the natural steady state. Instead, government spending shifts the Phillips line upward, and with it the field of trajectories. As a result, an increase in government spending connects the natural steady state to trajectories with higher consumption and inflation (panel A versus panel B).

It is typical of New Keynesian models that government spending operates by shifting the Phillips curve. The effect of government spending is clear in the linearized steady-state Phillips curve:

\[
\pi = -\frac{\varepsilon \kappa}{\delta \gamma a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)} \left[ (1 + \eta)(c - c^a) + \eta g \right].
\]

This is the equation of the Phillips line in the phase diagrams of figures 5 and 6. Since the disutility of labor is convex ($\eta > 0$), an increase in government spending shifts the line upward.

Just like forward guidance, government spending can have very strong effects in the NK model. For low levels of spending (panel B), the natural steady state is below the ZLB unstable line; it is therefore connected to trajectories coming from the southwest quadrant of the phase diagram—just as without government spending (panel A). Then, if the ZLB lasts longer, initial consumption and inflation fall lower. When government spending is high enough that the natural steady state is on the unstable line, the position at the beginning of the ZLB must also be on the unstable line (panel C). Finally, when government spending is even higher, the natural steady state moves above the unstable line, so it is connected to trajectories coming from the northeast quadrant of the phase diagram (panel D). As a result, initial output and inflation are much higher than previously. And as the ZLB lasts longer, initial output and inflation become higher, without bound.

The power of government spending at the ZLB is much weaker in the WUNK model, as shown in figure 6. After time $T$, the economy is at the natural steady state; prior to that comes the ZLB episode. Government spending improves the situation at the ZLB, as inflation and consumption tend to be higher than without spending. As the ZLB lasts longer, the position at the beginning of the ZLB converges to the ZLB steady state—unlike in the NK model, it does not go to infinity. So
equilibrium trajectories are bounded, and government spending cannot generate unbounded booms.

Based on these dynamics, we isolate two anomalies in the NK model, which are resolved in the WUNK model (appendix F fleshes out the proof):

**Proposition 5.** Consider a ZLB episode during \((0, T)\), accompanied by government spending \(g > 0\). Let \(c(t; g)\) and \(y(t; g)\) be private consumption and output at time \(t\); let \(s > 0\) be some incremental government spending; and let

\[
m(g, s) = \frac{y(0; g + s/2) - y(0; g - s/2)}{s} = 1 + \frac{c(0; g + s/2) - c(0; g - s/2)}{s}
\]

be the government-spending multiplier.

- In the NK model, there exists a government spending \(g^*\) such that the government-spending multiplier becomes infinitely large when the ZLB is infinitely long-lasting: for any \(s > 0\), \(\lim_{T \to \infty} m(g^*, s) = +\infty\). In addition, when government spending is above \(g^*\), a long-enough ZLB generates an arbitrarily large boom: for any \(g > g^*\), \(\lim_{T \to \infty} c(0; g) = +\infty\).

- In the WUNK model, in contrast, the multiplier always has a finite limit when the ZLB is infinitely long-lasting: for any \(g\) and \(s\), when \(T \to \infty\), \(m(g, s)\) converges to

\[
(11) \quad 1 + \frac{\eta}{\epsilon k} \cdot \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta).
\]

Moreover, for any ZLB duration, the economy remains bounded above: let \(c^g\) be private consumption in the ZLB steady state with government spending \(g\); then for any \(T\) and for all \(t \in (0, T)\), \(c(t; g) < \max\{c^g, c^n\}\).

The anomaly that a finite amount of government spending may generate an infinitely large boom as the ZLB becomes arbitrarily long is reminiscent of the findings by Christiano, Eichenbaum, and Rebelo (2011, fig. 2), Woodford (2011, fig. 2), and Cochrane (2017, fig. 5) that in the NK model, government spending is exceedingly powerful when the ZLB is long-lasting.
In the WUNK model, such anomaly vanishes. Diba and Loisel (2019) and Acharya and Dogra (2019) also obtain more realistic effects of government spending at the ZLB. Beside these papers, Bredemeier, Juessen, and Schabert (2018) obtain moderate multipliers at the ZLB by introducing an endogenous liquidity premium in the New Keynesian model.

5. Other New Keynesian properties at the ZLB

Beside the anomalous properties described above, the New Keynesian model has several other intriguing properties at the ZLB: the paradoxes of thrift, toil, and flexibility; and above-one government-spending multiplier. We now show that the WUNK model shares these properties.

In the NK model these properties are studied in the context of a temporary ZLB episode. In the WUNK model, we can simply work with a permanent ZLB episode: we assume that the natural rate of interest is permanently negative, and the central bank keeps the policy rate at zero forever. The only equilibrium is to be at the ZLB steady state, where the economy is in a slump: inflation is negative and output is below its natural level. The ZLB equilibrium is represented graphically in figure 7: it is the intersection of a Phillips line, describing the steady-state Phillips curve, and an Euler line, describing the steady-state Euler equation. When an unexpected and permanent shock occurs, the economy jumps to a new ZLB steady state; we use the graphs to study such jumps.

5.1. Paradox of thrift

We first study an increase in the marginal utility of wealth \((u'(0))\). The steady-state Phillips curve is unaffected, but the steady-state Euler equation does change. Using (5), we rewrite the steady-state Euler equation (8):

\[
\pi = -\delta + \sigma + u'(0)y.
\]

Hence increasing the marginal utility of wealth steepens the Euler line, which moves the economy inward along the Phillips line: output and inflation decrease (figure 7, panel A). The following
In panels A–C, the Euler and Phillips lines are the same as in panel D of figure 1. In panel D, the Euler and Phillips lines are the same as in figure 6. The ZLB equilibrium is at the intersection of the Phillips and Euler lines: output/consumption is below its natural level and inflation is negative. Panel A illustrates the paradox of thrift: increasing the marginal utility of wealth steepens the Euler line, which depresses output and inflation without changing relative wealth. Panel B illustrates the paradox of toil: reducing the disutility of labor moves the Phillips line outward, which depresses output, inflation, and hours worked. Panel C illustrates the paradox of flexibility: decreasing the price-adjustment cost rotates the Phillips line counterclockwise around the natural steady state, which depresses output and inflation. Panel D shows that the government-spending multiplier is above one: increasing government spending shifts the Phillips line upward, which raises private consumption and therefore increases output more than one-for-one.
proposition summarizes the results:

**Proposition 6.** At the ZLB in the WUNK model, the paradox of thrift holds: an unexpected and permanent increase in the marginal utility of wealth reduces output and inflation but does not affect relative wealth.

The paradox of thrift was first discussed by Keynes, but it also appears in the New Keynesian model (Eggertsson 2010, p. 16; Eggertsson and Krugman 2012, p. 1486). When the marginal utility of wealth is higher, people want to increase their wealth holdings relative to their peers, so they favor saving over consumption. But in equilibrium relative wealth is fixed at zero since everybody is the same; hence the only way to save more relative to consumption is to reduce consumption. In normal times the central bank would offset this reduction in aggregate demand by reducing nominal interest rates. This is not an option at the ZLB, so output falls.

5.2. Paradox of toil

Next we consider a reduction in the disutility of labor \((\kappa)\). In this case the steady-state Phillips curve changes while the steady-state Euler equation does not. Using (2), we rewrite the steady-state Phillips curve (3):

\[
\pi = \frac{\epsilon \kappa}{\delta y a} y - \frac{\epsilon - 1}{\delta y}.
\]

Hence reducing the disutility of labor flattens the Phillips line, which moves the economy inward along the Euler line: both output and inflation decrease (figure 7, panel B). Moreover, since hours worked and output are related by \(h = y/a\), hours fall as well. Proposition 7 states the results:

**Proposition 7.** At the ZLB in the WUNK model, the paradox of toil holds: an unexpected and permanent reduction in the disutility of labor reduces hours worked, output, and inflation.

The paradox of toil was discovered by Eggertsson (2010, p. 15) and Eggertsson and Krugman (2012, p. 1487). It operates as follows. With lower disutility of labor, real marginal costs are lower, and the natural level of output is higher: firms would like to produce and sell more. To increase
sales, firms tend to reduce their prices, reducing inflation. Away from the ZLB, the central bank would offset this reduction in inflation by lowering nominal interest rates. But this cannot happen at the ZLB, so the reduction in inflation raises real interest rates—which pushes households to save more. In equilibrium, this lowers output and hours worked.

As usual in this context, an increase in technology \((a)\) would have the same effect as a reduction in the disutility of labor: it would lower output and inflation.

### 5.3. Paradox of flexibility

We then examine a decrease in the price-adjustment cost \((\gamma)\). The steady-state Euler equation is not affected, but the steady-state Phillips curve is. Equation (3) shows that decreasing the price-adjustment cost leads to a counterclockwise rotation of the Phillips line around natural steady state, which moves the economy inward along the Euler line: both output and inflation decrease (figure 7, panel C). The following proposition records the results:

**Proposition 8.** At the ZLB in the WUNK model, the paradox of flexibility holds: an unexpected and permanent decrease in price-adjustment cost reduces output and inflation.

The paradox of flexibility was discovered by Werning (2011, pp. 13–14) and Eggertsson and Krugman (2012, pp. 1487–1488). Intuitively, with a lower price-adjustment cost, firms are keener to adjust their prices to bring production closer to the natural level of output, which accentuates the existing deflation. At the ZLB, lower inflation means higher real interest rate, which makes households more prone to save and less to consume. In equilibrium, this reduces output.

### 5.4. Above-one government-spending multiplier

We finally look at an increase in government spending \((g)\), using the model with government spending introduced in section 4.3. From (10) we see that increasing government spending shifts the Phillips line upward, which moves the economy upward along the Euler line: both private consumption and inflation increase (figure 7, panel D). Since private consumption increases when
public consumption does, the government-spending multiplier $dy/dg = 1 + dc/dg$ is greater than one. Proposition 9 gives the results:

**Proposition 9.** At the ZLB in the WUNK model, an unexpected and permanent increase in government spending raises private consumption and inflation. Hence the government-spending multiplier $dy/dg$ is above one; its value is given by (11).

The multiplier value (II) is derived in appendix F. Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), and Woodford (2011) first showed that at the ZLB in the New Keynesian model, the government-spending multiplier is above one. The intuition is the following. With higher government spending, real marginal costs for a given level of private consumption are higher, so firms would like to reduce their sales to households. Hence, firms tend to increase their prices, raising inflation. At the ZLB, the increase in inflation lowers real interest rates—as nominal interest rates are fixed—which makes households more prone to consume. In equilibrium this leads to higher private consumption and a multiplier above one.

### 6. Empirical assessment of the WUNK assumption

In the WUNK model the marginal utility of wealth is assumed to be high enough so that at the ZLB the steady-state Euler equation is steeper than the steady-state Phillips curve. We assess this assumption using empirical evidence for the United States.

As a first step, we re-express the WUNK assumption in terms of estimable statistics (see appendix G). We obtain the following condition, which is equivalent to (7) when the disutility of labor is linear, and to (9) when the disutility of labor is convex:

$$\delta - r^n > \frac{\lambda}{\delta},$$

where $\delta$ is the time discount rate, $r^n$ is the natural rate of interest, and $\lambda$ is the coefficient on output gap in a New Keynesian Phillips curve. The term $\delta - r^n$ measures the marginal rate of substitution between wealth and consumption, $u'(0)y^n$. It indicates how high the marginal utility of wealth is and
thus how steep the steady-state Euler equation is at the ZLB. The term $\lambda/\delta$ indicates how steep the steady-state Phillips curve is. The $\delta$ comes from the denominator of the slopes of the steady-state Phillips curves (3) and (10); the $\lambda$ measures the rest of the slope coefficients. We now survey the literature to obtain estimates of $r^n$, $\lambda$, and $\delta$.

### 6.1. Natural rate of interest

A large number of macroeconometric studies have estimated the natural rate of interest, using different statistical models, methodologies, and data. Recent studies obtain comparable estimates of the natural rate for the United States: around 2% per annum on average between 1985 and 2015 (Williams 2017, fig. 1). Accordingly, we use $r^n = 2\%$ as our estimate.

### 6.2. Output-gap coefficient in the New Keynesian Phillips curve

Many studies have estimated New Keynesian Phillips curves. Mavroeidis, Plagborg-Moller, and Stock (2014, sec. 5) offer a synthesis for the United States: they generate estimates of the New Keynesian Phillips curve using an array of US data, methods, and specifications found in the literature. There is significant uncertainty around the estimation, but in many cases the output-gap coefficient is positive and very small. Overall, their median estimate of the output-gap coefficient is $\lambda = 0.004$ (table 5, row 1), which we use as our estimate.

### 6.3. Time discount rate

Since the 1970s many studies have estimated time discount rates, using field and laboratory experiments and real-world behavior. Frederick, Loewenstein, and O’Donoghue (2002, table 1) survey 43 such studies. The estimates are quite dispersed, but the majority of them points to high discount rates, much higher than prevailing market interest rates. We compute the mean estimate in each of the studies covered by the survey, and then compute the median value of these means. We obtain an annual discount rate of $\delta = 35\%$.

There is one immediate limitation with the studies discussed by Frederick, Loewenstein, and
O’Donoghue: they use a single rate to exponentially discount future utility. But exponential discounting does not describe reality well because people seem to choose more impatiently for the present than for the future—they exhibit present-focused preferences (Ericson and Laibson 2019). Recent studies have moved away from exponential discounting and allowed for present-focused preferences, including quasi-hyperbolic ($\beta - \delta$) discounting. Andersen et al. (2014, table 3) survey 16 such studies, concentrating on experimental studies with real incentives. We compute again the mean estimate in each study, and then the median value of these means. We obtain an annual discount rate of $\delta = 43\%$: even after accounting for present-focus, time discounting remains high. Accordingly, we use $\delta = 43\%$ as our estimate. 10

6.4. Assessment

We now combine our estimates of $r^n$, $\lambda$, and $\delta$ to assess the WUNK assumption. Since $\lambda$ is estimated using quarter as a unit of time, we need to express $r^n$ and $\delta$ as quarterly rates: $r^n = 2\%/4 = 0.5\%$ per quarter, and $\delta = 43\%/4 = 10.8\%$ per quarter. We find that (13) comfortably holds: $\delta - r^n = 0.108 - 0.005 = 0.103$, which is much larger than $\lambda/\delta = 0.004/0.108 = 0.037$. Hence the WUNK assumption holds in US data.

The discount rate used here (43% per annum) is much higher than discount rates used in macroeconomic models (typically less than 5% per annum). This is because our discount rate is calibrated from microevidence, while the discount rate in macroeconomic models is calibrated to

---

10There are two potential issues with the experiments discussed in Andersen et al. (2014). First, many are run with university students instead of subjects representative of the general population. There does not seem to be systematic differences in discounting between student and non-student subjects, however (Cohen et al. 2019, sec. 6A). Hence, using students is unlikely to bias the estimates reported by Andersen et al.. Second, all the experiments elicit discount rates using financial flows, not consumption flows. As the goal is to elicit the discount rate on consumption, this could be problematic (Cohen et al. 2019, sec. 4B); the problems could be exacerbated if subjects derive utility from wealth. To assess this potential issue, suppose first (as in most of the literature) that monetary payments are consumed at the time of receipt, and that the utility function is locally linear. Then the experiments deliver estimates of the relevant discount rate (Cohen et al. 2019, sec. 4B). If these conditions do not hold, the experimental findings are more difficult to interpret. For instance, if subjects optimally smooth their consumption over time by borrowing and saving, then the experiments only elicit the interest rate faced by subjects, and reveal nothing about their discount rate (Cohen et al. 2019, sec. 4B). In that case, we should rely on experiments using time-dated consumption rewards instead of monetary rewards. Such experiments directly deliver estimates of the discount rate. Many such experiments have been conducted; a robust finding is that discount rates are systematically higher for consumption rewards than for monetary rewards (Cohen et al. 2019, sec. 3A). Hence, the estimates presented in Andersen et al. are, if anything, lower bounds on actual discount rates.
match observed real interest rates. This discrepancy occasions two remarks. First, an advantage of the wealth-in-the-utility assumption is that it explains why people are willing to save at single-digit interest rates while they exhibit double-digit discount rates in experimental studies. In the standard model, the real interest rate necessarily equals the discount rate in steady state, so the model could not have $\delta = 43\%$ and $r^n = 2\%$.

Second, the WUNK assumption would also hold with discount rates below 43%. Indeed, (13) holds for any annual discount rate above 27%, since $\delta - r^n = (0.27/4) - 0.005 = 0.062 > 0.059 = 0.004/(0.27/4) = \lambda/\delta$. And an annual discount rate of 27% is at the low end of available microestimates: in 11 of the 16 studies in Andersen et al. (2014, table 3), the bottom of the range of reported estimates is above 27%; and in 13 of the 16 studies, the mean estimate is above 27%.

Finally, the same discount rate appears in the right-hand side and left-hand side of (13). This is because our model omits firms and assumes that households are both producers and consumers. Of course in reality firms and households are often separate entities, so they could have different discount rates. With different discount rates, (13) would become

$$\delta^h - r^n > \frac{\lambda}{\delta^f},$$

where $\delta^h$ is the discount rate of households and $\delta^f$ is the discount rate of firms. Clearly if firms have a low discount rate, the WUNK assumption is less likely to be satisfied. If we use $\delta^h = 43\%$, $r^n = 2\%$, and $\lambda = 0.004$, as above, we find that the WUNK condition holds only if firms have an annual discount rate above 16%, since $\delta^h - r^n = 0.108 - 0.005 = 0.103 > 0.100 = 0.004/(0.16/4) = \lambda/\delta^f$.

This discount rate is only slightly above that reported by large US firms: in a survey of 228 CEOs, Poterba and Summers (1995) find an average annual real discount rate of 12.2%; and in a survey of 86 CFOs, Jagannathan et al. (2016, p. 447) find an average annual real discount rate of 12.7%.
7. Conclusion

This paper extends the New Keynesian model by introducing relative wealth into the utility function. When the marginal utility of wealth is sufficient high, our model is immune to the anomalies that plague the New Keynesian model at the ZLB: even when the ZLB is arbitrarily long-lasting, there is no collapse of inflation and output, and both forward guidance and government spending have limited, plausible effects. Yet our model retains other properties of the New Keynesian model at the ZLB: the paradoxes of thrift, toil, and flexibility; and above-one government-spending multiplier.

Beyond the assumption of wealth in the utility function, our analysis applies more generally to any New Keynesian model representable by a discounted Euler equation and a Phillips curve (for example, Del Negro, Giannoni, and Patterson 2015; Gabaix 2016; McKay, Nakamura, and Steinsson 2017; Campbell et al. 2017; Beaudry and Portier 2018; Angeletos and Lian 2018). Wealth in the utility function is a simple way to generate discounting; but any model with discounting would have similar phase diagrams and properties. Hence, for such model to behave well at the ZLB, there is only one requirement: that discounting is strong enough to make the steady-state Euler equation steeper than the steady-state Phillips curve at the ZLB; the source of discounting is unimportant. In the real world, several discounting mechanisms might operate at the same time and reinforce each other. A model blending these mechanisms would be more likely to behave well at the ZLB.

References


Macroeconomic Annual 2010, edited by Daron Acemoglu and Michael Woodford, chap. 2.
Chicago: University of Chicago Press.


Futagami, Koichi, and Akihisa Shibata. 1998. “Keeping One Step Ahead of the Joneses: Status, the


Appendix A. Formal derivation of Euler equation and Phillips curve

We derive the two differential equations that describe the equilibrium of our New Keynesian model: the Phillips curve, given by (1); and the Euler equation, given by (4).

Household’s problem

We begin by solving household $j$’s problem. The current-value Hamiltonian of the problem is

$$
H_j = \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \frac{\kappa}{\alpha} \gamma_j^d(p_j(t), t) - \frac{\gamma}{2} \pi_j(t)^2 + \mathcal{A}_j(t) \left[ I^h(t) b_j(t) + p_j(t) \gamma_j^d(p_j(t), t) - \int_0^1 p_k(t) c_{jk}(t) \, dk - \tau(t) \right] + \mathcal{B}_j(t) \pi_j(t) p_j(t),
$$

with control variables $c_{jk}(t)$ for all $k \in [0, 1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. Note that we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian.

The necessary conditions for a maximum to the household’s problem are given by Acemoglu (2009, theorem 7.9); we apply them here. These conditions form the basis for the model’s equilibrium conditions. (To ease notation we drop the time index $t$.)

The first set of optimality conditions are $\partial H_j/\partial c_{jk} = 0$ for all $k \in [0, 1]$. They yield

$$(A1) \quad \frac{1}{c_j} \left( \frac{c_{jk}}{c_j} \right)^{-1/\epsilon} = \mathcal{A}_j p_k.$$

Appropriately integrating (A1) over all $k \in [0, 1]$, and using the expressions for the consumption and price indices,

$$(A2) \quad c_j(t) = \left[ \int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} \, dk \right]^{\epsilon/(\epsilon-1)}$$

$$(A3) \quad p(t) = \left[ \int_0^1 p_j(t)^{1-\epsilon} \, dt \right]^{1/(1-\epsilon)},$$
we find

\[(A4) \quad \mathcal{A}_j = \frac{1}{pc_j}.\]

Combining (A1) and (A4), we then obtain

\[(A5) \quad c_{jk} = \left(\frac{p_k}{p}\right)^{-\varepsilon} c_j.\]

Integrating (A5) over all \(j \in [0, 1]\), we get the usual demand for good \(k\):

\[(A6) \quad y^d_k(p_k) = \int_0^1 c_{jk} \, dj = \left(\frac{p_k}{p}\right)^{-\varepsilon} c,\]

where \(c = \int_0^1 c_j \, dj\) is aggregate consumption. We use this expression for \(y^d_k(p_k)\) in household \(k\)’s Hamiltonian. We also obtain \(\int_0^1 p_k c_{jk} \, dk = pc_j\): the price of one unit of consumption index is \(p\).

The second optimality condition is \(\partial H_j/\partial b_j = \delta \mathcal{A}_j - \dot{\mathcal{A}}_j\), which gives

\[-\frac{\dot{\mathcal{A}}_j}{\mathcal{A}_j} = i^h + \frac{1}{p\mathcal{A}_j} \cdot u'(\frac{b_j - b}{p}) - \delta.\]

Using (A4) and \(i^h = i + \sigma\), we obtain the household’s Euler equation:

\[(A7) \quad \frac{\dot{c}_j}{c_j} = i + \sigma - \pi + c_j u'(\frac{b_j - b}{p}) - \delta.\]

This equation describes the optimal path for household \(j\)’s consumption.

The third optimality condition is \(\partial H_j/\partial \pi_j = 0\), which yields

\[(A8) \quad \mathcal{B}_j p_j = \gamma \pi_j.\]
Differentiating (A8) with respect to time, we obtain

\[ \frac{\dot{B}_j}{B_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j. \]  

(A9)

The fourth and last optimality condition is \( \partial H_j/\partial p_j = \delta B_j - \dot{B}_j \), which implies

\[ \frac{\kappa}{a} \cdot \frac{\epsilon y_j}{p_j} - (\epsilon - 1)A_j y_j + B_j \pi_j = \delta B_j - \dot{B}_j. \]

Reshuffling the terms, we obtain

\[ \pi_j - \frac{(\epsilon - 1) y_j A_j}{B_j p_j} \left( p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa}{a} \right) = \delta \frac{\dot{B}_j}{B_j}. \]

Then, using (A4), (A8), and (A9), we obtain the household’s Phillips curve:

\[ \frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j} \left( p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa c_j}{a} \right). \]  

(A10)

This equation describes the optimal path for the price set by household \( j \).

**Equilibrium**

All households face the same initial conditions, so in equilibrium they all behave the same. We therefore omit the subscripts \( j \) and \( k \). In particular, all households hold the same wealth, so relative wealth is zero: \( b_j = b \). Additionally, in equilibrium, production and consumption are equal: \( y = c \). Therefore the household’s Euler equation, given by (A7), simplifies to

\[ \frac{\dot{y}}{y} = r - r^n + u'(0)(y - y^n), \]

where \( r = i - \pi \) and \( r^n = \delta - \sigma - u'(0)y^n \). And the household’s Phillips curve, given by (A7), simplifies to

\[ \dot{\pi} = \delta \pi - \frac{\epsilon \kappa}{\gamma a} (y - y^n), \]
where \( y^n = (\varepsilon - 1)a/(\varepsilon \kappa) \). These differential equations are just (4) and (1).

**Appendix B. Heuristic derivation of Euler equation and Phillips curve**

Our model incorporates several nontraditional elements (wealth in the utility function, and to some extent continuous time and Rotemberg pricing), which alter Euler equation and Phillips curve. To help readers understand and interpret these two altered equations, we complement the formal derivations of appendix A with heuristic derivations (as in Blanchard and Fischer 1989, pp. 40–42).

**Euler equation**

The Euler equation says that households save in an optimal fashion: they cannot improve their situation by shifting consumption a little bit across time.

Consider a household delaying consumption of one unit of output from time \( t \) to time \( t + dt \). The unit of output, invested at a real interest rate \( r^h(t) \) during the interval, becomes \( 1 + r^h(t)dt \) at time \( t + dt \). Given log consumption utility, the marginal utility from consumption at time \( t \) is \( e^{-\delta t}/y(t) \). Hence, the foregone consumption utility at time \( t \) is \( e^{-\delta t}/y(t) \); and the extra consumption utility at time \( t + dt \) is \( e^{-\delta(t+dt)}[1 + r^h(t)dt]/y(t + dt) \).

In addition to financial returns, the one unit of output saved between \( t \) and \( t + dt \) provides hedonic returns, since people enjoy holding wealth. The marginal utility from real wealth at time \( t \) is \( e^{-\delta t}u'(0) \); hence the utility gained from holding extra wealth for a duration \( dt \) is \( e^{-\delta t}u'(0)dt \).

At the optimum, the consumption reallocation does not affect utility:

\[
0 = -\frac{e^{-\delta t}}{y(t)} + \left[ 1 + r^h(t)dt \right] \frac{e^{-\delta(t+dt)}}{y(t + dt)} + e^{-\delta t}u'(0)dt.
\]

After dividing by \( e^{-\delta t}/y(t) \), we obtain

\[
1 = [1 + r^h(t)dt]e^{-\delta dt}\frac{y(t)}{y(t + dt)} + u'(0)y(t)dt.
\]
Up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$,

$$
\frac{y(t + dt)}{y(t)} = 1 + \frac{\dot{y}(t)}{y(t)} dt,
$$

and $1/(1 + x dt) = 1 - x dt$ for any $x$. Hence, up to second-order terms, we have

$$
1 = \left[1 + r^h(t)dt\right] (1 - \delta dt) \left[1 - \frac{\dot{y}(t)}{y(t)} dt\right] + u'(0)y(t)dt
$$

$$
= 1 - \delta dt + r^h(t)dt - \frac{\dot{y}(t)}{y(t)} dt + u'(0)y(t)dt,
$$

which simplifies to

(A11) \quad \frac{\dot{y}(t)}{y(t)} = r^h(t) - \delta + u'(0)y(t).

Once we note that $r^h(t) = r(t) + \sigma$ and introduce $r^n = \delta - \sigma - u'(0)y^n$, we obtain (4).

Compared with the standard Euler equation, (A11) has an extra term: $u'(0)y(t)$. In the standard equation, consumption is governed by the cost of delaying consumption, given by the time discount rate $\delta$, and the return on saving, given by the real interest rate $r^h(t)$. With wealth in the utility function, the financial return on saving is supplemented by an hedonic return on saving, measured by the marginal rate of substitution between real wealth and consumption, $u'(0)y(t)$. Thus the total return on saving is $r^h(t) + u'(0)y(t)$ instead of $r^h(t)$, explaining the extra term in the equation.

The derivation also explains why in steady state consumption is a decreasing function of the real interest rate. When the real rate is higher, people have a financial incentive to save more and postpone consumption; they keep consumption constant only if the hedonic return on saving falls so as to offset the increase in financial return. The hedonic return is given by the marginal rate of substitution $u'(0)y$. Thus steady-state consumption $y$ must decline when the real rate rises.
Phillips curve

The Phillips curve says that households price optimally, so they cannot improve their situation by shifting inflation a little bit across time.

Consider a household delaying one percentage point of inflation from time $t$ to time $t + dt$. Given the quadratic price-change disutility, the marginal disutility from inflation at time $t$ is $e^{-\delta t} \gamma \pi(t)$. Hence, the foregone inflation disutility at time $t$ is $e^{-\delta t} \gamma \pi(t) \times 1\%$; and the extra inflation disutility at time $t + dt$ is $e^{-\delta (t+dt)} \gamma \pi(t + dt) \times 1\%$.

The one percentage point of inflation that is delayed reduces the price level between time $t$ and time $t + dt$ by $dp(t) = -1\% \times p(t)$, which then affects sales and hours worked. Since the price elasticity of demand is $-\epsilon$, sales increase by $dy(t) = -\epsilon \times -1\% \times y(t) = \epsilon y(t) \times 1\%$ during the same period. As a result, hours increase by $dh(t) = dy(t)/a = \epsilon[y(t)/a] \times 1\%$, raising the disutility of labor by $e^{-\delta t} \kappa \epsilon[y(t)/a] \times 1\%$ between $t$ and $t + dt$. The change in price and sales raises the revenue by $d(p(t)y(t)) = p(t)dy(t) + y(t)dp(t) = (\epsilon - 1)y(t)p(t) \times 1\%$. Since in equilibrium all prices are the same, the increase in revenue yields an increase in consumption $dy(t) = (\epsilon - 1)y(t) \times 1\%$, which raises consumption utility by $e^{-\delta t}dy(t)/y(t) = e^{-\delta t}(\epsilon - 1) \times 1\%$ between time $t$ and time $t + dt$.

At the optimum, shifting inflation across time does not affect utility:

$$0 = e^{-\delta t} \gamma \pi(t) \times 1\% - e^{-\delta (t+dt)} \gamma \pi(t + dt) \times 1\% - e^{-\delta t} \kappa \epsilon \frac{y(t)}{a} \times 1\% \times dt + e^{-\delta t}(\epsilon - 1) \times 1\% \times dt.$$

Up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$ and

$$\pi(t + dt) = \pi(t) + \dot{\pi}(t)dt.$$

Therefore, after dividing by $e^{-\delta t} \times 1\%$, we find that up to second-order terms,

$$0 = \gamma \pi(t) - (1 - \delta dt)\gamma \pi(t) + \dot{\pi}(t)dt - \kappa \epsilon \frac{y(t)}{a} dt + (\epsilon - 1) dt$$

$$= \delta \gamma \pi(t) dt - \gamma \dot{\pi}(t) dt - \kappa \epsilon \frac{y(t)}{a} dt + (\epsilon - 1) dt.$$
Dividing by $\gamma dt$, we obtain up to second-order terms:

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon\kappa}{y} \left[ y(t) - \frac{\epsilon - 1}{\epsilon} \cdot \frac{a}{\kappa} \right].$$

Once we introduce $y^n = [(\epsilon - 1)/\epsilon]a/\kappa$, we obtain (1).

The Phillips curve implies that in the absence of price-adjustment cost ($\gamma = 0$), households would like to produce at the natural level of output, $y^n$. This result comes from the monopolistic nature of competition. Without price-adjustment costs, it is optimal to charge a relative price that is a markup $\epsilon/(\epsilon - 1)$ over the real marginal cost—which is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. In equilibrium, all relative prices are 1, the marginal rate of substitution between labor and consumption is $\kappa/(1/y) = \kappa y$, and the marginal product of labor is $a$. Hence at the optimum $1 = [\epsilon/(\epsilon - 1)]\kappa y/a$, which implies $y = [(\epsilon - 1)/\epsilon]a/\kappa = y^n$.

Last, the derivation elucidates why in steady state inflation is positive whenever output is above its natural level. When inflation is positive, reducing inflation lowers the disutility from price adjustment. Since pricing is optimal, there must also be a cost to reducing inflation and hence increasing production. Therefore, production must already be excessive: output must be above its natural level.

**Appendix C. Euler equation and Phillips curve in discrete time**

We recast the model of section 3 in discrete time, and we rederive the Euler equation and Phillips curve. This reformulation is helpful to compare our model to the textbook New Keynesian model, which is presented in discrete time (see Woodford 2003; Gali 2008). The reformulation also shows that introducing wealth in the utility function yields a discounted Euler equation.
Assumptions

The discrete-time model is the same as the continuous-time model, except for government bonds. In discrete time, households trade one-period government bonds. Bonds purchased in period \( t \) have a price \( q(t) \) and pay one unit of money at maturity, in period \( t + 1 \). The nominal interest rate faced by households between \( t \) and \( t + 1 \) is defined as \( i_h(t) = -\ln(q(t)) \).

Household’s problem

We begin by solving household \( j \)'s problem. The household chooses sequences \( \{y_j(t), p_j(t), h_j(t), [c_{jk}(t)]_{k=0}, b_j(t)\}_{t=0}^{\infty} \) to maximize the discounted sum of instantaneous utilities

\[
\sum_{t=0}^{\infty} \beta^t \left\{ \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(\epsilon - 1)/\epsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \kappa h_j(t) - \frac{\gamma}{2} \left( \frac{p_j(t)}{p_j(t - 1)} - 1 \right)^2 \right\} \, dt,
\]

where \( \beta < 1 \) is the time discount factor. The maximization is subject to three constraints. First, there is the production function:

\[
y_j(t) = ah_j(t).
\]

Second, there is the demand for good \( j \), given by

\[
y_j(t) = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} c(t).
\]

The demand for good \( j \) is the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time. And third, there is a budget constraint:

\[
\int_0^1 p_k(t)c_{jk}(t) \, dk + q(t)b_j(t) + \tau(t) = p_j(t)y_j(t) + b_j(t - 1).
\]

Household \( j \) is also subject to a solvency constraint preventing Ponzi schemes. Lastly, household \( j \) takes as given the initial conditions \( b_j(-1) \) and \( p_j(-1) \), as well as the sequences of aggregate variables.
\[ \{p(t), q(t), c(t)\}_{t=0}^{\infty}. \]

The Lagrangian of the household’s problem is

\[
\mathcal{L}_j = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\epsilon}{\epsilon - 1} \ln\left( \int_0^1 c_{jk}(t)^{(e-1)/\epsilon} \, dk \right) + u\left( \frac{b_j(t) - b(t)}{p(t)} \right) - \frac{\kappa}{a} y_j^d(p_j(t), t) - \frac{\gamma}{2} \left[ \frac{p_j(t)}{p_j(t-1)} - 1 \right]^2 \\
+ \mathcal{A}_j(t) \left[ p_j(t) y_j^d(p_j(t), t) + b_j(t-1) - \int_0^1 p_k(t)c_{jk}(t) \, dk - q(t)b_j(t) - \tau(t) \right] \right\}
\]

where \( \mathcal{A}_j(t) \) is the Lagrange multiplier on the budget constraint in period \( t \), and

\[ y_j^d(p_j(t), t) = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} c(t) \]

is the demand for good \( j \) in period \( t \). Note that we have used the production and demand constraints to substitute \( y_j(t) \) and \( h_j(t) \) out of the Lagrangian.

The necessary conditions for a maximum to the household’s problem are standard first-order conditions: with respect to \( c_{jk}(t) \) for all \( k \in [0, 1] \), \( b_j(t) \), and \( p_j(t) \), for all \( t \geq 0 \).

The first set of first-order conditions is \( \frac{\partial \mathcal{L}_j}{\partial c_{jk}(t)} = 0 \) for all \( k \in [0, 1] \) and all \( t \). As in continuous time we obtain

\[ \mathcal{A}_j(t) = \frac{1}{p(t)c_j(t)}. \]

The second first-order condition is \( \frac{\partial \mathcal{L}_j}{\partial b_j(t)} = 0 \) for all \( t \), which gives

\[ q(t)\mathcal{A}_j(t) = \frac{1}{p(t)} u'\left( \frac{b_j(t) - b(t)}{p(t)} \right) + \beta \mathcal{A}_j(t+1). \]

Using (A13), we obtain the household’s Euler equation for consumption:

\[ q(t) = c_j(t)u'\left( \frac{b_j(t) - b(t)}{p(t)} \right) + \beta \frac{p(t)c_j(t)}{p(t+1)c_j(t+1)}. \]
The third first-order condition is \( \partial L_j/\partial p_j(t) = 0 \) for all \( t \), which yields

\[
0 = \frac{\kappa}{a} \cdot \frac{\epsilon y_j(t)}{p_j(t)} - \frac{\gamma}{p_j(t)} \left[ \frac{p_j(t)}{p_j(t-1)} - 1 \right] + (1 - \epsilon) A_j(t) y_j(t) + \beta \gamma \frac{p_j(t+1)}{p_j(t)^2} \left[ \frac{p_j(t+1)}{p_j(t)} - 1 \right].
\]

Multiplying this equation by \( p_j(t)/\gamma \) and using (A13), we obtain the household’s Phillips curve:

\[
(A15) \quad \frac{p_j(t)}{p_j(t-1)} \left[ \frac{p_j(t)}{p_j(t-1)} - 1 \right] = \beta \frac{p_j(t+1)}{p_j(t)} \left[ \frac{p_j(t+1)}{p_j(t)} - 1 \right] + \frac{\epsilon \kappa}{\gamma a \gamma^j(t)} - \frac{\epsilon - 1}{\gamma} \cdot \frac{p_j(t) y_j(t)}{p(t)c_j(t)}. \]

**Equilibrium**

In equilibrium, all households behave the same, so we drop the subscripts \( j \) and \( k \). Particularly, all households hold the same wealth, so relative wealth is zero: \( b_j(t) = b(t) \). As production and consumption are equal, we set \( y(t) = c(t) \). Then, from (A14), we obtain the Euler equation

\[
(A16) \quad q(t) = u'(0)y(t) + \beta \frac{p(t)y(t)}{p(t+1)y(t+1)}. \]

Similarly, using (A15), and using (2), we obtain the Phillips curve

\[
(A17) \quad \frac{p(t)}{p(t-1)} \left[ \frac{p(t)}{p(t-1)} - 1 \right] = \beta \frac{p(t+1)}{p(t)} \left[ \frac{p(t+1)}{p(t)} - 1 \right] + \frac{\epsilon - 1}{\gamma} \left[ \frac{y(t)}{y^n} - 1 \right]. \]

**Log-linearization**

To obtain standard expressions of the Euler equation and Phillips curve, we log-linearize them around the natural steady state—where \( y = y^n, \pi = 0, \) and \( i = r^n \). We introduce the log-deviation of output from its steady-state level: \( \hat{y}(t) = \ln(y(t)) - \ln(y^n) \). We also introduce the inflation rate between periods \( t \) and \( t + 1 \): \( \pi(t + 1) = \ln(p(t + 1)) - \ln(p(t)) \).

**Euler equation.** We start by log-linearizing the Euler equation (A16). We first take the log of the left-hand side of (A16). Using the discrete-time definition of the nominal interest rate faced by households, \( i^h(t) \), we obtain \( \ln(q(t)) = -i^h(t) \). At the natural steady state, the monetary-policy rate
is \( i = r^n \), so the interest rate faced by households satisfies \( i^h = r^n + \sigma \).

Next we take the log of the right-hand side of (A16) and obtain \( \Lambda \equiv \ln(\Lambda_1 + \Lambda_2) \), where

\[
\Lambda_1 \equiv u'(0)y(t) \\
\Lambda_2 \equiv \beta \frac{p(t)y(t)}{p(t+1)y(t+1)}.
\]

For future reference, we compute the values of \( \Lambda, \Lambda_1, \) and \( \Lambda_2 \) at the natural steady state. At the natural steady state, \( i^h = r^n + \sigma \), so the log of the left-hand side of (A16) equals \( -r^n - \sigma \), which implies that the log of the right-hand side of (A16) must also equal \( -r^n - \sigma \)—thus \( \Lambda = -r^n - \sigma \). Moreover, at the natural steady state, \( \Lambda_1 = u'(0)y^n \). And, since inflation is zero and output is constant at that steady state, \( \Lambda_2 = \beta \).

Using the results, we obtain a first-order approximation of \( \Lambda(\Lambda_1, \Lambda_2) \) around the natural steady state:

\[
\Lambda = -r^n - \sigma + \frac{\partial \Lambda}{\partial \Lambda_1} [\Lambda_1 - u'(0)y^n] + \frac{\partial \Lambda}{\partial \Lambda_2} [\Lambda_2 - \beta].
\]

Factoring out \( u'(0)y^n \) and \( \beta \), and using the definitions of \( \Lambda_1 \) and \( \Lambda_2 \), we obtain

\[
\Lambda = -r^n - \sigma + u'(0)y^n \cdot \frac{\partial \Lambda}{\partial \Lambda_1} \cdot \frac{y(t)}{y^n} - 1 + \beta \cdot \frac{\partial \Lambda}{\partial \Lambda_2} \cdot \left[ \frac{p(t)y(t)}{p(t+1)y(t+1)} - 1 \right].
\]

Since \( \Lambda = \ln(\Lambda_1 + \Lambda_2) \), we obviously have

\[
\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{\Lambda_1 + \Lambda_2}.
\]

In the first-order approximation, the derivatives are evaluated at the natural state, so their value is

\[
\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{u'(0)y^n + \beta}.
\]
Hence, (A18) becomes

\[(A19) \quad \Lambda = -r^n - \sigma + \frac{u'(0)y_n}{u'(0)y^n + \beta} \left[ \frac{y(t)}{y^n} - 1 \right] + \frac{\beta}{u'(0)y^n + \beta} \left[ \frac{p(t)y(t)}{p(t+1)y(t+1)} - 1 \right].\]

Last, since the first-order approximation of \(\ln(x)\) at \(x = 1\) is \(x - 1\), around the natural steady state we have

\[(A20) \quad \frac{y(t)}{y^n} - 1 = \ln \left( \frac{y(t)}{y^n} \right) = \hat{y}(t)\]

and

\[
\frac{p(t)y(t)}{p(t+1)y(t+1)} - 1 = \ln \left( \frac{p(t)y(t)}{p(t+1)y(t+1)} \right) = \ln \left( \frac{y(t)}{y^n} \right) - \ln \left( \frac{y(t+1)}{y^n} \right) - \ln \left( \frac{p(t+1)}{p(t)} \right) = \hat{y}(t) - \hat{y}(t+1) - \pi(t+1).
\]

Hence, we rewrite (A19) as

\[
\Lambda = -r^n - \sigma + \frac{u'(0)y_n}{u'(0)y^n + \beta}\hat{y}(t) + \frac{\beta}{u'(0)y^n + \beta} [\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)]
\]

\[
= -r^n - \sigma + (1 - \alpha)\hat{y}(t) + \alpha [\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)]
\]

where

\[
\alpha = \frac{\beta}{\beta + u'(0)y^n}.
\]

In conclusion, taking the log of the Euler equation (A16) yields

\[
-i^h(t) = -r^n - \sigma + (1 - \alpha)\hat{y}(t) + \alpha [\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)]
\]

This equation is valid up to terms that are second order around the natural steady state. Reshuffling
the terms, and noting that \( i^h(t) = i(t) + \sigma \), yields

\[
\hat{y}(t) = \alpha \hat{y}(t + 1) - [i(t) - r^n - \alpha \pi(t + 1)] .
\]

**Discounting.** Because the marginal utility of wealth \( u'(0) \) is positive, we have \( \alpha = \beta / [\beta + u'(0)y^n] < 1 \) in (A21). Thus the Euler equation is discounted: future output, \( \hat{y}(t + 1) \), appears discounted by the coefficient \( \alpha < 1 \). Such discounting may appear for a variety of reasons: overlapping generations (Del Negro, Giannoni, and Patterson 2015; Eggertsson, Mehrotra, and Robbins 2019); heterogeneous agents facing borrowing constraints and cyclical income risk (McKay, Nakamura, and Steinsson 2017; Acharya and Dogra 2019; Bilbiie 2019); consumers’ bounded rationality (Gabaix 2016); incomplete information (Angeletos and Lian 2018); bonds in the utility function (Campbell et al. 2017); and a cost of borrowing increasing in household debt (Beaudry and Portier 2018).

To make discounting more apparent, we solve the Euler equation forward:

\[
\hat{y}(t) = - \sum_{k=0}^{\infty} \alpha^k [i(t + k) - r^n - \alpha \pi(t + k + 1)] .
\]

The effect on current output of interest rates \( k \) periods in the future is discounted by \( \alpha^k < 1 \); hence, discounting is stronger for interest rates further in the future (McKay, Nakamura, and Steinsson 2017, p. 821).

**Phillips curve.** Next we log-linearize the Phillips curve (A17). We start with the left-hand side of (A17). Note that the first-order approximations of \( x(x - 1) \) and \( \ln(x) \) at \( x = 1 \) are both \( x - 1 \). This means that up to second-order terms around \( x = 1 \), we have \( x(x - 1) = \ln(x) \). Hence, up to second-order terms around the natural steady state,

\[
\frac{p(t)}{p(t - 1)} \left[ \frac{p(t)}{p(t - 1)} - 1 \right] = \ln\left( \frac{p(t)}{p(t - 1)} \right) = \pi(t) .
\]

We turn to the right-hand side of (A17). Following the same logic, up to second-order terms
around the natural steady state, we have

$$\beta \frac{p(t + 1)}{p(t)} \left[ \frac{p(t + 1)}{p(t)} - 1 \right] = \beta \ln \left( \frac{p(t + 1)}{p(t)} \right) = \beta \pi(t + 1).$$

Furthermore, using (A20), we find that up to second-order terms around the natural steady state,

$$\frac{\epsilon - 1}{\gamma} \left[ \frac{\gamma(t)}{\gamma^n} - 1 \right] = \frac{\epsilon - 1}{\gamma} \dot{y}(t).$$

Combining all these results, we find that the Phillips curve (A17) implies

(A22) $$\pi(t) = \beta \pi(t + 1) + \frac{\epsilon - 1}{\gamma} \dot{y}(t).$$

This equation is valid up to terms that are second order around the natural steady state.

**Appendix D. Proofs**

We provide alternative proofs of propositions 1 and 2. These proofs are not graphical but algebraic; they are closer to the proofs found in the literature. We also complement the proof of proposition 4 presented in the main text.

**Alternative proof of proposition 1**

We study the properties of the dynamical system generated by the Euler equation (4) and Phillips curve (1) in normal times. The natural rate of interest is positive and monetary policy imposes

$$r(\pi) = r^n + (\phi - 1)\pi.$$ 

**Steady state.** A steady state must satisfy the steady-state Phillips curve (3) and steady-state Euler equation (6). These equations form a linear system:

$$\pi = \frac{\epsilon \kappa}{\delta \gamma a} (y - \gamma^n).$$
\[(\phi - 1)\pi = -u'(0)(y - y^n).\]

As \([y = y^n, \pi = 0]\) satisfies both equations, it is a steady state. Furthermore the two equations are non-parallel. In the NK model this is obvious since \(u'(0) = 0\). In the WUNK model the slope of the second equation is \(-u'(0)/(\phi - 1)\): if \(\phi > 1\), the slope is negative; if \(\phi \in [0, 1)\), the slope is positive and greater than \(u'(0)\) and thus than \(\epsilon \kappa / (\delta y a)\), as (7) holds; in both cases the two equations have different slope. We conclude that \([y^n, 0]\) is the unique steady state. The monetary-policy rate at \([y^n, 0]\) is given by \(i = r^n + \phi \times 0 = r^n > 0\).

**Linearization.** The Euler-Phillips system is nonlinear, so we determine its dynamical properties by linearizing it around its steady state. For linearization, we write The Phillips curve as \(\hat{\pi}(t) = P(y(t), \pi(t))\), where \(P(y, \pi) = \delta \pi - \epsilon \kappa (y - y^n)/(y a)\), and the Euler equation as \(\hat{y}(t) = E(y(t), \pi(t))\), where \(E(y, \pi) = y[(\phi - 1)\pi + u'(0)(y - y^n)]\). The Euler-Phillips system is linearized as follows:

\[
\begin{bmatrix}
\hat{y}(t) \\
\hat{\pi}(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial E}{\partial y} & \frac{\partial E}{\partial \pi} \\
\frac{\partial P}{\partial y} & \frac{\partial P}{\partial \pi}
\end{bmatrix}
\begin{bmatrix}
y(t) - y^n \\
\pi
\end{bmatrix},
\]

where the derivatives are evaluated at \([y = y^n, \pi = 0]\). We have \(\partial E / \partial y = y^n u'(0), \partial E / \partial \pi = y^n (\phi - 1), \partial P / \partial y = -\epsilon \kappa /(y a)\), and \(\partial P / \partial \pi = \delta\). Accordingly the linearized system is

\[(\text{A23})
\begin{bmatrix}
\hat{y}(t) \\
\hat{\pi}(t)
\end{bmatrix} =
\begin{bmatrix}
u'(0)y^n & (\phi - 1)y^n \\
-\epsilon \kappa / (y a) & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) - y^n \\
\pi(t)
\end{bmatrix}.
\]

We denote by \(M\) the matrix in (A23), and by \(\mu_1 \in \mathbb{C}\) and \(\mu_2 \in \mathbb{C}\) the two eigenvalues of \(M\), assumed to be distinct.
Solution with two real eigenvalues. We begin by solving (A23) when \( \mu_1 \) and \( \mu_2 \) are real and nonzero. Without loss of generality, we assume \( \mu_1 < \mu_2 \). Then the solution takes the form

\[
\begin{bmatrix}
y(t) - y^n \\
\pi(t)
\end{bmatrix} = x_1 e^{\mu_1 t} \mathbf{v}_1 + x_2 e^{\mu_2 t} \mathbf{v}_2,
\]

where \( \mathbf{v}_1 \in \mathbb{R}^2 \) and \( \mathbf{v}_2 \in \mathbb{R}^2 \) are the linearly independent eigenvectors respectively associated with the eigenvalues \( \mu_1 \) and \( \mu_2 \), and \( x_1 \in \mathbb{R} \) and \( x_2 \in \mathbb{R} \) are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, p. 35).

From (A24), we see that the Euler-Phillips system is a source around the steady state if \( \mu_1 > 0 \) and \( \mu_2 > 0 \). Moreover, the trajectories are tangent to \( \mathbf{v}_1 \) when \( t \to -\infty \) and are parallel to \( \mathbf{v}_2 \) when \( t \to +\infty \). The system is a saddle if \( \mu_1 < 0 \) and \( \mu_2 > 0 \); in that case, the vector \( \mathbf{v}_1 \) gives the direction of the stable line (saddle path) while the vector \( \mathbf{v}_2 \) gives the direction of the unstable line. Lastly, if \( \mu_1 < 0 \) and \( \mu_2 < 0 \), the system is a sink. (See Hirsch, Smale, and Devaney 2013, pp. 40–44.)

Solution with two complex eigenvalues. Next we solve (A23) when \( \mu_1 \) and \( \mu_2 \) are complex conjugates. We write the eigenvalues as \( \mu_1 = \theta + i\zeta \) and \( \mu_2 = \theta - i\zeta \). We also write the eigenvector associated with \( \mu_1 \) as \( \mathbf{v}_1 + i \mathbf{v}_2 \), where the vectors \( \mathbf{v}_1 \in \mathbb{R}^2 \) and \( \mathbf{v}_2 \in \mathbb{R}^2 \) are linearly independent. Then the solution takes a more complicated form:

\[
\begin{bmatrix}
y(t) - y^n \\
\pi(t)
\end{bmatrix} = e^{\theta t} [\mathbf{v}_1, \mathbf{v}_2] \begin{bmatrix}
\cos(\zeta t) & \sin(\zeta t) \\
-\sin(\zeta t) & \cos(\zeta t)
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]

where \( [\mathbf{v}_1, \mathbf{v}_2] \in \mathbb{R}^{2\times2} \) is a 2 \( \times \) 2 matrix whose columns are respectively the real and imaginary components of an eigenvector of \( M \), and \( x_1 \in \mathbb{R} \) and \( x_2 \in \mathbb{R} \) are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, pp. 44–55).

These solutions wind periodically around the steady state, either moving toward it (\( \theta < 0 \)) or away from it (\( \theta > 0 \)). Hence, the Euler-Phillips system is a spiral source if the \( \theta > 0 \) and a spiral sink if \( \theta < 0 \). In the special case \( \theta = 0 \), the solutions just circle the steady state: the Euler-Phillips
system is a center. (See Hirsch, Smale, and Devaney 2013, pp. 44–47.)

Classification. We classify the Euler-Phillips system from the trace and determinant of $M$ (see Hirsch, Smale, and Devaney 2013, pp. 61–64). The classification relies on the property that $\text{tr}(M) = \mu_1 + \mu_2$ and $\det(M) = \mu_1 \mu_2$. The following situations will occur in the NK and WUNK models:

- $\det(M) < 0$: the Euler-Phillips system is a saddle. This is because $\det(M) < 0$ indicates that $\mu_1$ and $\mu_2$ are real, nonzero, and of opposite sign. (Indeed, if $\mu_1$ and $\mu_2$ were real and of the same sign, $\det(M) = \mu_1 \mu_2 > 0$; and if they were complex conjugates, $\det(M) = \mu_1 \overline{\mu_1} = \text{Re}(\mu_1)^2 + \text{Im}(\mu_1)^2 > 0$.)

- $\det(M) > 0$ and $\text{tr}(M) > 0$: the Euler-Phillips system is a source. This is because $\det(M) > 0$ indicates that $\mu_1$ and $\mu_2$ are real, nonzero, and of the same sign; or complex conjugates. Since in addition $\text{tr}(M) = \delta > 0$, $\mu_1$ and $\mu_2$ must be real and positive, or complex with positive real part. (Indeed, if $\mu_1$ and $\mu_2$ were real and negative, $\text{tr}(M) = \mu_1 + \mu_2 < 0$; if they were complex conjugates with negative real part, $\text{tr}(M) = \mu_1 + \overline{\mu_1} = 2 \text{Re}(\mu_1) < 0$.)

We now compute the trace and determinant of $M$ to classify the Euler-Phillips system:

\[
\text{tr}(M) = \delta + u'(0)y^n
\]
\[
\det(M) = \delta u'(0)y^n + (\phi - 1) \frac{\varepsilon \kappa}{\gamma a} y^n.
\]

In the NK model, $u'(0) = 0$, so $\text{tr}(M) = \delta$ and $\det(M) = (\phi - 1)y^n \varepsilon \kappa / (\gamma a)$. If $\phi > 1$, $\det(M) > 0$ and $\text{tr}(M) > 0$, so the system is a source. If $\phi < 1$, $\det(M) < 0$, so the system is a saddle.

In the WUNK model, since $\phi - 1 \geq -1$ and (7), we have

\[
\det(M) \geq \delta u'(0)y^n - \frac{\varepsilon \kappa}{\gamma a} y^n = \delta y^n \left[ u'(0) - \frac{\varepsilon \kappa}{\delta \gamma a} \right] > 0.
\]

Furthermore, $\text{tr}(M) > \delta > 0$. Since $\det(M) > 0$ and $\text{tr}(M) > 0$, the system is a source.
Alternative proof of proposition 2

We study the properties of the dynamical system generated by the Euler equation (4) and Phillips curve (1) at the ZLB. The natural rate of interest is negative and monetary policy imposes $r(\pi) = -\pi$.

**Steady state.** A steady state must satisfy the steady-state Phillips curve (3) and the steady-state Euler equation (6). These equations form a linear system:

\begin{align*}
(A25) & \quad \pi = \frac{\varepsilon\kappa}{\delta\gamma a} (y - y^n) \\
(A26) & \quad \pi = -r^n + u'(0)(y - y^n).
\end{align*}

A solution to this system with positive output is a steady state.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

\begin{align*}
(A27) & \quad \pi^z = -r^n \\
(A28) & \quad y^z = y^n - \frac{\delta\gamma a}{\varepsilon\kappa} r^n.
\end{align*}

Since $r^n < 0$, the solution satisfies $y^z > y^n > 0$: the solution has positive output so it is a steady state. Hence the NK model admits a unique steady state at the ZLB, where $\pi^z > 0$ (since $r^n < 0$) and $y^z > y^n$.

In the WUNK model, since (7) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Substituting $y - y^n$ out of (A26) using (A25), we find that inflation in that solution is given by

\begin{equation}
(A29) \quad \pi^z = \frac{r^n}{u'(0)\delta\gamma a / (\varepsilon\kappa) - 1}.
\end{equation}

Condition (7) implies that $u'(0)\delta\gamma a / (\varepsilon\kappa) > 1$, so $\pi^z$ has the sign of $r^n$. Since $r^n < 0$, we conclude that $\pi^z < 0$.

Next, using (A25) and the value of $\pi$ given by (A29), we find that output in the unique solution
is given by

\[(A30)\]

\[y^z = y^n + r^n \frac{u'(0) - \epsilon \kappa / (\delta y a)}{u'(0) - \epsilon \kappa / (\delta y a)}.\]

Since (7) holds and \(r^n < 0\), we infer that \(y^z < y^n\).

The last step is to verify that \(y^z > 0\). Using (A30), we need

\[y^n > \frac{-r^n}{u'(0) - \epsilon \kappa / (\delta y a)}.\]

Since \(-r^n = u'(0)y^n - \delta\) and \(u'(0) - \epsilon \kappa / (\delta y a) > 0\) (from (5) and (7)), this is equivalent to

\[
\left[ u'(0) - \frac{\epsilon \kappa}{\delta y a} \right] y^n > u'(0)y^n - \delta.\]

Eliminating \(u'(0)y^n\) on both sides, we find that this is equivalent to

\[-\frac{\epsilon \kappa y^n}{\delta y a} > -\delta,\]

or

\[\frac{\epsilon \kappa y^n}{\gamma a} < \delta^2.\]

Using (2), we have \((\epsilon \kappa y^n)/(\gamma a) = (\epsilon - 1)/\gamma\). So we need to verify that \(\delta > \sqrt{(\epsilon - 1)/\gamma}\). But we have imposed \(\delta > \sqrt{(\epsilon - 1)/\gamma}\) in the WUNK model (to accommodate a positive natural rate of interest). Thus we conclude that \(y^z > 0\): the solution to the system has positive output, so it is a steady state.

**Linearization.** The Euler-Phillips system is nonlinear, so we determine its dynamical properties by linearizing it around its steady state. The linearized system is

\[(A31)\]

\[
\begin{bmatrix}
\dot{y}(t) \\
\dot{\pi}(t)
\end{bmatrix} =
\begin{bmatrix}
u'(0)y^z & -y^z \\
-\epsilon \kappa / (\gamma a) & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) - y^z \\
\pi(t) - \pi^z
\end{bmatrix}.
\]
This matrix, denoted $M$, is obtained from the matrix in (A23) by setting $\phi = 0$ and replacing $y^n$ by $y^z$.

**Classification.** We classify the Euler-Phillips system by computing the trace and determinant of $M$. We have $\text{tr}(M) = \delta + u'(0)y^z > 0$ and

$$\det(M) = \delta y^2 \left[ u'(0) - \frac{\varepsilon \kappa}{\gamma a} \right].$$

In the NK model, $u'(0) = 0$ so $\det(M) < 0$, which implies that the Euler-Phillips system is a saddle. In the WUNK model, (7) implies that $\det(M) > 0$; since in addition $\text{tr}(M) > 0$, the Euler-Phillips system is a source. In fact, in the WUNK model, the discriminant of the characteristic equation of $M$ is strictly positive:

$$\text{tr}(M)^2 - 4 \det(M) = \delta^2 + [u'(0)y^n]^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{\varepsilon \kappa}{\gamma a} y^n = \left[ \delta - u'(0)y^n \right]^2 + 4\frac{\varepsilon \kappa}{\gamma a} y^n > 0.$$ 

Hence the eigenvalues of $M$ are real, not complex: the Euler-Phillips system is a nodal source, not a spiral source.

**Complement to proof of proposition 4**

We characterize the forward-guidance duration $\Delta^*$ in the NK model, and the ZLB duration $T^*$ in the WUNK model.

In the NK model, the forward-guidance duration $\Delta^*$ is the duration that brings the economy on the ZLB unstable line at time $T$ (panel C of figure 3). With longer forward guidance, the economy is above the unstable line at time $T$, and so it is connected to ZLB trajectories coming from the northeast quadrant of the phase diagram (panel D of figure 3). Then, during the ZLB and forward guidance, inflation is positive and output is above its natural level. Moreover, since the position at the end of the ZLB is unaffected by the duration of the ZLB, continuously increasing the duration of the ZLB when $\Delta > \Delta^*$ will lead initial output and inflation to be infinitely high.
In the WUNK model, for any forward-guidance duration, the economy at the beginning of forward guidance is bound to be in the right-hand green triangle of figure 4, panel D. All the points in that triangle are connected to ZLB trajectories that flow from the ZLB steady state, through the left-hand green triangle of figure 4, panel D. For any of these trajectories, initial inflation $\pi(0)$ converges from above to the ZLB steady state’s inflation $\pi^z$ as the ZLB duration $T$ goes to infinity. Since $\pi^z < 0$, we infer that for each trajectory, there is a $\hat{T}$, such that for any $T > \hat{T}$, $\pi(0) < 0$. (Furthermore, as showed in panel D of figure 4, $y(0) < y^n$ whenever $\pi(0) < 0$.) Then we have $T^* = \max\{\hat{T}\}$. The maximum exists because the right-hand green triangle is a closed and bounded subset of $\mathbb{R}^2$, so the set $\{\hat{T}\}$ is a closed and bounded subset of $\mathbb{R}$, and so this set admits a maximum. We know that the set $\{\hat{T}\}$ is closed and bounded because the function that maps a position at the beginning of forward guidance to a threshold $\hat{T}$ is continuous.

**Appendix E. Model with government spending**

We introduce government spending into the model of section 3. We compute the model’s Euler equation and Phillips curve, and use them to construct the phase diagrams of figures 5 and 6.

**Assumptions**

The government purchases quantities $g_k(t)$ of the goods $k \in [0, 1]$. These quantities are aggregated into an index of public consumption

\[
(A32) \quad g(t) \equiv \left[ \int_0^1 g_k(t)^{(e-1)/e} \, dk \right]^{e/(e-1)}.
\]

Public consumption $g(t)$ enters separately into households’ utility functions. Government expenditure is financed with lump-sum taxation.

Additionally, the disutility of labor is convex—which implies a finite Frisch elasticity of labor supply. Household $j$ incurs disutility

\[
\frac{\kappa^{1+\eta}}{1+\eta} h_j(t)^{1+\eta}.
\]
from working, where $\eta > 0$ is the inverse of the Frisch elasticity. The utility function is altered to ensure that government spending affects inflation and private consumption.

**Euler equation and Phillips curve**

We begin by computing the government’s spending on each good. At any time $t$ the government chooses the amounts $g_j(t)$ of each good $j \in [0, 1]$ to minimize the expenditure

$$\int_0^1 p_j(t) g_j(t) \, dj$$

subject to the constraint of providing an amount of public consumption $g$:

$$\left[ \int_0^1 g_j(t)^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} = g(t).$$

To solve the government’s problem at time $t$, we set up a Lagrangian:

$$\mathcal{L} = \int_0^1 p_j(t) g_j(t) \, dj + C \cdot \left\{ g - \left[ \int_0^1 g_j(t)^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} \right\},$$

where $C$ is the Lagrange multiplier on the public-consumption constraint. We then follow the same steps as in the derivation of (A6). The first-order conditions with respect to $g_j(t)$ for all $j \in [0, 1]$ are

$$\frac{\partial \mathcal{L}}{\partial g_j} = 0.$$ 

These conditions imply

$$(A33) \quad p_j(t) = C \cdot \left[ \frac{g_j(t)}{g(t)} \right]^{-1/\epsilon}.$$ 

Appropriately integrating (A33) over all $j \in [0, 1]$, and using (A3) and (A32), we find

$$(A34) \quad C = p(t).$$
Combining (A33) and (A34), we obtain the government’s demand for good \( j \):

\[
g_j(t) = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} g(t).
\]

Next we determine household \( j \)'s saving and pricing. The current-value Hamiltonian of the household’s problem is

\[
\mathcal{H}_j = \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(1-\epsilon)/\epsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \frac{1}{1 + \eta} \left[ \frac{\rho_j}{a} y_j^d(p_j(t), t) \right]^{1+\eta} - \frac{\eta}{2} \pi_j(t)^2
+ \mathcal{A}_j(t) \left[ t^b(t) b_j(t) + \int_1^t \left( b_j(t) - b(t) \right) \frac{\pi_j(t)}{\pi_j(0)} \, dk - \tau(t) \right] + \mathcal{B}_j(t) \pi_j(t) p_j(t).
\]

The Hamiltonian’s terms including the consumption levels \( c_{jk}(t) \) are the same as in appendix A, so the optimality conditions \( \partial \mathcal{H}_j / \partial c_{jk} = 0 \) remain the same, which implies that (A1), (A4), and (A5) remain valid.

Adding the government’s demand, given by (A35), to households’ demand, given by (A5), we obtain the total demand for good \( j \) at time \( t \):

\[
y_j^d(p_j(t), t) = g_j(t) + \int_0^1 c_{jk}(t) \, dk = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} y(t),
\]

where \( y(t) \equiv g(t) + \int_0^1 c(t) \, dj \) measures total consumption. The expression for \( y_j^d(p_j(t), t) \) enters into the Hamiltonian \( \mathcal{H}_j \).

The Hamiltonian’s terms including bond holdings \( b_j(t) \) are the same as in appendix A; therefore, the optimality condition \( \partial \mathcal{H}_j / \partial b_j = \delta \mathcal{A}_j - \mathcal{A}_j \) remains the same, implying that the Euler equation (A7) remains valid.

The Hamiltonian’s terms including inflation \( \pi_j(t) \) are also the same as in appendix A, so the optimality condition \( \partial \mathcal{H}_j / \partial \pi_j = 0 \) is unchanged. Equations (A8) and (A9) therefore remain valid.

Last, because the disutility from labor is convex, the optimality condition \( \partial \mathcal{H}_j / \partial p_j = \delta \mathcal{B}_j - \mathcal{B}_j \) is
modified. The condition now yields
\[
\frac{\epsilon}{p_j} \left( \frac{\kappa}{a} y_j \right)^{1+\eta} + (1 - \epsilon) \mathcal{A}_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \hat{\mathcal{B}}_j,
\]
which can be rewritten
\[
\pi_j - \frac{(\epsilon - 1) y_j \mathcal{A}_j}{\mathcal{B}_j p_j} \left[ p_j - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} \frac{y_j}{\mathcal{A}_j} \right] = \delta - \mathcal{B}_j.
\]
Then, using (A4), (A8), and (A9), we obtain the household’s Phillips curve:
\[
(A36) \quad \frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j} \left[ \frac{p_j}{p} - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} \frac{y_j}{\mathcal{A}_j} \right].
\]
In equilibrium all households behave the same, so we can simplify the Euler equation (A7) to
\[
(A37) \quad \frac{\dot{c}}{c} = r - \delta + \sigma + u'(0)c.
\]
Similarly, the Phillips curve (A36) reduces to
\[
(A38) \quad \dot{\pi} = \delta \pi + \frac{(\epsilon - 1)(c + g)}{\gamma c} \left[ 1 - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} (c + g) \eta c \right],
\]
where \( c + g = y \) is aggregate output. The interpretation of the Phillips curve remains the same as in the basic model, except that the real marginal cost—the term after \( \epsilon/(\epsilon - 1) \) in the curly brackets—takes a more complicated form, because the marginal disutility of labor is more complex:
\[
\kappa^{1+\eta} \eta^n = \kappa^{1+\eta} \left( \frac{y}{a} \right)^\eta = \kappa^{1+\eta} \left( \frac{c + g}{a} \right)^\eta.
\]

Linearized Euler-Phillips system

We now linearize the Euler-Phillips system around the natural steady state. This steady state has zero inflation and no government spending. Since \( \dot{\pi} = \pi = g = 0 \) at the natural steady state, (A38)
implies that the natural level of consumption is

\[ c^n = \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{1/(1+\eta)} \frac{a}{\kappa}. \]

And since \( \dot{c} = 0 \) and \( c = c^n \) at the natural steady state, (A37) implies that the natural rate of interest is

\[ r^n = \delta - \sigma - u'(0)c^n. \]

**Euler equation.** We begin by linearizing the Euler equation (A37) around the point \([c = c^n, \pi = 0]\). We consider two different monetary-policy rules. First, when monetary policy is normal, \( r(\pi) = r^n + (\phi - 1)\pi \). Then the Euler equation is

\[ \dot{c} = E(c, \pi), \]

where

\[ E(c, \pi) = c [(\phi - 1)\pi + u'(0)(c - c^n)]. \]

The linearized version is

\[ \dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi} \pi, \]

where the derivatives are evaluated at \([c = c^n, \pi = 0]\). We have \( E(c^n, 0) = 0 \) and

\[ \frac{\partial E}{\partial c} = c^n u'(0), \quad \frac{\partial E}{\partial \pi} = c^n (\phi - 1). \]

So the linearized Euler equation with normal monetary policy is

(A39) \[ \dot{c} = c^n [(\phi - 1)\pi + u'(0)(c - c^n)]. \]

In steady state, with normal monetary policy, the linearized Euler equation becomes

(A40) \[ \pi = -\frac{u'(0)}{\phi - 1} (c - c^n). \]

Second, when monetary policy is at the ZLB, \( r(\pi) = -\pi \). Then the Euler equation can be written
\[ \dot{c} = E(c, \pi) \]

where

\[ E(c, \pi) = c \left[-r^n - \pi + u'(0)(c - c^n)\right]. \]

The linearized version is

\[ \dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi} \pi, \]

where the derivatives are evaluated at \([c = c^n, \pi = 0]\). We have \(E(c^n, 0) = -c^n r^n\) and

\[ \frac{\partial E}{\partial c} = c^n u'(0), \quad \frac{\partial E}{\partial \pi} = -c^n. \]

So the linearized Euler equation at the ZLB is

\[ (A41) \quad \dot{c} = c^n \left[-r^n - \pi + u'(0)(c - c^n)\right]. \]

In steady state, at the ZLB, the linearized Euler equation becomes

\[ (A42) \quad \pi = -r^n + u'(0)(c - c^n). \]

**Phillips curve.** Next we linearize the Phillips curve (A38) around the point \([c = c^n, \pi = 0, g = 0]\).

The Phillips curve can be written \(\dot{\pi} = P(c, \pi, g)\) where

\[ P(c, \pi, g) = \delta \pi + \frac{(\epsilon - 1)(c + g)}{\gamma c} \left[ 1 - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1 + \eta} (c + g)^\eta c \right]. \]

The linearized version is

\[ \dot{\pi} = P(c^n, 0, 0) + \frac{\partial P}{\partial c}(c - c^n) + \frac{\partial P}{\partial \pi} \pi + \frac{\partial P}{\partial g} g, \]

where the derivatives are evaluated at \([c = c^n, \pi = 0, g = 0]\). We have \(P(c^n, 0, 0) = 0\) and

\[ \frac{\partial P}{\partial c} = -\frac{\epsilon}{\gamma} \left( \frac{\kappa}{a} \right)^{1 + \eta} (1 + \eta)(c^n)^\eta = -(1 + \eta) \frac{\epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1 + \eta)} \]
\[
\frac{\partial P}{\partial \pi} = \delta \\
\frac{\partial P}{\partial g} = -\frac{\varepsilon}{\gamma a} \left( \frac{\kappa}{a} \right)^{1+\eta} \eta c^n = -\eta \frac{\varepsilon \kappa}{\gamma a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)} 
\]

Hence, the linearized Phillips curve is

(A43) \[ \pi = \delta \pi - \frac{\varepsilon \kappa}{\gamma a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)} \left[ (1 + \eta)(c - c^n) + \eta g \right]. \]

In steady state, the linearized Phillips curve becomes

(A44) \[ \pi = -\frac{\varepsilon \kappa}{\delta \gamma a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)} \left[ (1 + \eta)(c - c^n) + \eta g \right]. \]

**Properties of the linearized Euler-Phillips system**

**Normal times.** We first study the linearized Euler-Phillips system in normal times: the natural rate of interest is positive, monetary policy follows \( i(\pi) = r^n + \phi \pi \), and government spending is zero. The system is composed of the linear differential equations (A39) and (A43).

A phase diagram with private consumption \( c \) on the horizontal axis and inflation \( \pi \) on the vertical axis would look exactly like the phase diagrams in panels A and B of figure 1. Such diagram can be constructed with the same methodology as in section 3. Therefore, in normal times, the linearized Euler-Phillips system shares the properties of proposition 1.

An algebraic approach confirms this result. In normal times the linearized Euler-Phillips system can be written

(A45) \[
\begin{bmatrix}
\dot{c} \\
\dot{\pi}
\end{bmatrix} =
\begin{bmatrix}
u'(0)c^n & (\phi - 1)c^n \\
-(1 + \eta)\frac{\varepsilon \kappa}{\gamma a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)} & \delta
\end{bmatrix}
\begin{bmatrix}
c - c^n \\
\pi
\end{bmatrix}.
\]

We denote the above matrix \( M \). We can classify the Euler-Phillips system using the trace and
determinant of $M$:

$$\text{tr}(M) = \delta + u'(0)c^n$$

$$\det(M) = \delta c^n \left[ u'(0) + (\phi - 1)(1 + \eta) \frac{\epsilon \kappa}{\delta \gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} \right].$$

In the NK model, $u'(0) = 0$ so the sign of $\det(M)$ is given by the sign of $\phi - 1$. Accordingly, when monetary policy is active ($\phi > 1$), $\det(M) > 0$; since $\text{tr}(M) = \delta > 0$, the Euler-Phillips system is a source. In contrast, when monetary policy is passive ($\phi < 1$), $\det(M) < 0$, indicating that the Euler-Phillips system is a saddle.

In the WUNK model, since $\phi - 1 \geq -1$ for any $\phi \geq 0$, we have

$$\det(M) \geq \delta c^n \left[ u'(0) - (1 + \eta) \frac{\epsilon \kappa}{\delta \gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} \right]$$

Moreover, the WUNK assumption (9) says that the term in square brackets is positive, so $\det(M) > 0$. Since we also have $\text{tr}(M) > \delta > 0$, we conclude that the Euler-Phillips system is a source whether monetary policy is active or passive.

**ZLB.** We turn to the linearized Euler-Phillips system at the ZLB: the natural rate of interest is negative, monetary policy sets $i = 0$, and government spending is $g > 0$. The system is composed of the linear differential equations (A41) and (A43).

A phase diagram with private consumption $c$ on the horizontal axis and inflation $\pi$ on the vertical axis would look like the phase diagrams in panels C and D of figure 1, with one difference: the Phillips line shifts upward when government spending is positive, so it lies above the point $[c = c^n, \pi = 0]$. While this shift does not affect the classification of the Euler-Phillips system (source versus saddle), it may change the properties of its steady state.

In fact, at the ZLB, the linearized Euler-Phillips system shares the properties of proposition 2,
except that the steady-state private consumption and inflation are given by

\[ c^g = c^n + \frac{r^n + \frac{\varepsilon K}{\delta y a} \left( \frac{\varepsilon - 1}{\varepsilon} \right) \eta}{\left( u'(0) - (1 + \eta) \frac{\varepsilon K}{\delta y a} \left( \frac{\varepsilon - 1}{\varepsilon} \right) \eta \right)^{(1+\eta)}} \]

(A46)

\[ \pi^g = \frac{(1 + \eta) r^n + u'(0) \eta g}{u'(0) \frac{\delta y a}{\varepsilon} \left( \frac{\varepsilon - 1}{\varepsilon} \right) \eta \left( 1 + \eta \right)} - (1 + \eta). \]

(A47)

The values \( c^g \) and \( \pi^g \) are obtained by solving the linear system given by (A42) and (A44). These expressions indicate that whether steady-state consumption is above or below natural consumption depends on the amount of government spending. Furthermore, in the WUNK model, whether inflation is positive or negative also depends on the amount of government spending.

An algebraic approach confirms the classification of the linearized Euler-Phillips system. Once rewritten it in canonical form, the system becomes

\[
\begin{bmatrix}
\dot{c} \\
\dot{\pi}
\end{bmatrix} =
\begin{bmatrix}
u'(0) c^n & -c^n \\
-(1 + \eta) \frac{\varepsilon K}{\delta y a} \left( \frac{\varepsilon - 1}{\varepsilon} \right) \eta \left( 1 + \eta \right) \delta & \delta
\end{bmatrix}
\begin{bmatrix}
c - c^g \\
\pi - \pi^g
\end{bmatrix}.
\]

(A48)

We denote the above matrix \( M \). The trace and determinant of \( M \) are

\[ \text{tr}(M) = \delta + u'(0) c^n \]

\[ \det(M) = \delta c^n \left[ u'(0) - (1 + \eta) \frac{\varepsilon K}{\delta y a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{(1+\eta)} \right]. \]

In the NK model, \( u'(0) = 0 \) so \( \det(M) < 0 \), indicating that the Euler-Phillips system is a saddle. In the WUNK model, since we assume (9), \( \det(M) > 0 \); since we also have \( \text{tr}(M) > 0 \), we conclude that the Euler-Phillips system is a source. We can also show that \( \text{tr}(M)^2 - 4 \det(M) > 0 \), which indicates that the system is a nodal source, not a spiral source.
Appendix F. Proofs in the model with government spending

We complement the graphical proofs of propositions 5 and 9 developed in the main text. These propositions pertain to the model with government spending.

Complement to proof of proposition 5

We characterize the amount \( g^* \) in the NK model, and we compute the limit of the government-spending multiplier in the WUNK model.

In the NK model, the amount \( g^* \) of government spending is the amount that makes the unstable line of the dynamical system go through the natural steady state. With a bit less spending than \( g^* \) (panel B of figure 5), the natural steady state is below the unstable line and is connected to trajectories coming from the southwest quadrant of the phase diagram. Hence, for \( g < g^* \), \( \lim_{T \to \infty} c(0; g) = -\infty \).

With a bit more spending than \( g^* \) (panel D of figure 5), the natural steady state is above the unstable line and is connected to trajectories coming from the northeast quadrant of the phase diagram. Hence, for \( g > g^* \), \( \lim_{T \to \infty} c(0; g) = +\infty \). Accordingly, for any \( s > 0 \), \( \lim_{T \to \infty} m(g^*, s) = +\infty \).

In the WUNK model, when the ZLB is infinitely long-lasting, the economy jumps to the ZLB steady state at time 0: \( \lim_{T \to \infty} c(0; g) = c^d(g) \), where \( c^d(g) \) is given by (A46). The steady-state consumption \( c^d(g) \) is linear in government spending \( g \), with a coefficient in front of \( g \) of

\[
\frac{\eta}{u'(0) \frac{\delta y a}{ek} \left( \frac{e}{e-1} \right)^{\eta/(1+\eta)} - (1 + \eta)}.
\]

Accordingly, for any \( s > 0 \), we have

\[
\lim_{T \to \infty} m(g, s) = 1 + \frac{\lim_{T \to \infty} c(0; g - s/2) - \lim_{T \to \infty} c(0; g + s/2)}{s} = 1 + \frac{c^d(g + s/2) - c^d(g - s/2)}{s} = 1 + \frac{\eta}{u'(0) \frac{\delta y a}{ek} \left( \frac{e}{e-1} \right)^{\eta/(1+\eta)} - (1 + \eta)},
\]
which yields (11).

**Complement to proof of proposition 9**

We compute the government-spending multiplier at the ZLB in the WUNK model. Private consumption and inflation at the ZLB steady state are determined by (A46) and (A47). The coefficients in front of government spending in these expressions are

\[
\frac{\eta}{u'(0) \delta \gamma a \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta)} \quad \text{and} \quad \frac{u'(0) \eta}{u'(0) \delta \gamma a \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta)}.
\]

Since (9) holds, both coefficients are positive. Hence, an increase in $g$ raises private consumption and inflation. Moreover, $dc/dg$ is given by the first of these coefficient; this immediately gives us the expression for the multiplier $dy/dg = 1 + dc/dg$.

**Appendix G. Expressing the WUNK assumption with estimable statistics**

We re-express the WUNK assumption in terms of estimable statistics. We first work on the model with linear disutility of labor, in which the WUNK assumption is given by (7). We then turn to the model with convex disutility of labor, in which the WUNK assumption is given by (9).

**Linear disutility of labor**

When the disutility of labor is linear, the WUNK assumption is given by (7). Multiplying (7) by $y^n$, we obtain

\[
u'(0)y^n > \frac{1}{\delta} \cdot \frac{y^n \epsilon \kappa}{\gamma a}.
\]

We aim to assess the condition in normal times (several of the parameters could vary over the business cycle). The time discount rate $\delta$ has been estimated in numerous studies. We therefore only need to express $u'(0)y^n$ and $(y^n \epsilon \kappa)/(\gamma a)$ in terms of estimable statistics.

First, the definition of the natural rate of interest, given by (5), implies that $u'(0)y^n = \delta - \sigma - r^n$. Following Woodford (2011, p. 20), and in line with the New Keynesian literature, we set the
financial-intermediation spread to σ = 0 in normal times. Hence, in normal times, \( u'(0)y^n = \delta - r^n \).

This shows how to measure \( u'(0)y^n \): by estimating the gap between discount rate \( \delta \) and the average natural rate of interest \( r^n \)—both of which have been estimated by many studies.

Second, we show that \( (y^n\epsilon k)/(ya) \) can be measured from estimates of the New Keynesian Phillips curve. To establish this, we compute the discrete-time New Keynesian Phillips curve arising from our continuous-time model. We start from the first-order approximation \( \pi(t) = \pi(t + dt) - \hat{\pi}(t + dt)dt \) and use (1) to measure \( \hat{\pi}(t + dt) \). We obtain

\[
\pi(t) = \pi(t + dt) - \delta \pi(t + dt)dt + \frac{y^n\epsilon k}{ya} \cdot \frac{y(t) - y^n}{y^n} dt.
\]

(We have replaced \( y(t + dt)dt \) by \( y(t)dt \) since the difference between the two is of second order.)

Setting the unit of time to one quarter (as in the empirical literature) and \( dt = 1 \), we obtain

\[
(A49) \quad \pi(t) = (1 - \delta)\pi(t + 1) + \frac{y^n\epsilon k}{ya} x(t),
\]

where \( \pi(t) \) is quarterly inflation at time \( t \), \( \pi(t + 1) \) is quarterly inflation at time \( t + 1 \),

\[
x(t) = \frac{y(t) - y^n}{y^n}
\]

is the output gap at time \( t \). Equation (A49) is a typical New Keynesian Phillips curve, which has been estimated many times (see Mavroeidis, Plagborg-Moller, and Stock 2014). Hence we can measure \( (y^n\epsilon k)/(ya) \) by estimating the coefficient on output gap in a standard New Keynesian Phillips curve.

To sum up, we rewrite the WUNK assumption as \( \delta - r^n > \lambda/\delta \), where \( \lambda \) is the output-gap coefficient in a standard New Keynesian Phillips curve. This is just (13).
Convex disutility of labor

When the disutility of labor is convex, the WUNK assumption is given by (9):

\[ u'(0)y^n > \frac{1}{\delta} \cdot \frac{y^n \epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} (1 + \eta). \]

Following the previous steps, we can rewrite the WUNK assumption as \( \delta - r^n > \lambda / \delta \), where \( \delta \) is the time discount rate, \( r^n \) is the average natural rate of interest, and \( \lambda \) is the output-gap coefficient in a standard New Keynesian Phillips curve.

The only difference with the previous derivation occurs when computing the discrete-time New Keynesian Phillips curve arising from the continuous-time model. To measure \( \dot{\pi}(t + dt) \), we use (A43) with \( g = 0 \) and so \( c = y \). As a result, (A49) becomes

\[ \pi(t) = (1 - \delta)\pi(t + 1) + \frac{y^n \epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} (1 + \eta)x(t), \]

where \( \pi(t) \) and \( \pi(t + 1) \) are quarterly inflation rates and \( x(t) \) is the output gap. This again is just a typical New Keynesian Phillips curve. Hence we can measure

\[ \frac{y^n \epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} (1 + \eta) \]

by estimating the output-gap coefficient in a standard New Keynesian Phillips curve.

References


