Resolving New Keynesian Anomalies with Wealth in the Utility Function

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The New Keynesian model makes several anomalous predictions at the zero lower bound: collapse of output and inflation, and implausibly large effects of forward guidance and government spending. To resolve these anomalies, we introduce wealth into the utility function. The justification is that wealth is a marker of social status, and people value social status. Since people save not only for future consumption but also to accrue social status, the Euler equation is modified. As a result, when the marginal utility of wealth is sufficiently large, the dynamical system representing the equilibrium at the zero lower bound becomes a source instead of a saddle—which resolves all the anomalies.

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1. Introduction

A current issue in monetary economics is that the New Keynesian model makes several anomalous predictions when the zero lower bound on nominal interest rates (ZLB) is binding: collapse of output and inflation to implausibly low levels (Eggertsson and Woodford 2004; Eggertsson 2011; Cochrane 2017), and implausibly large effects of forward guidance and government spending (Del Negro, Giannoni, and Patterson 2015; Carlstrom, Fuerst, and Paustian 2015; Christiano, Eichenbaum, and Rebelo 2011; Woodford 2011; Cochrane 2017).

The anomalies of the New Keynesian model can be traced to the behavior of the dynamical system representing the equilibrium at the ZLB. This dynamical system is composed of two differential equations: the Phillips curve, describing how prices are set, and the Euler equation, describing how consumers save. Mathematically, the diagnostic is straightforward: the anomalies arise because the dynamical system is a saddle at the ZLB. (In normal times, in contrast, the system is source, so there are no anomalies.) In economic terms, the anomalies arise at the ZLB because the response of households' consumption to the real interest rate is stronger than the response of firms' production to the inflation rate. Indeed, since the only motive for saving is future consumption, households are very forward-looking, and their response to interest rates is too strong for the model to behave well.

In this paper, we propose to remedy these anomalies by introducing relative wealth into the utility function. The justification is that relative wealth is a marker of social status, and people value high social status. Under this assumption, people save not only for future consumption but also because they accrue social status from their wealth. Hence, they are less forward-looking than in the standard model, and as a result their consumption responds less to interest rates, which resolves the anomalies. Mathematically, with enough marginal utility of wealth, the Euler equation is sufficiently modified that the dynamical system representing the ZLB equilibrium becomes a source instead of a saddle. At that point, the response to a temporary shock, however long-lasting, is a muted version of the response to a permanent shock, which is finite, so the anomalies disappear.

Indeed, our New Keynesian model with wealth in the utility function makes reasonable predictions at the ZLB, even when the ZLB is arbitrarily long-lasting. First, output and inflation are bounded below by the ZLB steady state, so they never completely collapse. Second, forward guidance and government spending have limited effects on output and inflation, even when the ZLB is very long-lasting. Our model has another advantage: it admits a well-behaved, determinate ZLB steady state; in the standard New Keynesian model, in contrast, the ZLB steady state is always indeterminate.

Beside its anomalous properties, the New Keynesian model has several other intriguing properties at the ZLB (Eggertsson 2010; Eggertsson and Krugman 2012). Some of these properties are labeled “paradoxes”: not because they are anomalous, but because they defy usual economic logic. We find that our model shares these other properties. First, the paradox of thrift holds: when all households desire to save more than their neighbors, the economy contracts and they end up saving the same amount as the neighbors. The paradox of toil also holds: when all households desire to work more, the economy contracts and they end up working less. The paradox of flexibility is present too: the economy contracts when prices become more flexible. Finally, the government-spending multiplier is above one, so public spending stimulates private consumption.

Related literature. Other papers have developed variants of the New Keynesian model in which the ZLB anomalies disappear. Eggertsson and Mehrotra (2014) introduce overlapping generations and downward nominal wage rigidity. Gabaix (2016) introduces bounded rationality. Diba and Loisel (2019) let the central bank determine the amount of bank reserves and the interest rate on these reserves. Cochrane (2018) takes into account a fiscal theory of the price level. Finally, Bilbiie (2019) and Acharya and Dogra (2019) introduce heterogeneous agents facing income risk. These variants tend to be more complex than the standard model, however. Overlapping generations, bounded rationality, and heterogeneity increase the complexity of the derivations. Bank reserves and fiscal theory of the price level add one differential equation to the dynamical system describing the equilibrium, which grows from two to three dimensions and becomes harder to analyze.

It may therefore be useful to strip the logic to the bone. We do so by keeping the deviation from the textbook New Keynesian model to a minimum: the sole difference is that relative wealth enters the utility function. As a result, the equilibrium system remains 2-dimensional, and the derivations exactly parallel those of the standard model—there only is an extra term in the Euler equation. Furthermore, to better understand where the ZLB anomalies come from and why they are resolved with wealth in the utility function, we veer away from numerical simulations and instead establish our results using phase diagrams.

Our paper is also related to the work of Ono and Yamada (2018), Michaillat and Saez (2014), and Michau (2018). These authors develop non-New-Keynesian models with wealth in the utility function that feature a well-behaved ZLB steady state. The results are not portable to the New Keynesian model, however, because they require strong forms of wage or price rigidity: Ono and Yamada assume that wages follow an exogenous time path; Michaillat and Saez assume that inflation is fixed; and Michau assumes that nominal wages are downward rigid.[1]

[1]Other papers have inserted wealth in the utility function to study long-run growth (Kurz 1968; Konrad 1992;

Last, our paper relates to the work of Fisher (2015), Campbell et al. (2017), Rannenberg (2019), Hagedorn (2018), Kaplan and Violante (2018), Auclert, Rognlie, and Straub (2018), and Aurissergues (2018), who build New Keynesian models with government bonds in the utility function.[2] The assumptions are slightly different: we assume that what enters the utility function is relative wealth, while these papers assume it is absolute wealth.[3] The assumptions are also applied differently. We use our assumption to resolve ZLB anomalies. These papers use theirs to generate risk-premium shocks (Fisher), alleviate the forward-guidance puzzle (Campbell et al.; Rannenberg), determinate the price level (Hagedorn), mimic the behavior of heterogeneous-agent models (Kaplan and Violante; Auclert, Rognlie, and Straub), and study the implications of nonseparable preferences over consumption and wealth (Aurissergues).

Justification for wealth in the utility function. Analytically, introducing wealth into the utility function only leads to a small departure from the standard model. Conceptually, however, the departure is significant, as the standard model assumes that people only save to smooth consumption over time. But it has long been recognized that people save for many reasons. For instance, observing the behavior of the European upper class in the early 20th century, Keynes (1919, chap. 2) noted that "The duty of saving became nine-tenths of virtue and the growth of the cake the object of true religion. . . . Saving was for old age or for your children; but this was only in theory—the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you." A few years later, Irving Fisher observed that "A man may include in the benefits of his wealth . . . the social standing he thinks it gives him, or political power and influence, or the mere miserly sense of possession, or the satisfaction in the mere process of further accumulation" (Fisher 1930, p. 17). Fisher's perspective is particularly interesting since he is often credited for the modern theory of saving based on consumption smoothing.

Neuroscientific evidence also supports the view that people derive utility from accumulating wealth independently of future consumption. Camerer, Loewenstein, and Prelec (2005, p. 32) note that "brain-scans conducted while people win or lose money suggest that money activates similar reward areas as do other 'primary reinforcers' like food and drugs, which implies that money confers direct utility, rather than simply being valued only for what it can buy."

Zou 1994; Corneo and Jeanne 1997; Futagami and Shibata 1998; Corneo and Jeanne 2001), attitudes to risk (Robson 1992; Clemens 2004), asset pricing (Bakshi and Chen 1996; Gong and Zou 2002; Kamihigashi 2008; Michau, Ono, and Schlegl 2018), life-cycle consumption (Zou 1995; Carroll 2000; Francis 2009), social stratification (Long and Shimomura 2004), international macroeconomics (Fisher 2005; Fisher and Hof 2005), financial crises (Kumhof, Ranciere, and Winant 2015), and capital taxation (Saez and Stantcheva 2018). The work by Cole, Mailath, and Postlewaite (1992) is also related: they provide a microfoundation for relative wealth in the utility function.

[2]The bonds-in-the-utility-function assumption was initially developed by Fair and Malkiel (1971), Poterba and Rotemberg (1987), and Krishnamurthy and Vissing-Jorgensen (2012).

[3]The assumptions overlap when aggregate real wealth is fixed.

In modeling the utility from wealth, we postulate that people enjoy wealth because it brings social status, and we therefore introduce relative wealth into the utility function. This assumption is convenient: in equilibrium, since there is no heterogeneity, everybody's relative wealth is zero. The assumption also seems plausible. Adam Smith, Ricardo, John Rae, J.S. Mill, Marshall, Veblen, and Frank Knight all supported the idea that people accumulate wealth to attain high social status (Steedman 1981). More recently, a broad literature documents that people seek to achieve high social status, and that accumulating wealth is a prevalent pathway to do so (Weiss and Fershtman 1998; Heffetz and Frank 2011; Fiske 2010; Anderson, Hildreth, and Howland 2015; Cheng and Tracy 2013; Ridgeway 2014; Mattan, Kubota, and Cloutier 2017).

Finally, an infinitesimal marginal utility of wealth is generally not sufficient for our results. The marginal utility of wealth needs to be larger than the slope of the Phillips curve. To assess this condition, we develop a method to measure the marginal utility of wealth based on the insight that when people derive utility from wealth, they are willing to save even if the real interest rate is below their time discount rate. In fact, we recover the marginal utility of wealth from the gap between the time discount rate and real interest rate. Since the time discount rate obtained from direct experimental evidence is typically significantly above market interest rates, we infer that the marginal utility of wealth is quite high. On the other hand, the Phillips curve is estimated to be quite flat. Hence, our results hold for a preponderance of available estimates.

2. Model

We present a New Keynesian model in which households derive utility not only from consumption and leisure but also from relative wealth. To simplify derivations, we follow Benhabib, Schmitt-Grohe, and Uribe (2001) and base our analysis on a slightly modified version of the New Keynesian model. Our version uses continuous time instead of discrete time; self-employed households instead of separate firms and households; and Rotemberg (1982) pricing instead of Calvo (1983) pricing. Our model is also inspired by the continuous-time formulation of the New Keynesian model proposed by Werning (2011), and adopted by Cochrane (2017). In continuous time, we can use phase diagrams to illustrate the anomalies of the New Keynesian model at the ZLB, clarify how they operate, and explain how they can be cured with wealth in the utility function.

2.1. Assumptions

The economy is composed of a measure 1 of self-employed households. Each household $j \in [0, 1]$ produces $y_j(t)$ units of a good j at time t, sold to other households at a price $p_j(t)$. The household's production function is

$$y_j(t) = ah_j(t), \tag{1}$$

where $a > 0$ represents the level of technology, and $h_j(t)$ is hours of work. Working causes a disutility

$$\kappa h_j(t),$$

where $\kappa > 0$ governs the disutility of labor.

The goods produced by households are imperfect substitutes for one another, so each household exercises some monopoly power. Moreover, households face a quadratic cost when they change their price: changing price at a rate $\pi_j(t) = \dot{p}_j(t)/p_j(t)$ causes a disutility

$$\frac{\gamma}{2}\pi_j(t)^2.$$

The parameter $\gamma > 0$ governs the cost to change prices and thus price rigidity.

Each household consumes goods produced by other households. Household j buys quantities $c_{jk}(t)$ of the goods $k \in [0, 1]$. These quantities are aggregated into a consumption index

$$c_j(t) = \left[\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right]^{\epsilon/(\epsilon-1)}, \tag{2}$$

where $\epsilon > 1$ is the elasticity of substitution between goods. The consumption index yields utility $\ln(c_j(t))$. Given the consumption index, the relevant price index is

$$p(t) = \left[\int_0^1 p_j(t)^{1-\epsilon}\, di\right]^{1/(1-\epsilon)}. \tag{3}$$

When households optimally allocate their consumption expenditure across goods, $p(t)$ is the price of one unit of consumption index. The inflation rate is defined as $\pi(t) = \dot{p}(t)/p(t)$.

Households save using government bonds. Since we postulate that people derive utility from their relative real wealth, and since bonds are the only store of wealth in the model, holding bonds directly yields utility. Formally, household j holds a nominal quantity of bonds $b_j(t)$, which yields utility

$$u\left(\frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)}\right),$$

where the function u is increasing and concave, $b(t) = \int_0^1 b_k(t)\, dk$ is average nominal wealth, and $[b_j(t) - b(t)]/p(t)$ is the household's relative real wealth.

Bonds earn a nominal interest rate $i(t)$, determined by monetary policy. The law of motion of bond holdings is

$$\dot{b}_j(t) = i(t)b_j(t) + p_j(t)y_j(t) - \int_0^1 p_k(t)c_{jk}(t)\, dk - \tau(t). \tag{4}$$

The term $i(t)b_j(t)$ is interest income; $p_j(t)y_j(t)$ is labor income; $\int_0^1 p_k(t)c_{jk}(t)\, dk$ is consumption expenditure; and $\tau(t)$ is a lump-sum tax (used among other things to service government debt).

Then, taking as given aggregate variables, initial wealth $b_j(0)$, and initial price $p_j(0)$, household j chooses time paths for $y_j(t)$, $p_j(t)$, $h_j(t)$, $\pi_j(t)$, $c_{jk}(t)$ for all $k \in [0, 1]$, and $b_j(t)$ to maximize the discounted sum of instantaneous utilities

$$\int_0^\infty e^{-\delta t} \left[\ln\big(c_j(t)\big) + u\left(\frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)}\right) - \kappa h_j(t) - \frac{\gamma}{2}\pi_j(t)^2 \right] dt, \tag{5}$$

where $\delta > 0$ is the time discount rate. The household faces four constraints: the production function (1); the budget constraint (4); the law of motion $\dot{p}_j(t) = \pi_j(t)p_j(t)$; and the demand for good j coming from other households' maximization:

$$y_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t), \tag{6}$$

where $c(t) = \int_0^1 c_k(t)\, dk$ is aggregate consumption. The household also faces a borrowing constraint preventing Ponzi schemes.

Finally, the central bank aims to maintain the economy at the natural steady state, where inflation is zero and output is at its natural level. The natural level of output, y^n, is the level of output at which producers keep their prices constant. The real interest rate at which households consume a quantity y^n is the natural rate of interest, r^n.

In normal times, the natural rate of interest is positive, and the central bank follows a simple interest-rate rule: $i(\pi) = r^n + \phi\pi$, which implies a real interest rate $r(\pi) = r^n + (\phi - 1)\,\pi$. The parameter $\phi \geq 0$ governs the response of monetary policy to inflation: monetary policy is active when $\phi > 1$ and passive when $\phi < 1$. This policy will allow the central bank to maintain the economy at the natural steady state.

When the natural rate of interest is negative, however, the natural steady state cannot be achieved: it would require a negative nominal interest rate, which would violate the ZLB. In that case, the central bank moves to the ZLB: $i = 0$ and $r = -\pi$.

2.2. Equilibrium

We assume that all households face the same initial conditions, so they all behave the same. The equilibrium can be described by two differential equations: a Phillips curve and an Euler equation. These two equations determine inflation $\pi(t)$ and aggregate output $y(t)$. The formal derivations of the equations are relegated to appendix A, but we provide heuristic derivations in the next section. Appendix B provides the discrete-time version of the equations.

The Phillips curve arises from households' optimal pricing decisions. It is not affected by the presence of wealth in the utility function:

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon\kappa}{\gamma a}\left[y(t) - y^n\right], \tag{7}$$

where the natural level of output satisfies

$$y^n = \frac{\epsilon - 1}{\epsilon} \cdot \frac{a}{\kappa}. \tag{8}$$

The steady-state Phillips curve, obtained by setting $\dot{\pi} = 0$ in (7), describes inflation as a linearly increasing function of output:

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}(y - y^n). \tag{9}$$

The steady-state Phillips curve shows that inflation is zero when output is at its natural level, and that output would be at the natural level if prices were flexible ($\gamma = 0$).

The Euler equation arises from households' optimal saving decisions. As people save not only for future consumption but also because they enjoy holding wealth, a new term appears in the equation:

$$\frac{\dot{y}(t)}{y(t)} = r(\pi(t)) - r^n + u'(0)\left[y(t) - y^n\right], \tag{10}$$

where the natural rate of interest satisfies

$$r^n = \delta - u'(0)y^n. \tag{11}$$

The term $r(\pi)$ describes the real interest rate set by the central bank as a function of inflation. The marginal utility of wealth, u', is evaluated at zero because in equilibrium relative wealth is zero, since all households hold the same wealth. At the household level the Euler equation is expressed in terms of consumption, but since consumption and output are equal in equilibrium, we can express the aggregate Euler equation in terms of output.

The steady-state Euler equation, obtained by setting $\dot{y} = 0$ in (10), describes output as a linear function of inflation:

$$u'(0)(y - y^n) = r^n - r(\pi). \tag{12}$$

As long as $u'(0) > 0$, output is a decreasing function of the real interest rate—as in the old-fashioned Keynesian IS curve. Further, output is at its natural level when the real interest rate equals the natural rate of interest.

We will compare two submodels:

Definition 1. *The New Keynesian (NK) model has zero marginal utility of wealth: $u'(0) = 0$. The wealth-in-the-utility New Keynesian (WUNK) model has sufficient marginal utility of wealth:*

$$u'(0) > \frac{\epsilon\kappa}{\delta\gamma a}. \tag{13}$$

The NK model is the textbook model; the WUNK model is the alternative proposed in this paper. When prices are completely fixed ($\gamma \to \infty$), condition (13) simply becomes $u'(0) > 0$; when prices are perfectly flexible ($\gamma = 0$), condition (13) becomes $u'(0) > \infty$. Hence, at the fixed-price limit, the WUNK model only requires an infinitesimal marginal utility of wealth; at the flexible-price limit, the WUNK model is not well-defined. In the WUNK model we also impose $\delta > \sqrt{(\epsilon - 1)/\gamma}$, so the model accommodates positive natural rates of interest while respecting (13).

2.3. Heuristic derivations

Our model incorporates several nontraditional elements (wealth in the utility function, continuous time, Rotemberg pricing). Thus, before delving into the analysis, we provide heuristic derivations of the Euler equation and Phillips curve (following Blanchard and Fischer 1989, pp. 40–42), and we review their interpretations.

Euler equation. The Euler equation says that households save in an optimal fashion: they cannot improve their situation by shifting consumption a little bit across time.

Consider a household delaying consumption of one unit of output from time t to time $t + dt$. The unit of output, invested at a real rate $r(t)$ during the interval, becomes $1 + r(t)dt$ at time $t + dt$. Given the utility function (5), the marginal utility from consumption at time t is $e^{-\delta t}/y(t)$.

Hence, the foregone consumption utility at time t is $e^{-\delta t}/y(t)$; and the extra consumption utility at time $t + dt$ is $e^{-\delta(t+dt)}[1 + r(t)dt]/y(t + dt)$.

In addition to financial returns, the one unit of output saved between t and $t + dt$ provides hedonic returns, since people enjoy holding wealth. The marginal utility from real wealth at time t is $e^{-\delta t}u'(0)$; hence the utility gained from holding extra wealth for a duration dt is $e^{-\delta t}u'(0)dt$.

At the optimum, the consumption reallocation does not affect utility:

$$0 = -\frac{e^{-\delta t}}{y(t)} + [1 + r(t)dt]\,\frac{e^{-\delta(t+dt)}}{y(t + dt)} + e^{-\delta t}u'(0)dt.$$

After dividing by $e^{-\delta t}/y(t)$, we obtain

$$1 = [1 + r(t)dt]e^{-\delta dt}\frac{y(t)}{y(t + dt)} + u'(0)y(t)dt.$$

Up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$,

$$\frac{y(t + dt)}{y(t)} = 1 + \frac{\dot{y}(t)}{y(t)}dt,$$

and $1/(1 + xdt) = 1 - xdt$ for any x. Hence, up to second-order terms, we have

$$\begin{aligned} 1 &= [1 + r(t)dt]\,(1 - \delta dt)\left[1 - \frac{\dot{y}(t)}{y(t)}dt\right] + u'(0)y(t)dt \\ &= 1 - \delta dt + r(t)dt - \frac{\dot{y}(t)}{y(t)}dt + u'(0)y(t)dt, \end{aligned}$$

which simplifies to

$$\frac{\dot{y}(t)}{y(t)} = r(t) - \delta + u'(0)y(t). \tag{14}$$

Once we introduce $r^n = \delta - u'(0)y^n$, this equation yields (10).

Compared with the standard Euler equation, (14) has an extra term: $u'(0)y(t)$. In the standard equation, consumption is governed by the cost of delaying consumption, given by the time discount rate δ, and the return on saving, given by the real interest rate $r(t)$. With wealth in the utility function, the financial return on saving is supplemented by an hedonic return on saving, measured by the marginal rate of substitution between real wealth and consumption, $u'(0)y(t)$. Thus the total return on saving is $r(t) + u'(0)y(t)$ instead of $r(t)$, explaining the extra term in the equation. Because consumption depends not only on interest rates but also on the marginal rate of substitution between wealth and consumption, future interest rates have less impact on today's consumption. In fact, appendix B shows that the discrete-time Euler equation is discounted exactly as in McKay, Nakamura, and Steinsson (2017), and that discounting is stronger with higher marginal utility of wealth.

The derivation also explains why in steady state consumption is a decreasing function of the real interest rate. When the real rate is higher, people have a financial incentive to save more and postpone consumption. People keep consumption constant only if the hedonic return on saving falls so as to compensate the increase in the financial return. The hedonic return is given by the marginal rate of substitution $u'(0)y$. In steady state consumption y must therefore decline when the real rate increases.

Phillips curve. The Phillips curve says that households price optimally, so they cannot improve their situation by shifting inflation a little bit across time.

Consider a household delaying one percentage point of inflation from time t to time $t + dt$. Given the utility function (5), the marginal disutility from inflation at time t is $e^{-\delta t}\gamma\pi(t)$. Hence, the foregone inflation disutility at time t is $e^{-\delta t}\gamma\pi(t) \times 1\%$; and the extra inflation disutility at time $t + dt$ is $e^{-\delta(t+dt)}\gamma\pi(t + dt) \times 1\%$.

The one percentage point of inflation that is delayed reduces the price level between time t and time $t + dt$ by $dp(t) = -1\% \times p(t)$, which then affects sales and hours worked. Since the price elasticity of demand is $-\epsilon$, sales increase by $dy(t) = -\epsilon \times -1\% \times y(t) = \epsilon y(t) \times 1\%$ during the same period. As a result, hours increase by $dh(t) = dy(t)/a = \epsilon[y(t)/a] \times 1\%$, raising the disutility of labor by $e^{-\delta t}\kappa\epsilon[y(t)/a] \times 1\%$ between t and $t + dt$. The change in price and sales raises the revenue by $d(p(t)y(t)) = p(t)dy(t) + y(t)dp(t) = (\epsilon - 1)y(t)p(t) \times 1\%$. Since in equilibrium all prices are the same, the increase in revenue yields an increase in consumption $dy(t) = (\epsilon - 1)y(t) \times 1\%$, which raises consumption utility by $e^{-\delta t}dy(t)/y(t) = e^{-\delta t}(\epsilon - 1) \times 1\%$ between time t and time $t + dt$.

At the optimum, shifting inflation across time does not affect utility:

$$0 = e^{-\delta t}\gamma\pi(t) \times 1\% - e^{-\delta(t+dt)}\gamma\pi(t + dt) \times 1\% - e^{-\delta t}\kappa\epsilon\frac{y(t)}{a} \times 1\% \times dt + e^{-\delta t}(\epsilon - 1) \times 1\% \times dt.$$

Up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$ and

$$\pi(t + dt) = \pi(t) + \dot{\pi}(t)dt.$$

Therefore, after dividing by $e^{-\delta t} \times 1\%$, we find that up to second-order terms,

$$
\begin{aligned}
0 &= \gamma\pi(t) - (1-\delta dt)\gamma\left[\pi(t) + \dot{\pi}(t)dt\right] - \kappa\epsilon\frac{y(t)}{a}dt + (\epsilon - 1)dt \\
&= \delta\gamma\pi(t)dt - \gamma\dot{\pi}(t)dt - \kappa\epsilon\frac{y(t)}{a}dt + (\epsilon-1)dt.
\end{aligned}
$$

Dividing by γdt, we obtain up to second-order terms:

$$
\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon\kappa}{\gamma a}\left[y(t) - \frac{\epsilon-1}{\epsilon}\cdot\frac{a}{\kappa}\right].
$$

Once we introduce $y^n = [(\epsilon-1)/\epsilon]a/\kappa$, this equation yields (7).

The Phillips curve implies that in the absence of price-adjustment cost ($\gamma = 0$), households would like to produce at the natural level of output, y^n. This result comes from the monopolistic nature of competition. Without price-adjustment costs, it is optimal to charge a relative price that is a markup $\epsilon/(\epsilon-1)$ over the real marginal cost—which is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. In equilibrium, all relative prices are 1, the marginal rate of substitution between labor and consumption is $\kappa/(1/y) = \kappa y$, and the marginal product of labor is a. Hence at the optimum $1 = [\epsilon/(\epsilon-1)]\kappa y/a$, which implies $y = [(\epsilon-1)/\epsilon]a/\kappa = y^n$.

Last, the derivation elucidates why in steady state, inflation is positive whenever output is above its natural level. When inflation is positive, reducing inflation lowers the disutility from price adjustment. Since pricing is optimal, there must also be a cost to reducing inflation and hence increasing production. Therefore, production must already be excessive: output must be above its natural level.

2.4. *Dynamics*

We now describe the dynamics of the NK and WUNK models. We first describe normal times: the natural rate of interest is positive ($r^n > 0$), and monetary policy follows $i(\pi) = r^n + \phi\pi$.

Proposition 1. *Consider the NK and WUNK models in normal times. They admit a unique steady state, where output is at its natural level ($y = y^n$), inflation is zero ($\pi = 0$), and the ZLB is not binding ($i = r^n > 0$). Around this natural steady state, dynamics are governed by the linear dynamical system*

$$
\begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} u'(0)y^n & (\phi-1)y^n \\ -\epsilon\kappa/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix}. \tag{15}
$$

In the NK model, the dynamical system is a source when monetary policy is active ($\phi > 1$) and a saddle when monetary policy is passive ($\phi < 1$). In the WUNK model, the dynamical system is a source whether monetary policy is active or passive.

We next turn to dynamics at the ZLB: the natural rate of interest is negative ($r^n < 0$), and monetary policy is $i = 0$.[4]

Proposition 2. *Consider the NK and WUNK models at the ZLB. They admit a unique steady state, where output and inflation are given by*

$$y^z = y^n + \frac{r^n}{u'(0) - \epsilon\kappa/(\delta\gamma a)} \tag{16}$$

$$\pi^z = \frac{r^n}{u'(0)\delta\gamma a/(\epsilon\kappa) - 1}. \tag{17}$$

Around the steady state, dynamics are governed by the linear dynamical system

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} u'(0)y^z & -y^z \\ -\epsilon\kappa/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y(t) - y^z \\ \pi(t) - \pi^z \end{bmatrix}. \tag{18}$$

In the NK model, the steady state has positive inflation ($\pi^z = -r^n > 0$) and above-natural output ($y^z > y^n$), and the dynamical system is a saddle. In the WUNK model, the steady state has deflation ($\pi^z < 0$) and below-natural output ($y^z < y^n$), and the dynamical system is a source.

To illustrate the economic mechanisms behind the results, we offer a graphical proof based on the diagrams in figure 1. To reduce the number of diagrams, we only discuss an active monetary policy in normal times; a passive monetary policy can be analyzed similarly. Appendix C presents an alternative, algebraic proof that is closer to the eigenvalue-based proofs found in the literature.

We begin with the steady states. A steady state satisfies both the steady-state Phillips curve, given by (9), and the steady-state Euler equation, given by (12). These steady-state equations are represented by the lines labeled "Phillips" and "Euler" in figure 1. The Phillips line is the same in the NK and WUNK models, and in normal times and at the ZLB: it is upward-sloping and goes through the point $[y = y^n, \pi = 0]$. The Euler line, on the other hand, is different in each case.

[4] To represent the ZLB in the NK model, we follow the literature and assume that the time discount rate in the Euler equation (denoted r^n) is negative, while the time discount rate in the Phillips curve (denoted δ) remains positive. This assumption has been justified in various ways: firm managers have constant discount rate while households' discount rate fluctuates (McKay, Nakamura, and Steinsson 2017, pp. 826–827); or financial intermediation creates a fluctuating spread between the central bank's interest rate and the interest rate used by households for saving decisions (Woodford 2011, p. 16). In the WUNK model, things are more straightforward. To obtain a negative natural rate of interest while retaining a positive time discount rate, we assume that the marginal utility of wealth is high enough: $u'(0) > \delta/y^n$.

In the NK model the Euler line is horizontal because when $u'(0) = 0$, (12) imposes that $r(\pi) = r^n$, which determines inflation independently of output (panels A and C). In the WUNK model, the Euler line is not horizontal because when $u'(0) > 0$, (12) makes output a decreasing function of the real rate $r(\pi)$. When monetary policy is active ($\phi > 1$), $r(\pi) = r^n + (\phi - 1)\pi$ is increasing in π, so the Euler line slopes downward (panel B); at the ZLB, $r(\pi) = -\pi$ is decreasing in π, so the Euler line slopes upward (panel D).

The Euler line also changes between normal times and the ZLB because the natural rate of interest and monetary policy change. In the NK model the Euler line shifts up from $\pi = 0$ in normal times to $\pi = -r^n > 0$ at the ZLB (panels A and C). In the WUNK model, in normal times with active monetary policy, the Euler line is

$$\pi = -\frac{u'(0)}{\phi - 1}(y - y^n),$$

so it is downward sloping and goes through the point $[y = y^n, \pi = 0]$ (panel B). At the ZLB the Euler line is

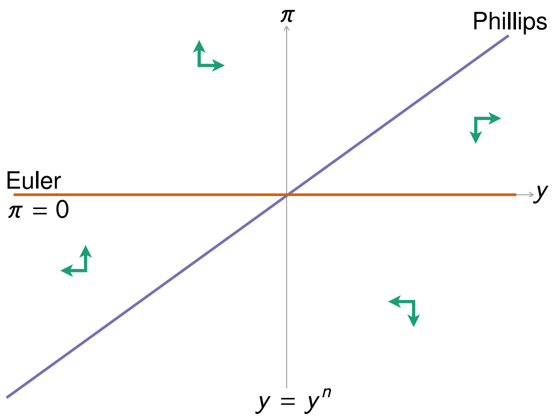
$$\pi = -r^n + u'(0)(y - y^n), \tag{19}$$

so it is upward sloping and goes through the point $[y = y^n + r^n/u'(0), \pi = 0]$ (panel D). Since $r^n \leq 0$ at the ZLB, the ZLB Euler line is always inward of the point $[y = y^n, \pi = 0]$—all the more so when r^n is lower—explaining why the central bank is not able to achieve the natural steady state. Further, since the slope of the Phillips line is $\epsilon\kappa/(\delta\gamma a)$ and the slope of the ZLB Euler line is $u'(0)$, the WUNK condition (13) ensures that the ZLB Euler line is steeper than the Phillips line.

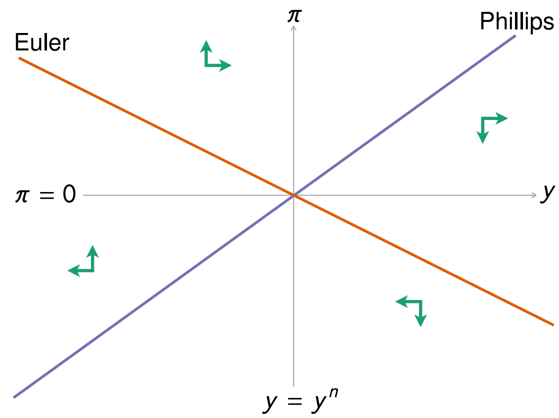
In normal times, in both models, the Phillips and Euler lines intersect at the point $[y = y^n, \pi = 0]$; therefore, the models admit a unique steady state with zero inflation and natural output (panels A and B). At the ZLB, in both models, the Phillips and Euler lines have a unique intersection, so the steady state exists and is unique.[5] In the NK model, the steady state has positive inflation and above-natural output (panel C); in the WUNK model, the steady state has negative inflation and below-natural output (panel D).

Next, we turn to the dynamics of the NK and WUNK models. The global dynamics are governed by the dynamical system generated by the Phillips curve (7) and Euler equation (10), but around the steady states the dynamics can be obtained from linearized versions of this system. The linearization is particularly straightforward because (7) is already linear, and (10) can

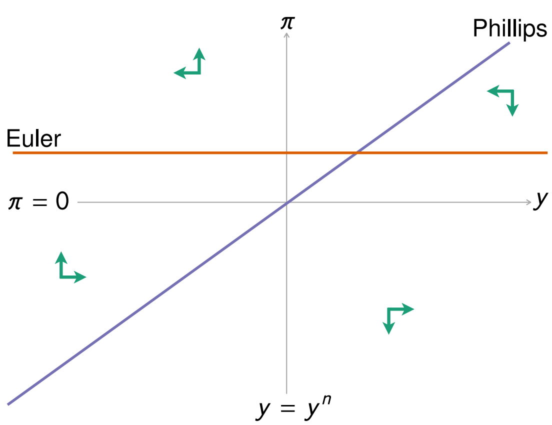
[5]In the WUNK model we also need to check that the intersection has positive output; appendix C shows that that is always the case.



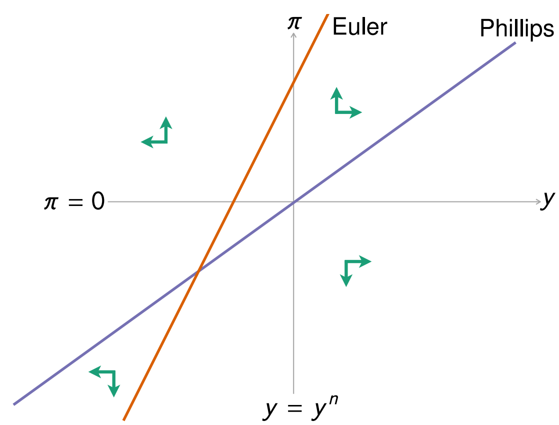
A. NK model in normal times with active monetary policy: source



B. WUNK model in normal times with active monetary policy: source



C. NK model at the ZLB: saddle



D. WUNK model at the ZLB: source

FIGURE 1. Phase diagrams in normal times and at the ZLB in the NK and WUNK models

Notation: π is inflation; y is output; y^n is the natural level of output; the Euler line is the y-nullcline (the locus $\dot{y} = 0$); and the Phillips line is the π-nullcline (the locus $\dot{\pi} = 0$). The NK model is the textbook New Keynesian model. The WUNK model is the same as the NK model, except that wealth enters the utility function, and the marginal utility of wealth is sufficiently large to satisfy (13). In normal times, the natural rate of interest (r^n) is positive, and the nominal interest rate is given by $i = r^n + \phi\pi$; when monetary policy is active, $\phi > 1$. At the ZLB, the natural rate of interest is negative, and the nominal interest rate is zero. The phase diagrams in panels A and B represent the dynamical system (15), which is obtained by linearizing the Phillips curve and Euler equation around the natural steady state. The phase diagrams in panels C and D represent the dynamical system (18), which is obtained by linearizing the Phillips curve and Euler equation around the ZLB steady state. The phase diagrams are constructed using standard methods for planar linear systems, as explained in the text. Panel A shows that in the NK model, in normal times, the dynamical system is a source when monetary policy is active. Panel C shows that in the NK model the dynamical system is a saddle at the ZLB. Panels B and D show that in the WUNK model the dynamical system is a source both in normal times and at the ZLB.

be immediately linearized as $\dot{y} = \bar{y}\,[r(\pi) - r^n + u'(0)(y - y^n)]$, where $\bar{y}$ stands for steady-state output. Combining these two linear differential equations and the appropriate expressions for $r(\pi)$ and $\bar{y}$, we obtain the linear dynamical systems (15) and (18).

The dynamics produced by these linear systems are described by their phase diagrams. The first step in constructing these diagrams is to plot the y-nullcline (the locus $\dot{y} = 0$) and the π-nullcline (the locus $\dot{\pi} = 0$) given by (15) and (18). As the π-nullcline reduces to the steady-state Phillips curve (9), and the y-nullcline to the steady-state Euler equation (12), these nullclines are given by the Phillips and Euler lines in figure 1.

Second, we plot in figure 1 the arrows giving the directions of the trajectories in the phase diagrams. We first determine the sign of $\dot{\pi}$ in the linear systems. The differential equation giving $\dot{\pi}$ is just the Phillips curve (7); it shows that any point above the Phillips line (where $\dot{\pi} = 0$) has $\dot{\pi} > 0$, and any point below the line has $\dot{\pi} < 0$. So in all the panels of figure 1 we indicate that inflation is rising above the Phillips line and falling below it.

Next we examine the sign of $\dot{y}$ in the linear systems (15) and (18). In the NK model, in normal times with active monetary policy (panel A), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}}{y^n} = (\phi - 1)\pi$$

with $\phi > 1$. Hence any point above the Euler line (where $\pi = 0$) has $\dot{y} > 0$, and any point below the line has $\dot{y} < 0$. Accordingly, in the four quadrants delimited by the Phillips and Euler lines, the trajectories move away from the steady state: the dynamical system a source.

In the NK model at the ZLB (panel C), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}(t)}{y^z} = -\pi - r^n.$$

Hence any point above the Euler line (where $\pi = -r^n$) has $\dot{y} < 0$, and any point below the line has $\dot{y} > 0$. We conclude that the dynamical system is a saddle, because in the southwest and northeast quadrants the trajectories move toward the steady state, while in the southeast and northwest quadrants the trajectories move away from the steady state.

In the WUNK model, in normal times, and with active monetary policy (panel B), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}}{y^n} = (\phi - 1)\pi + u'(0)\,(y - y^n)$$

with $\phi > 1$. Accordingly, any point above the Euler line (where $\dot{y} = 0$) has $\dot{y} > 0$, and any point below the line has $\dot{y} < 0$. As in all four quadrants the trajectories move away from the steady state, the dynamical system is a source.

Last, in the WUNK model at the ZLB (panel D), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}}{y^z} = -\pi - r^n + u'(0)\,(y - y^n)\,.$$

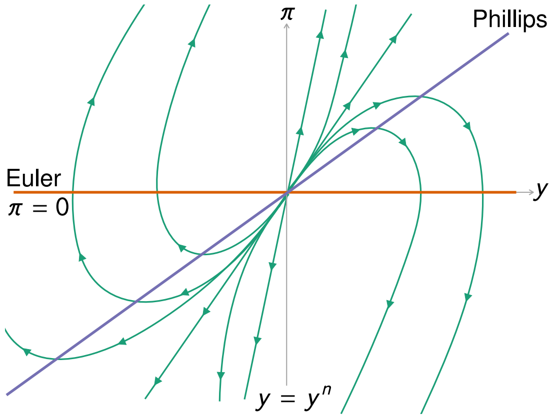
Any point above the Euler line (where $\dot{y} = 0$) has $\dot{y} < 0$, and any point below the line has $\dot{y} > 0$. Thus in all four quadrants the trajectories move away from the steady state: the dynamical system remains a source.

Overall, the key difference between the NK and WUNK models is that at the ZLB, the dynamical system representing the equilibrium remains a source in the WUNK model, whereas it becomes a saddle in the NK model. This difference will explain why the WUNK model does not suffer from the anomalies plaguing the NK model at the ZLB.

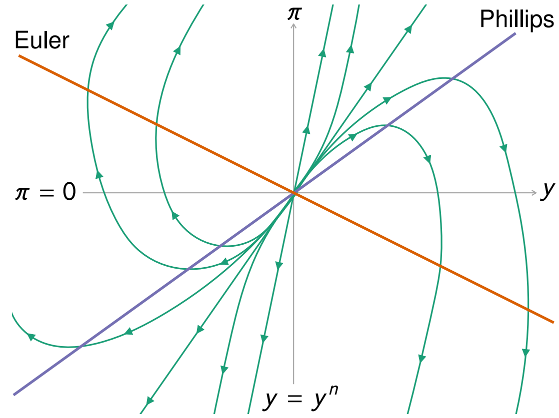
To complement the phase diagrams of figure 1, we plot the trajectories associated with these phase diagrams in figure 2. The plots show how the trajectories escape the steady state when the system is a source, and how they converge to the steady state along the saddle path when the system is a saddle. When the system is a source, the trajectories are organized around two unstable lines—trajectories that move away from the steady state in a straight line. At $t \to -\infty$, all the trajectories leave the steady state and are tangent to the same unstable line. At $t \to +\infty$, all the trajectories move to infinity and are parallel to the other unstable line. When the system is a saddle, there is one stable line, which goes to the steady state in a straight line, and one unstable line, which moves away from the steady state in a straight line. All trajectories are parallel to the stable line when $t \to -\infty$ and are parallel to the unstable line when $t \to +\infty$. Appendix C explains how the stable and unstable lines are linked to the eigenvectors and eigenvalues of the matrices in (15) and (18).

These phase diagrams illustrate the origin of the WUNK condition (13). The dynamical system for the WUNK model remains a source at the ZLB as long as the Euler line is steeper than the Phillips line (panels D of figures 1 and 2). The slope of the Euler line at the ZLB is the marginal utility of wealth, $u'(0)$, so the marginal utility of wealth is required to be above a certain level—this level is given by (13).

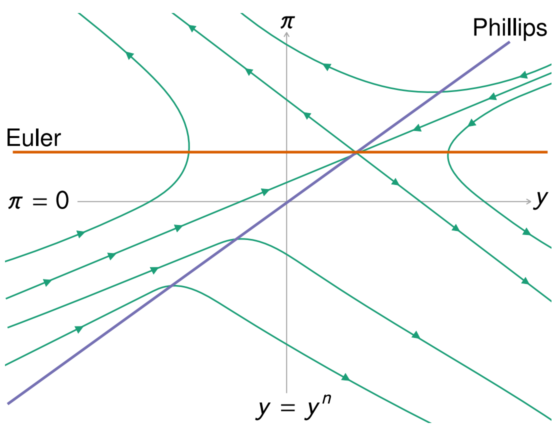
Propositions 1 and 2 have several implications. First, they have implications for equilibrium determinacy. When the system is a source, the equilibrium is determinate: the only equilibrium trajectory in the vicinity of the steady state is to jump to the steady state and stay there; if the economy jumped somewhere else, output or inflation would diverge following a trajectory similar to those plotted in panels A, B, and D of figure 2. When the system is a saddle, in contrast,



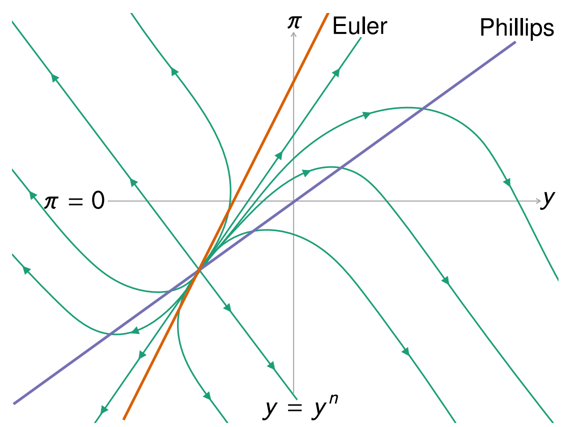
A. NK model in normal times with active monetary policy: source



B. WUNK model in normal times with active monetary policy: source



C. NK model at the ZLB: saddle



D. WUNK model at the ZLB: source

FIGURE 2. Trajectories in normal times and at the ZLB in the NK and WUNK models

Notation: π is inflation; y is output; y^n is the natural level of output; the Euler line is the y-nullcline; and the Phillips line is the π-nullcline. The NK model is the textbook New Keynesian model. The WUNK model is the same as the NK model, except that wealth enters the utility function, and the marginal utility of wealth is sufficiently large to satisfy (13). In normal times, the natural rate of interest (r^n) is positive, and the nominal interest rate is given by $i = r^n + \phi\pi$; when monetary policy is active, $\phi > 1$. At the ZLB, the natural rate of interest is negative, and the nominal interest rate is zero. The trajectories in panels A and B solve the dynamical system (15), which is obtained by linearizing the Phillips curve and Euler equation around the natural steady state. The trajectories in panels C and D solve the dynamical system (18), which is obtained by linearizing the Phillips curve and Euler equation around the ZLB steady state. All the trajectories are constructed using standard methods for planar linear systems, as explained in the text. Panel A shows that in the NK model, in normal times, the dynamical system is a source when monetary policy is active. Panel C shows that in the NK model the dynamical system is a saddle at the ZLB. Panels B and D show that in the WUNK model the dynamical system is a source both in normal times and at the ZLB. In panels A and B, we have plotted a nodal source, but the system could also be a spiral source, depending on the value of ϕ; in panel D the system is always a nodal source.

the equilibrium is indeterminate: any trajectory jumping somewhere on the saddle path and converging to the steady state is an equilibrium, as illustrated in panel C of figure 2. Hence, in the NK model, the ZLB equilibrium is indeterminate. This indeterminacy has forced researchers, starting with Krugman (1998) and Eggertsson and Woodford (2003), to focus on temporary ZLB episodes—as we do in section 3. In the WUNK model, on the other hand, the ZLB equilibrium is determinate: the economy jumps to the ZLB steady state and remains there. Hence the model can be used to study permanent ZLB episodes, as showcased in section 4.

Second, the propositions imply that the Taylor principle does not hold in the WUNK model, unlike in the NK model. Thus, monetary policy plays quite a different role in the NK and WUNK models. In the NK model, the Taylor principle holds: the equilibrium is determinate only when monetary policy is active. Hence, the central bank must adhere to an active monetary policy to avoid indeterminacy. In the WUNK model, the equilibrium is determinate whether monetary policy is active or passive. Hence, the central bank does not need to worry about how strongly its interest-rate rule responds to inflation: it can simply use an interest-rate peg.

The NK results in the propositions are well-known (for example, Woodford 2001). The WUNK results share many similarities with the results obtained by Gabaix (2016). The dynamical system representing the equilibrium in his model has the same structure as ours: it is composed of two equations, with two jump variables. Furthermore, he finds that when bounded rationality is strong enough, the dynamical system is a source even at the ZLB. He also finds that when prices are more flexible, more bounded rationality is required to maintain the source property. The same is true in our model: higher marginal utility of wealth is required for condition (13) to hold when the price-adjustment cost γ is lower. The logic is illustrated in panel D of figure 1. The system remains a source at the ZLB only when the Euler line is steeper that the Phillips line. As prices are more flexible, the Phillips line becomes steeper, and the required steepness for the Euler line increases. As the slope of the Euler line is determined by bounded rationality in the Gabaix model and by marginal utility of wealth in our model, these need to be larger when prices are more flexible.

3. New Keynesian anomalies at the ZLB

This section illustrates the anomalies of the NK model at the ZLB: collapse of output and inflation when the ZLB is long lasting; and implausibly strong effects of both forward guidance and government spending. It then shows that these anomalies do not exist in the WUNK model.

3.1. Output and inflation collapse

We consider a temporary ZLB episode, as in Werning (2011) and Cochrane (2017). Between times 0 and $T > 0$, a negative aggregate demand shock hits the economy, bringing the natural rate of interest below zero. In the NK model, this shock would be an increase in the time discount rate that enters the Euler equation; in the WUNK model, it would be an increase in the marginal utility of wealth; these shocks affect the Euler equation but not the Phillips curve. In response to the negative shock, the central bank maintains the nominal interest rate at zero. Then, after time T, the natural rate becomes positive again, and the central bank returns to the normal monetary-policy rule. To ensure determinacy, monetary policy must be active in the NK model, but it can be active or passive in the WUNK model.

We analyze the ZLB episode by going backward in time. After time T, monetary policy maintains the economy at the natural steady state. Since equilibrium trajectories must be continuous, the economy must be at the natural steady state at the end of the ZLB, when $t = T$. We then move back to the ZLB episode, when $t < T$. The corresponding phase diagrams are shown in figure 3; the diagrams are the same as in panels C and D of figure 1.

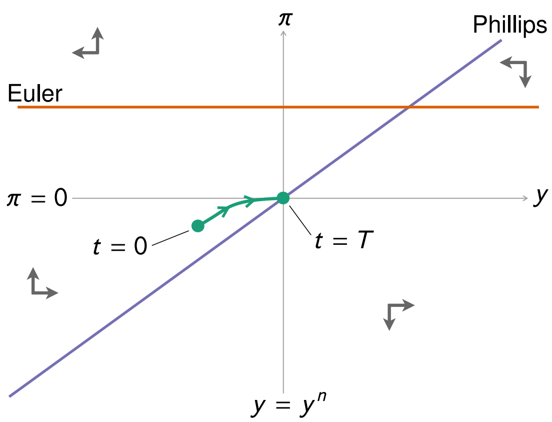
We start with the NK model. Initially inflation and output jump down to $\pi(0) < 0$ and $y(0) < y^n$. After their initial collapse, inflation and output recover following the unique trajectory leading to $[y = y^n, \pi = 0]$ at time T. The ZLB therefore creates a slump, with below-natural output and deflation (panel A). Output and inflation become unboundedly low as the ZLB lasts longer (panel C).

In the WUNK model, output and inflation never collapse during the ZLB. Initially inflation and output jump down toward the ZLB steady state, but not all the way: $\pi^z < \pi(0) < 0$ and $y^z < y(0) < y^n$. Then they recover, following the unique trajectory leading to $[y = y^n, \pi = 0]$ at time T. Thus, the ZLB episode creates a slump (panel B), which is deeper when the ZLB lasts longer (panel D). But unlike in the NK model, output and inflation are bounded below: irrespective of the length of the ZLB, they always fall less than if the ZLB were permanent. Moreover, if the natural rate of interest is negative but close to 0, such that π^z is close to 0 and y^z to y^n, output and inflation barely deviate from the natural steady state during the ZLB, even if the ZLB lasts a very long time.

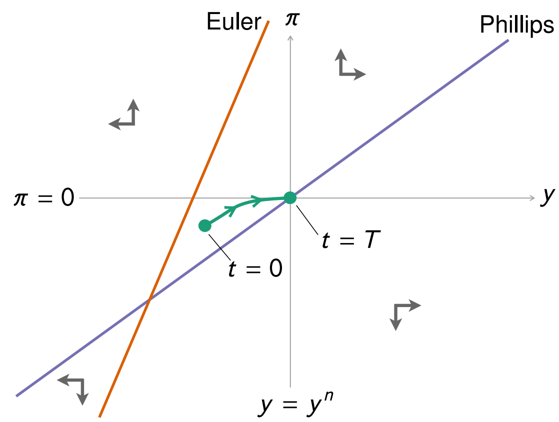
The following proposition records these results:[6]

PROPOSITION 3. *Consider a ZLB episode between times* 0 *and* T *in the NK and WUNK models.*

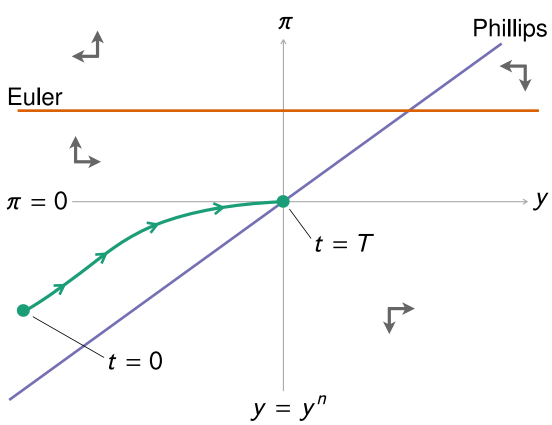
[6]The result that output becomes infinitely negative when the ZLB becomes infinitely long should not be interpreted literally. It is obtained from the dynamical system (18), which is a local approximation of the dynamical system given by (7) and (10). The interpretation is that output falls much below its natural level—so much so that the local approximation stops being valid. The global dynamical system guarantees that output remains positive.



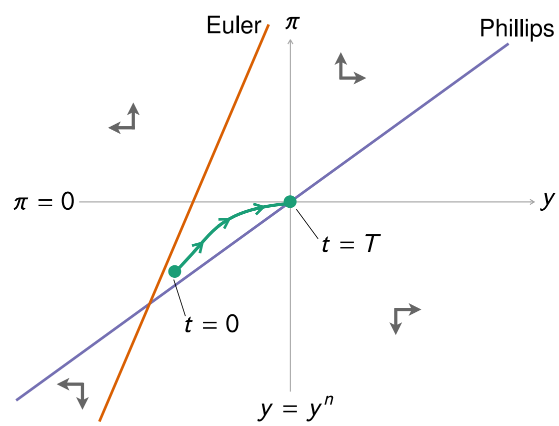
A. NK model: short ZLB



B. WUNK model: short ZLB



C. NK model: long ZLB



D. WUNK model: long ZLB

FIGURE 3. ZLB episodes of various durations in the NK and WUNK models

The ZLB is binding between times 0 and T; then, at time T, the central bank brings the economy to the natural steady state, where inflation is zero and output is at its natural level. The phase diagrams describe dynamics at the ZLB; they come from panels C and D in figure 1. The equilibrium trajectories are the unique trajectories reaching the natural steady state at time T. In the NK and WUNK models, the economy slumps during the ZLB: inflation is negative and output is below its natural level. In the NK model, the initial slump becomes unboundedly large as the ZLB becomes longer. In the WUNK model, there is no such output and inflation collapse: the initial slump is bounded below by the ZLB steady state.

The economy enters a slump: $y(t) < y^n$ *and* $\pi(t) < 0$ *for all* $t \in (0, T)$*. In the NK model, the slump becomes infinitely severe as the ZLB becomes infinitely long:* $\lim_{T\to\infty} y(0) = -\infty$ *and* $\lim_{T\to\infty} \pi(0) = -\infty$*. In the WUNK model, in contrast, the slump is bounded below by the ZLB steady state:* $y(t) > y^z$ *and* $\pi(t) > \pi^z$ *for all* $t \in (0, T)$*; in fact, the slump approaches the ZLB steady state as the ZLB becomes infinitely long:* $\lim_{T\to\infty} y(0) = y^z$ *and* $\lim_{T\to\infty} \pi(0) = \pi^z$.

In the NK model, output and inflation collapse when the ZLB is long-lasting—which is well-known (Eggertsson and Woodford 2004, fig. 1; Eggertsson 2011, fig. 1; Cochrane 2017, fig. 1). This collapse is difficult to reconcile with real-world observations. The ZLB episode that started in 1995 in Japan has been lasting for more than twenty years without sustained deflation. The ZLB episode that started in 2009 in the euro area has been lasting for more than 10 years; it did not yield sustained deflation either. The same is true of the ZLB episode that occurred in the United States between 2008 and 2015.

In the WUNK model, in contrast, the ZLB slump is bounded below by the ZLB steady state. So inflation and output never collapse at the ZLB, even if the ZLB lasts a very long time. Instead, as the duration of the ZLB increases, the economy converges to the ZLB steady state. And the ZLB steady state may not be far from the natural steady state: if the natural rate of interest is only slightly negative at the ZLB, steady-state inflation is only slightly below zero and steady-state output only slightly below its natural level (see (16) and (17)).

Gabaix (2016) obtains the results closest to those in proposition 3. In his model output and inflation also converge to the ZLB steady state as the ZLB becomes arbitrarily long.

3.2. Forward guidance

We turn to the effects of forward guidance at the ZLB. We consider a three-stage scenario, as in Cochrane (2017). Between times 0 and T, there is a ZLB episode, exactly as in the previous section. To alleviate the ZLB, the central bank makes a forward-guidance promise at time 0: that it will maintain the nominal interest rate at zero for a duration Δ once the ZLB is over.[7] At time T, the negative shock that brought the economy to the ZLB recedes, and the natural rate of interest returns above zero. Then, between times T and $T + \Delta$, the central bank abides by its forward-guidance promise and keeps the nominal interest rate at zero. Finally, after time $T + \Delta$, the natural rate of interest remains positive, and monetary policy returns to normal.

Let's begin with the NK model. Forward guidance is analyzed in figure 4 by going backward in time. After time $T + \Delta$, monetary policy maintains the economy at the natural steady state, and

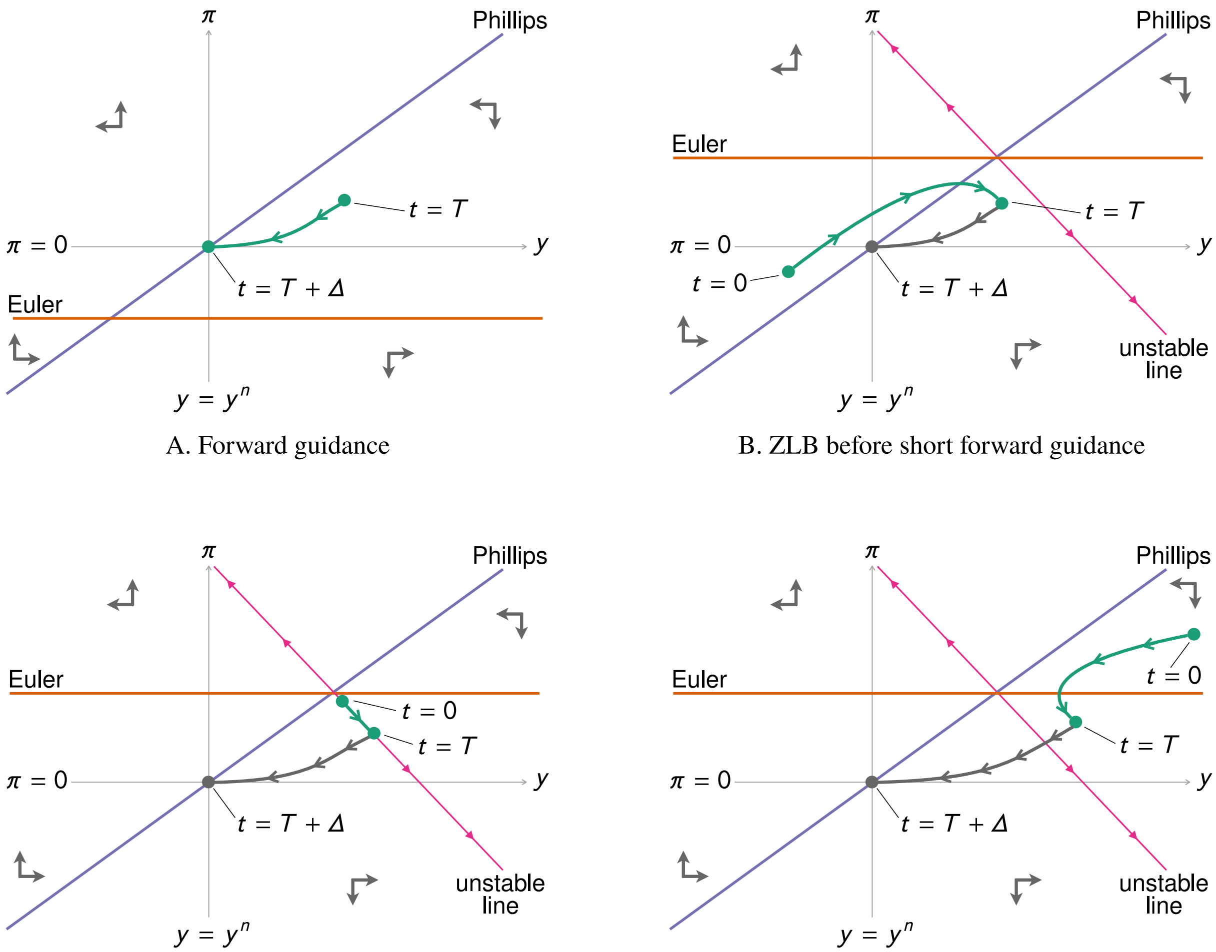
[7] Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), and Werning (2011) show that such promise is optimal in this situation.

so the economy must be at the natural steady state at the end of forward guidance, when $t = T + \Delta$. Between times T and $T + \Delta$, the economy is in forward guidance, as depicted in panel A. The phase diagram is similar to that in panel C of figure 1, except that $r^n > 0$ instead of $r^n < 0$. The Euler line, given by $\pi = -r^n$, is lower but the system remains a saddle. Following the logic of figure 3, we find that at time T, inflation must be positive and output above its natural level. They then decrease over time, following the unique trajectory leading to the natural steady state at time $T + \Delta$. Accordingly, the economy booms during forward guidance: inflation is always positive and output above its natural level. Furthermore, as the duration of forward guidance increases, inflation and output at the beginning of forward guidance become higher.

We look next at the ZLB episode, between times 0 and T. This episode is depicted in panels B, C, and D; the three panels differ by the duration of forward guidance after the ZLB episode. The phase diagram in these panels is the same as in panels C of figures 1 and 2. Since equilibrium trajectories are continuous, the economy must be at the same point at the end of the ZLB and at the beginning of forward guidance. Because of the boom engineered during forward guidance, then, the situation is improved at the ZLB. Instead of reaching the natural steady state at time T, the economy reaches a point with positive inflation and above-natural output, so at any time before T, inflation and output tend to be higher than without forward guidance.

Forward guidance can have tremendously strong effects. For small durations of forward guidance, the position at the beginning of forward guidance is on the left-hand side of the unstable line of the dynamical system representing the ZLB equilibrium. It is therefore connected to trajectories coming from the southwest, slumpy quadrant of the phase diagram (panel B). As the ZLB lasts longer, initial output and inflation collapse. When the duration of forward guidance is such that the position at the beginning of forward guidance is exactly on the unstable line, then the position at the beginning of the ZLB must be on the unstable line as well (panel C). As the ZLB lasts longer, the initial position inches closer to the ZLB steady state. For even longer forward guidance, the position at the beginning of forward guidance is on the right-hand side of the unstable line, so it is connected to trajectories coming from the northeast, boomy quadrant of the phase diagram (panel D). From here, as the ZLB lasts longer, initial output and inflation become higher and higher. As a result, if the duration of forward guidance is long enough, a deep slump can be transformed into a roaring boom. Moreover, the effect of such policy are larger when the ZLB lasts longer, although the required duration of forward guidance is independent of the ZLB duration.

The power of forward guidance is much more subdued in the WUNK model, as illustrated in figure 5. After time $T + \Delta$, the economy is at the natural steady state. Between times T and $T + \Delta$, forward guidance is in operation, as shown in panel A. The phase diagram is similar to that in



A. Forward guidance

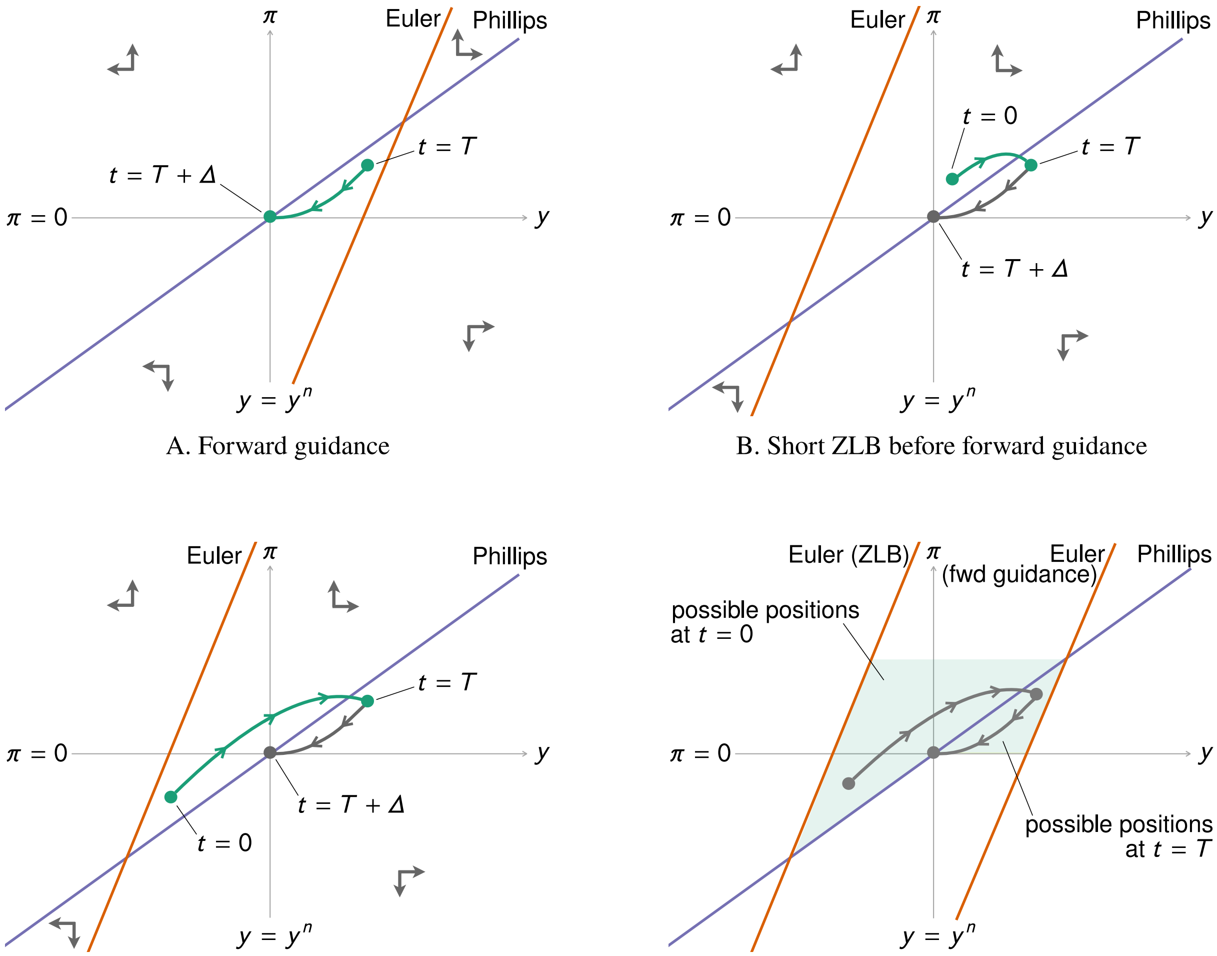
B. ZLB before short forward guidance

C. ZLB before medium forward guidance

D. ZLB before long forward guidance

FIGURE 4. NK model: ZLB followed by forward-guidance episodes of various durations

The ZLB episode lasts between times 0 and T. It is followed by forward guidance between times T and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. Then, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagram in panel A describes dynamics during forward guidance; it is similar to the diagram in panel C of figure 1 but with $r^n > 0$. The phase diagrams in panels B, C, and D describe dynamics at the ZLB; they come from panels C of figures 1 and 2. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time $T + \Delta$. The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time T. The NK model suffers from a major anomaly: when forward guidance lasts sufficiently to bring $[y(T), \pi(T)]$ on the right-hand side of the unstable line, any ZLB episode—however long—-will be a boom (panel B versus panel D).



A. Forward guidance

B. Short ZLB before forward guidance

C. Long ZLB before forward guidance

D. Possible trajectories

FIGURE 5. WUNK model: ZLB episodes of various durations followed by forward guidance

The ZLB episode lasts between times 0 and T. It is followed by forward guidance between times T and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. Then, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagram in panel A describes dynamics during forward guidance; it is similar to the diagram in panel D of figure 1 but with $r^n > 0$. The phase diagrams in panels B and C describe dynamics at the ZLB; they come from panel D of figure 1. Panel D is a generic version of panels B and C, describing any duration of ZLB and forward guidance. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time $T + \Delta$. The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time T. The anomaly of the NK model disappears in the WUNK model, as a long-enough ZLB always leads to a slump (panels C and D versus panel B).

panel D of figure 1, except that $r^n > 0$ instead of $r^n < 0$, so the Euler line (19) is shifted outward; yet, the dynamical system remains a source. Inflation must be positive and output must be above its natural level at time T; then they decrease over time, following the unique trajectory leading to the natural steady state at time $T + \Delta$. The economy booms during forward guidance; but unlike in the NK model, output and inflation are bounded above by the forward-guidance steady state.

Before forward guidance comes the ZLB episode. It is depicted in panels B and C; the phase diagrams are the same as in panel D of figure 1. Because of the boom engineered by forward guidance, the situation is improved at the ZLB: inflation and output tend to be higher than without forward guidance. Yet, unlike in the NK model, output during the ZLB episode is always below its level at the beginning of forward guidance. So forward guidance cannot generate unbounded booms in the WUNK model. The ZLB cannot generate unbounded slumps either, since output and inflation are bounded below by the ZLB steady state. These properties are summarized in panel F. Finally, for any forward-guidance duration, as the ZLB lasts longer, the economy converges to the ZLB steady state at time 0; thus, forward guidance can never prevent a slump when the ZLB lasts long enough.

Based on these dynamics, we isolate two anomalies in the NK model, which are resolved in the WUNK model:

Proposition 4. *Consider a ZLB episode during* $(0, T)$ *followed by a forward-guidance episode during* $(T, T + \Delta)$. *Let* $y(t; T, \Delta)$ *and* $\pi(t; T, \Delta)$ *be output and inflation at time* t.

- *In the NK model, there exists a threshold* Δ^*, *such that any forward guidance longer than* Δ^* *transforms a ZLB of any duration into a boom: let* $\Delta > \Delta^*$; *then for any* T, $y(t; T, \Delta) > y^n$ *and* $\pi(t; T, \Delta) > 0$ *for all* $t \in (0, T + \Delta)$. *In addition, when the forward guidance is longer than* Δ^*, *a long-enough forward guidance or a long-enough ZLB generates an arbitrarily large boom: for any* T, $\lim_{\Delta\to\infty} y(0; T, \Delta) = +\infty$ *and* $\lim_{\Delta\to\infty} \pi(0; T, \Delta) = +\infty$; *and for any* $\Delta > \Delta^*$, $\lim_{T\to\infty} y(0; T, \Delta) = +\infty$ *and* $\lim_{T\to\infty} \pi(0; T, \Delta) = +\infty$.
- *In the WUNK model, in contrast, there exists a threshold* T^*, *such that any ZLB longer than* T^* *generates a slump, irrespective of the duration of forward guidance: let* $T > T^*$; *then for any* Δ, $y(0; T, \Delta) < y^n$ *and* $\pi(0; T, \Delta) < 0$. *In addition, the economy is always bounded above by the forward-guidance steady state, denoted* $[y^f, \pi^f]$: *for any* T *and* Δ, $y(t; T, \Delta) < y^f$ *and* $\pi(t; T, \Delta) < \pi^f$ *for all* $t \in (0, T + \Delta)$.

The anomalies identified in the proposition correspond to the instances of the forward-guidance puzzle described by Carlstrom, Fuerst, and Paustian (2015, fig. 1) and Cochrane (2017, fig. 6). These papers also find that a long-enough forward guidance transforms a ZLB slump into a boom whose amplitude increases with the duration of forward guidance.

In the WUNK model, such anomalous patterns vanish. Like us, Gabaix (2016), Diba and Loisel (2019), Acharya and Dogra (2019), and Bilbiie (2019) address the forward-guidance puzzle; there is also a literature devoted solely to the forward-guidance puzzle.[8] Among these papers, ours belongs to the group that uses discounted Euler equations—in which future interest rates have less effect on today's consumption than in the standard equation.[9] For example, Del Negro, Giannoni, and Patterson (2015) generate discounting by introducing overlapping generations; McKay, Nakamura, and Steinsson (2016) and Bilbiie (2017) by introducing heterogeneous agents facing borrowing constraints; Angeletos and Lian (2018) by introducing incomplete information. More related to our approach, Campbell et al. (2017) and Rannenberg (2019) generate discounting by introducing households who derive utility from government bonds.

3.3. Government spending

We now assume that the government purchases quantities $g_k(t)$ of the goods $k \in [0, 1]$. (The derivations are relegated to appendix D.) These quantities are aggregated into an index of public consumption

$$g(t) \equiv \left[\int_0^1 g_k(t)^{(\epsilon-1)/\epsilon} dk\right]^{\epsilon/(\epsilon-1)}. \tag{20}$$

Public consumption $g(t)$ enters separately into households' utility functions. Government expenditure is financed with lump-sum taxation.

We also assume that the disutility of labor is convex—which implies a finite Frisch elasticity of labor supply. Household j incurs disutility

$$\frac{\kappa^{1+\eta}}{1+\eta} h_j(t)^{1+\eta}$$

from working, where $\eta > 0$ is the inverse of the Frisch elasticity. The utility function is altered to ensure that government spending affects inflation and private consumption.

In this extended model, once it is expressed in terms of private consumption, the Euler

[8]The forward-guidance puzzle takes several forms. Their common element is that future monetary policy has an implausibly strong effect on output and inflation today.

[9]Other approaches to address the forward-guidance puzzle include modifying the Phillips curve (Carlstrom, Fuerst, and Paustian 2015), considering the economy in a temporary equilibrium with agents having "reflective" expectations (Garcia-Schmidt and Woodford 2019), combining bounded rationality and incomplete markets (Farhi and Werning 2017), or introducing an endogenous liquidity premium (Bredemeier, Kaufmann, and Schabert 2018).

equation is the same as before:

$$\frac{\dot{c}(t)}{c(t)} = r(\pi(t)) + u'(0)c(t) - \delta. \tag{21}$$

On the other hand, the Phillips curve is modified:

$$\dot{\pi}(t) = \delta\pi(t) + \frac{(\epsilon-1)\left[c(t)+g(t)\right]}{\gamma c(t)}\left\{1 - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta}\left[c(t)+g(t)\right]^{\eta} c(t)\right\}. \tag{22}$$

The interpretation of the Phillips curve remains the same, except that the real marginal cost—the term after $\epsilon/(\epsilon-1)$ in the curly brackets—takes a more complicated form. This is because because the marginal disutility of labor takes a more complicated shape: $\kappa^{1+\eta}h(t)^{\eta} = \kappa^{1+\eta}[y(t)/a]^{\eta} = \kappa^{1+\eta}[c(t)+g(t)]^{\eta}/a^{\eta}$.

Since the Phillips curve changes, we adjust the WUNK assumption. We replace (13) by

$$u'(0) > (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}. \tag{23}$$

For $\eta = 0$, this assumption reduces to (13); for larger values of η, the assumption requires a larger value of $u'(0)$ than (13).

Despite the extension, the model retains virtually the same properties: propositions similar to propositions 1 and 2 hold (propositions A1 and A2 in the appendix). The main novelty is that additional steady states may appear, because of the nonlinearity of the Phillips curve. In that case, we follow the literature and concentrate on the steady state closest to the natural steady state.

To analyze government spending at the ZLB, we construct phase diagrams (figures 6 and 7). The phase diagrams represent the linear dynamical system obtained by linearizing the Euler equation (21) and Phillips curve (22) around the natural steady state without government spending. The linearized Euler equation is

$$\frac{\dot{c}(t)}{c^n} = u'(0)\left[c(t) - c^n\right] - \pi(t) - r^n; \tag{24}$$

and the linearized Phillips curve is

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\left[(1+\eta)(c(t)-c^n) + \eta g\right], \tag{25}$$

where the natural level of consumption c^n and natural rate of interest r^n are given by

$$c^n = \left(\frac{\epsilon - 1}{\epsilon}\right)^{1/(1+\eta)} \frac{a}{\kappa}$$
$$r^n = \delta - u'(0)c^n.$$

We use these linear differential equations to determine the directions of the trajectories in the phase diagrams. In steady state, the linearized Euler equation and Phillips curve become

$$\pi = -\delta + u'(0)c \tag{26}$$
$$\pi = -\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} \left[(1+\eta)(c - c^n) + \eta g\right]. \tag{27}$$

We use these steady-state equations to draw the c-nullcline and π-nullcline in the phase diagrams. As is typical in New Keynesian models, government spending operates by shifting the Phillips curve (see Werning 2011; Cochrane 2017).[10]

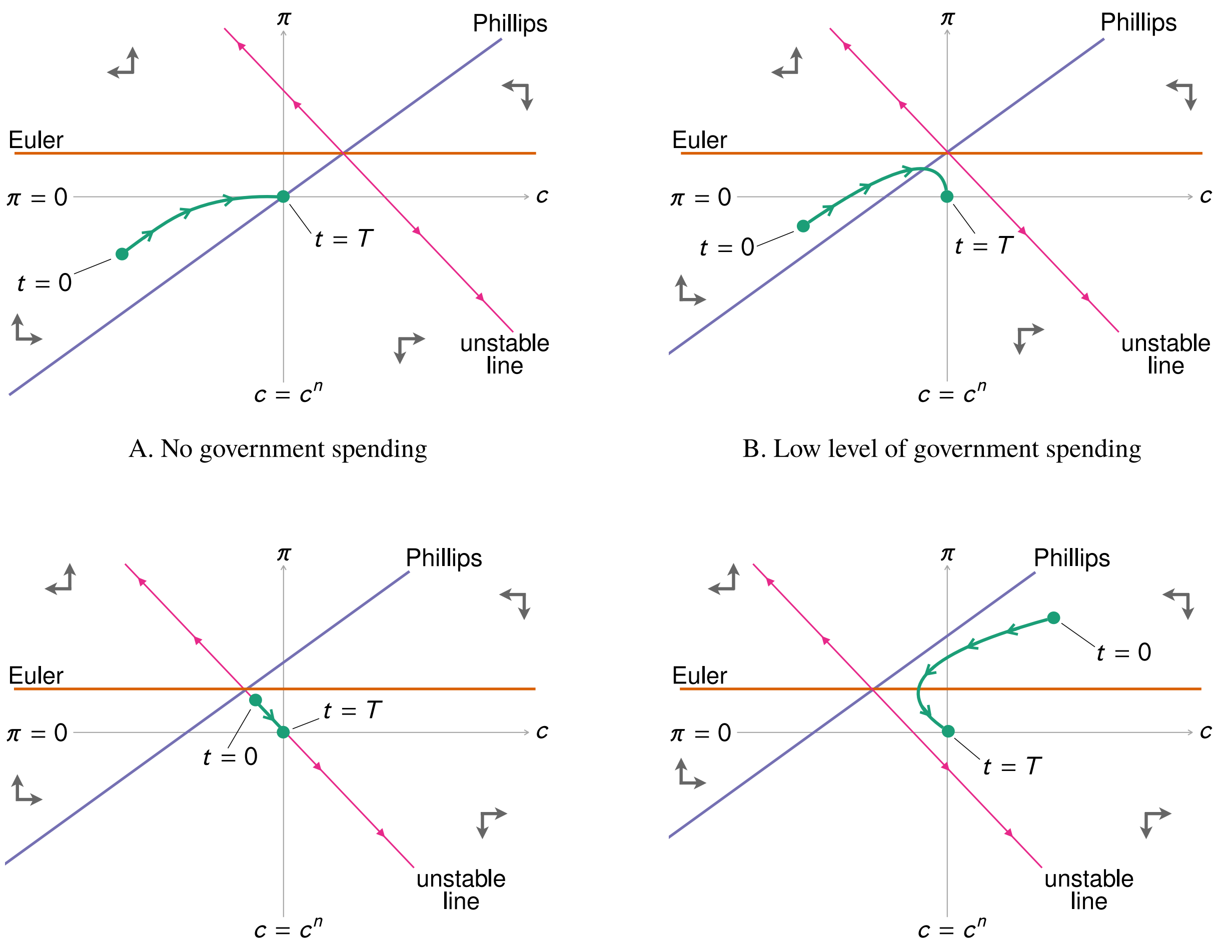
We now study a ZLB episode during which the government purchases goods to stimulate the economy, as in Cochrane (2017). Between times 0 and T, the ZLB binds: a negative aggregate demand shock makes the natural rate of interest negative, which prompts the central bank to keep the nominal interest rate at zero. To further alleviate the situation, the government provides an amount $g > 0$ of public consumption.[11] After time T, the natural rate becomes positive again, so monetary policy returns to normal, and government spending stops: the economy returns to the natural steady state, where inflation is zero and private consumption is c^n.

We begin with the NK model in figure 6. The phase diagrams are the same as in panels C of figures 1 and 2—except that the Phillips line shifts inward when government spending is higher, as shown by (27). We construct the paths of private consumption and inflation by going backward in time, given that at time T, monetary policy brings the economy to the natural steady state.

During the ZLB, government spending improves the situation, but through a different mechanism than forward guidance. Forward guidance betters the situation at the end of the ZLB, which pulls up the economy during the entire ZLB. With government spending the end of the ZLB is unchanged: the economy reaches the natural steady state. But by shifting the Phillips line and with it the field of trajectories, an increase in government spending connects the natural steady state to trajectories with higher consumption and inflation.

[10] When the disutility of labor is linear ($\eta = 0$), government spending has no first-order effect on the Phillips curve (since $\eta g = 0$) and government spending does not affect private consumption and inflation. This is why we assumed a convex disutility of labor ($\eta > 0$).

[11] Woodford (2011) shows that increasing government spending is optimal in such a situation.



A. No government spending

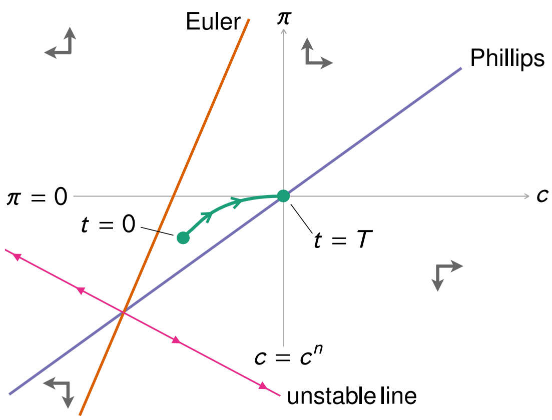
B. Low level of government spending

C. Medium level of government spending

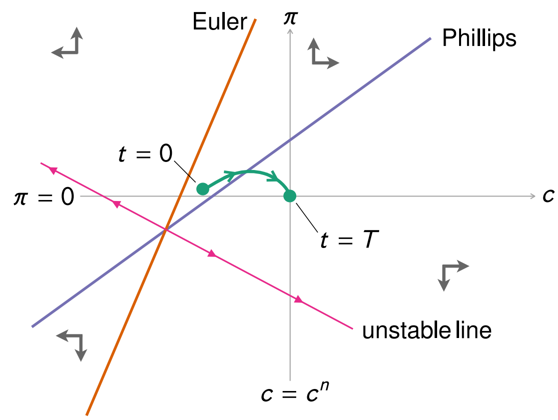
D. High level of government spending

FIGURE 6. NK model: ZLB episodes with various levels of government spending

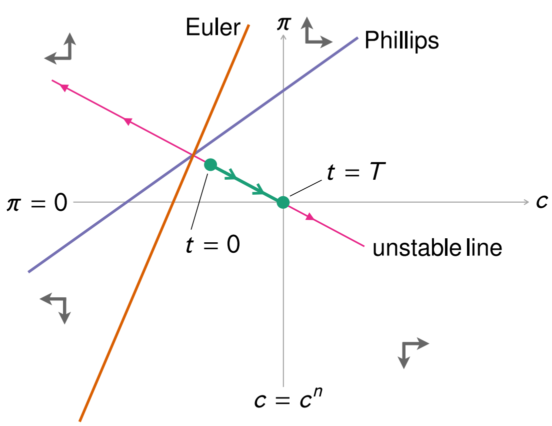
Notation: π is inflation; c is private consumption; c^n is the natural level of private consumption; the Euler line is the c-nullcline; and the Phillips line is the π-nullcline. The phase diagrams represent the linear dynamical system composed by (24) and (25) with $u'(0) = 0$; this system is obtained by linearizing the Phillips curve and Euler equation around the natural steady state without government spending. The phase diagrams have the same properties as those in panels C of figures 1 and 2, except that the Phillips line shifts inward as government spending increases. The ZLB episode lasts between times 0 and T. During the ZLB, government spending is positive. Then, at time T, government spending stops and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time T. The NK model suffers from two anomalies: when government spending brings the unstable line from the right to the left of the natural steady state, any long-enough ZLB episode will see an arbitrarily large increase in output (panel B versus panel D); and then, any long-enough ZLB episode will experience an unboundedly large boom.



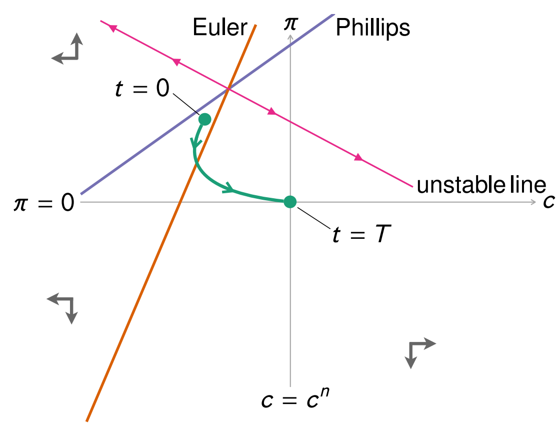
A. No government spending



B. Low level of government spending



C. Medium level of government spending



D. High level of government spending

FIGURE 7. WUNK model: ZLB episodes with various levels of government spending

Notation: π is inflation; c is private consumption; c^n is the natural level of private consumption; the Euler line is the c-nullcline; and the Phillips line is the π-nullcline. The phase diagrams represent the linear dynamical system composed by (24) and (25) with $u'(0)$ satisfying (23); this system is obtained by linearizing the Phillips curve and Euler equation around the natural steady state without government spending. The phase diagrams have the same properties as those in panels D of figures 1 and 2, except that the Phillips line shifts inward as government spending increases. The ZLB episode lasts between times 0 and T. During the ZLB, government spending is positive. Then, at time T, government spending stops and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time T. The anomalies of the NK model disappear in the WUNK model: output multipliers are finite when the ZLB becomes arbitrarily long-lasting; and irrespective of the duration of the ZLB, the equilibrium trajectories are always bounded.

Just like forward guidance, government spending can have very strong effects in the NK model. For low levels of spending (panel B), the natural steady state is on the left of the unstable line of the dynamical system representing the ZLB equilibrium; it is therefore connected to trajectories coming from the southwest, slumpy quadrant of the phase diagram—just as without government spending (panel A). Then, if the ZLB lasts longer, initial consumption and inflation fall lower. When government spending is higher enough that the natural steady state is on the unstable line, the position at the beginning of the ZLB must also be on the unstable line (panel C). Then, as the ZLB lasts longer, the initial position moves closer to the ZLB steady state. Finally, when government spending is even higher, the natural steady state moves to the right of the unstable line, so it is connected to trajectories coming from the northeast, boomy quadrant of the phase diagram (panel D). As a result, initial output and inflation are much higher than previously. And as the ZLB lasts longer, initial output and inflation become higher, without any upper bound.

The power of government spending at the ZLB is much weaker in the WUNK model, as illustrated in figure 7. The phase diagrams are the same as in panels D of figures 1 and 2—except that the Phillips line shifts inward when government spending is higher. After time T, the economy is at the natural steady state; prior to that comes the ZLB episode. Government spending improves the situation at the ZLB, as inflation and consumption tend to be higher than without spending. As the ZLB lasts longer, the position at the beginning of the ZLB converges to the ZLB steady state—unlike in the NK model, it does not diverge to infinity. So equilibrium trajectories are bounded, and government spending cannot generate unbounded booms.

Based on these dynamics, we isolate two anomalies in the NK model, which are resolved in the WUNK model (appendix E fleshes out the proof):

Proposition 5. *Consider a ZLB episode during $(0, T)$ accompanied by government spending $g > 0$. Let $c(t; T, g)$ and $y(t; T, g)$ be private consumption and output at time t, let $s > 0$ be some incremental government spending, and let*

$$\lambda(T, g, s) = \frac{y(0; T, g + s/2) - y(0; T, g - s/2)}{s} = 1 + \frac{c(0; T, g + s/2) - c(0; T, g - s/2)}{s}$$

be the government-spending multiplier.

- *In the NK model, there exists a government spending g^* such that the government-spending multiplier becomes infinitely large when the ZLB is infinitely long-lasting: for any $s > 0$, $\lim_{T\to\infty} \lambda(T, g^*, s) = +\infty$. In addition, when government spending is above g^*, a long-enough ZLB generates an arbitrarily large boom: for any $g > g^*$, $\lim_{T\to\infty} c(0; T, g) = +\infty$.*

- *In the WUNK model, in contrast, when the ZLB is infinitely long-lasting, the multiplier always* *has a finite limit: for any g and s, when $T \to \infty$, $\lambda(T, g, s)$ converges to*

$$1 + \frac{\eta}{\frac{u'(0)\delta\gamma a}{\epsilon\kappa} \cdot \left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)}. \tag{28}$$

Moreover, for any ZLB duration, the economy remains bounded: let c^g be private consumption in the ZLB steady state with government spending g; then for any T, $c(t; T, g) < \max\{c^g, c^n\}$ for all $t \in (0, T)$.

The anomaly that a finite amount of government spending may generate an infinitely large boom as the ZLB becomes arbitrarily long is reminiscent of the findings by Christiano, Eichenbaum, and Rebelo (2011, fig. 2), Woodford (2011, fig. 2), and Cochrane (2017, fig. 5) that in the NK model, government spending is exceedingly powerful when the ZLB is long-lasting.

In the WUNK model, such anomaly vanishes. Diba and Loisel (2019) and Acharya and Dogra (2019) also obtain more realistic effects of government spending at the ZLB. Beside these papers, Bredemeier, Juessen, and Schabert (2018) obtain moderate multipliers at the ZLB by introducing an endogenous liquidity premium in the New Keynesian model.

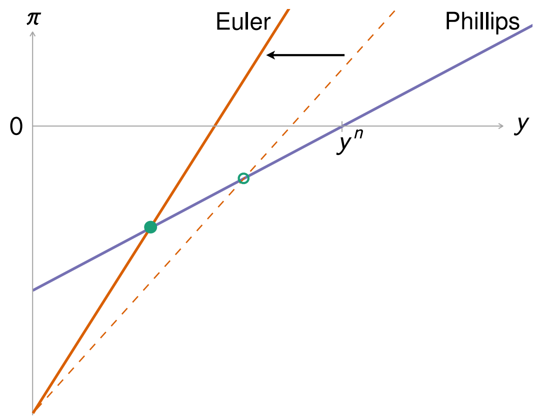
4. Other New Keynesian properties at the ZLB

Beside the anomalous properties described in the previous section, the New Keynesian model has several other intriguing properties at the ZLB: paradox of thrift, paradox of toil, paradox of flexibility, and above-one government-spending multiplier. Here we show that the WUNK model shares these properties.[12]

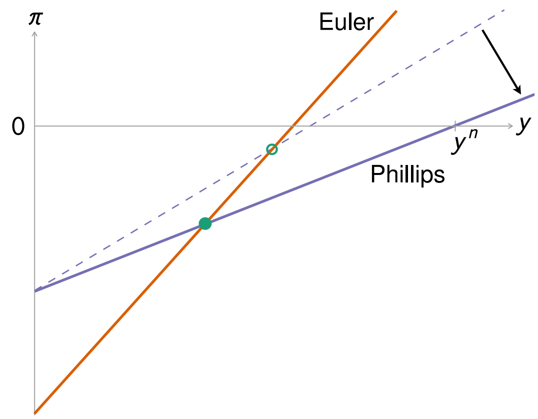
Since the ZLB equilibrium is determinate in the WUNK model, we are not forced to introduce a temporary ZLB episode: we simply work with a permanent ZLB. We assume that the natural rate of interest is negative and the central bank maintains the nominal interest rate at zero. The only equilibrium is to be at the ZLB steady state, where the economy is in a slump: inflation is negative and output is below its natural level. When an unexpected and permanent shock occurs, the equilibrium immediately jumps to the new ZLB steady state.

The ZLB equilibrium is represented graphically in figure 8: it is given by the intersection of a Phillips line, describing the steady-state Phillips curve (9), and an Euler line, describing the steady-state Euler equation (19). We use these graphs to establish the properties of the WUNK model at the ZLB.

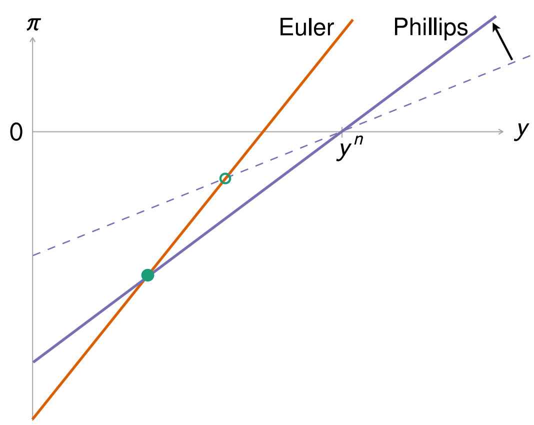
[12]Among the variants of the New Keynesian model without ZLB anomalies, the paradoxes are sometimes preserved and sometimes not. In the model of Eggertsson and Mehrotra (2014), the paradoxes of thrift, toil, and flexibility hold. But the paradoxes of toil and flexibility disappear in the model of Diba and Loisel (2019).



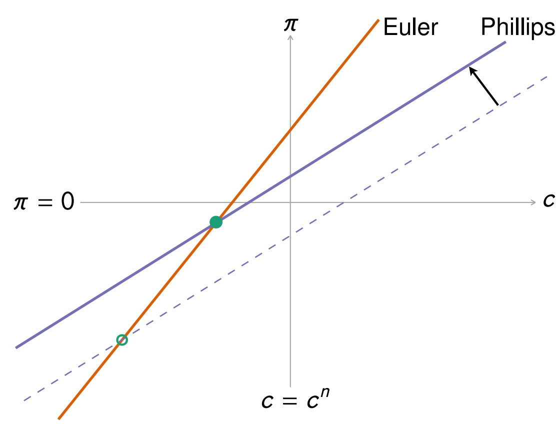
A. Paradox of thrift: higher marginal utility of wealth



B. Paradox of toil: lower disutility of labor



C. Paradox of flexibility: lower price-adjustment cost



D. Above-one government-spending multiplier

FIGURE 8. WUNK model: other properties at the ZLB

Notation: π is inflation; y is output; y^n is natural output; c is private consumption; c^n is natural consumption; the Phillips line represents the steady-state Phillips curve (9) in panels A–C, and the steady-state Phillips curve (27) in panel D; the Euler line represents the steady-state Euler equation (29) in panels A–C and (26) in panel D. The ZLB equilibrium is at the intersection of the Phillips and Euler lines: output/consumption is below its natural level and inflation is negative. Panel A illustrates the paradox of thrift: increasing the marginal utility of wealth steepens the Euler line, which depresses output and inflation without changing relative wealth. Panel B illustrates the paradox of toil: reducing the disutility of labor moves the Phillips line outward, which depresses output, inflation, and hours worked. Panel C illustrates the paradox of flexibility: decreasing the price-adjustment cost rotates the Phillips line counterclockwise around the natural steady state, which depresses output and inflation. Panel D shows that the government-spending multiplier is above one: increasing government spending shifts the Phillips line inward, which raises private consumption and therefore increases output more than one-for-one.

4.1. *Paradox of thrift*

We first study the effect of an increase in the marginal utility of wealth ($u'(0)$). The steady-state Phillips curve is unaffected, but the steady-state Euler equation does change. Using (11), we rewrite the steady-state Euler equation (19):

$$\pi = -\delta + u'(0)y. \tag{29}$$

Hence increasing the marginal utility of wealth steepens the Euler line, which moves the economy inward along the Phillips line: output and inflation decrease (figure 8, panel A). The following proposition summarizes the result:

PROPOSITION 6. *At the ZLB in the WUNK model, the paradox of thrift holds: an unexpected and permanent increase in the marginal utility of wealth reduces output and inflation but does not affect relative wealth.*

The paradox of thrift was first discussed by Keynes, but it also appears in the New Keynesian model (Eggertsson 2010, p. 16). When the marginal utility of wealth is higher, people want to increase their wealth holdings relative to their peers, so they favor saving over consumption. But in equilibrium relative wealth is fixed at zero since everybody is the same; hence the only way to save more relative to consumption is to reduce consumption. In normal times the central bank would offset this reduction in aggregate demand by reducing the nominal interest rate. This is not an option at the ZLB, so output falls.

4.2. *Paradox of toil*

Next we examine the effect of a reduction in the disutility of labor (κ). In this case the steady-state Phillips curve changes while the steady-state Euler equation does not. Using (8), we rewrite the steady-state Phillips curve (9):

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}y - \frac{\epsilon - 1}{\delta\gamma}.$$

Hence reducing the disutility of labor flattens the Phillips line, which moves the economy inward along the Euler line: both output and inflation decrease (figure 8, panel B). Moreover, since hours worked and output are related by $h = y/a$, hours fall as well. The following proposition states the result:

PROPOSITION 7. *At the ZLB in the WUNK model, the paradox of toil holds: an unexpected and permanent reduction in the disutility of labor reduces hours worked, output, and inflation.*

The paradox of toil was discovered by Eggertsson (2010). It operates as follows. With lower disutility of labor, real marginal costs are lower, and the natural level of output is higher: firms would like to produce and sell more. To increase sales, firms tend to reduce their prices, reducing inflation. Away from the ZLB, the central bank would offset this reduction in inflation by lowering the nominal interest rate. But this cannot happen at the ZLB, so the reduction in inflation raises the real interest rate—as the nominal interest rate is at zero—which makes households more prone to save. In equilibrium, this lowers output and hours worked.

As usual in this context, an increase in technology (a) would have the same effect as a reduction in the disutility of labor: it would lower output and inflation.

4.3. Paradox of flexibility

We study the effect of a decrease in the price-adjustment cost (γ). The steady-state Euler equation is not affected, but the steady-state Phillips curve is. Equation (9) shows that decreasing the price-adjustment cost leads to a counterclockwise rotation of the Phillips line around natural steady state, which moves the economy inward along the Euler line: both output and inflation decrease (figure 8, panel C). Proposition 8 records the results:

Proposition 8. *At the ZLB in the WUNK model, the paradox of flexibility holds: an unexpected and permanent decrease in price-adjustment cost reduces output and inflation.*

The paradox of flexibility was discovered in the New Keynesian model by Eggertsson and Krugman (2012). Intuitively, with a lower price-adjustment cost, firms are keener to adjust their prices to bring production closer to the natural level of output, which accentuates the existing deflation. Hence, the real interest rate rises, which results in lower output.

4.4. Above-one government-spending multiplier

We finally turn to the effect of an increase in government spending (g). We use the model with government spending introduced in section 3.3. From (27) we see that increasing government spending shifts the Phillips line inward, which moves the economy upward along the Euler line: both private consumption and inflation increase (figure 8, panel D). Since private consumption increases when public consumption does, the government-spending multiplier $dy/dg = 1 + dc/dg$ is greater than one. Proposition 9 gives the results:

Proposition 9. *At the ZLB in the WUNK model, an unexpected and permanent increase in government spending raises private consumption and inflation. Hence the government-spending multiplier dy/dg is above one; its value is given by* (28).

The multiplier value (28) is derived in appendix E. Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), and Woodford (2011) first showed that at the ZLB in the New Keynesian model, the government-spending multiplier is above one. The intuition is the following. With higher government spending, real marginal costs for a given level of private consumption are higher, so firms would like to reduce their sales to households. Hence, firms tend to increase their prices, raising inflation. At the ZLB, the increase in inflation lowers the real interest rate—as the nominal interest rate is at zero—which makes households more prone to consume. This, in equilibrium, leads to higher private consumption and a multiplier above one.

5. Empirical support for the WUNK assumption

In the WUNK model the marginal utility of wealth is above the threshold specified in (13). We now use empirical evidence to assess this assumption.

As a first step, we rewrite (13) to express it with estimable statistics. Multiplying the inequality (13) by δy^n, we obtain $\delta u'(0)y^n > y^n\epsilon\kappa/(\gamma a)$. The definition of the natural rate of interest, given by (11), implies that $u'(0)y^n = \delta - r^n$. This shows how to measure the marginal rate of substitution between wealth and consumption: by estimating the gap between time discount rate and natural rate of interest. The expression for natural output, given by (8), implies that $y^n\epsilon\kappa/(\gamma a) = (\epsilon - 1)/\gamma$. The statistic $(\epsilon - 1)/\gamma$ has been estimated before because, as shown by equation (A19) in the appendix, it is the coefficient on the output gap in the discrete-time Phillips curve. Hence, we rewrite (13) as

$$\delta \times (\delta - r^n) > \frac{\epsilon - 1}{\gamma}. \tag{30}$$

We need estimates of three statistics to assess the WUNK assumption: the natural rate of interest r^n, the time discount rate δ, and the output-gap coefficient in the Phillips curve, $(\epsilon - 1)/\gamma$. We now survey the literatures estimating these statistics.

5.1. *Natural rate of interest*

The natural rate is usually estimated by computing slow-moving trends of the real interest rate. While using different methodology, specification, and data, existing studies all obtain fairly similar estimates of the natural rate of interest in the US (see Williams 2017, fig. 1 and Holston, Laubach, and Williams 2017, fig. 1). For instance, using both reduced-form and structural approaches, Del Negro et al. (2017, fig. 1) find that the US natural rate of interest averages 2% over the 1960–2016 period. So we use $r^n = 2\%$ as our estimate.

5.2. *Slope of Phillips curve*

The slope of the Phillips curve has been estimated by a large literature, reviewed in Mavroeidis, Plagborg-Moller, and Stock (2014, fig. 3). For a specification similar to (A19), estimates of $(\epsilon - 1)/\gamma$ vary between 0.005 and 0.08, with a median estimate across 16 studies of $(\epsilon - 1)/\gamma = 0.03$. Mavroeidis, Plagborg-Moller, and Stock (2014, table 3) also propose their own estimates of the Phillips curve. They are slightly lower than the median: 0.026 and 0.018.

As an alternative, we can measure $(\epsilon - 1)/\gamma$ from microestimates of ϵ and γ. Using firm-level data, De Loecker and Eeckhout (2017, fig. 1) attempt to measure the goods-market markup in the United States. They find that the average markup $\epsilon/(\epsilon - 1)$ hovers between 1.2 and 1.3 between 1950 and 1980 before continuously rising to 1.7 between 1980 and 2014. Since 1990 the average markup is around 1.5, implying an average elasticity $\epsilon = 3$, which we use as our midrange estimate. Next, following Michaillat (2014, p. 206), we calibrate the price-adjustment cost to $\gamma = 61$. This cost is obtained from the microevidence provided by Zbaracki et al. (2004), who describe the pricing process of a large industrial US firm and measure the physical, managerial, and customer costs of changing prices. Combining the estimates for ϵ and γ yields $(\epsilon - 1)/\gamma = (3 - 1)/61 = 0.033$. This number is close to the estimate from the Phillips-curve literature, so we use $(\epsilon - 1)/\gamma = 0.03$ as a midpoint estimate.

5.3. *Time discount rate*

A vast literature has attempted to estimate time discount rates. Frederick, Loewenstein, and O'Donoghue (2002, table 1) survey the estimates obtained from real-world behavior and elicitation in field or laboratory experiments. There is a lot of variation in the estimates, but the majority of them points to high time discounting, much higher than prevailing market interest rates. We compute the mean estimate in each of the 43 studies covered by the survey, and then compute the median value of these means. We obtain an annual discount rate of $\delta = 0.35$.

One immediate problem with most of the studies discussed by Frederick, Loewenstein, and O'Donoghue is that they assume that people use a single rate to exponentially discount future utility. This exponential-discounting model is subject to many anomalies, however. One issue is particularly problematic here: people seem to choose more impatiently for the present than for the future—they exhibit present-focused preferences (Ericson and Laibson 2019). One simple way to address this issue and estimate the exponential, long-run discount rate is to focus on studies with long horizon. Frederick, Loewenstein, and O'Donoghue (2002, fig. 1B) reviews studies with an horizon above one year. There the average annual discount factor is $1/(1 + \delta) = 0.8$, implying an annual discount rate of $\delta = 0.25$.

Many studies published after the survey by Frederick, Loewenstein, and O'Donoghue move away from the exponential-discounting model and allow for more general discounting—including hyperbolic and quasi-hyperbolic (β-δ) discounting. Andersen et al. (2014, table 3) survey 16 such studies. Computing again the mean estimate in each study, and then the median value of these means, we obtain an annual discount rate of $\delta = 0.43$.

There are three potential issues with the studies surveyed by Frederick, Loewenstein, and O'Donoghue and Andersen et al.—which could explain why they find such high discount rates. First, some of these studies use hypothetical choices instead of real, monetary incentives to elicit subjects' preferences. However, the literature does not find systematic differences between the discount rates in hypothetical-choice and incentivized-choice experiments (Cohen et al. 2019, sec. 4C). Second, many of the studies are run with university students instead of subjects representative of the general population. Here again, there does not seem to be systematic differences in discounting between student and non-student subjects (Cohen et al. 2019, sec. 6A). Hence, using hypothetical choices and students subjects is unlikely to bias the estimates reported in the surveys.

A third potential issue is that the discount rates in Frederick, Loewenstein, and O'Donoghue and Andersen et al. are elicited from experiments using financial flows instead of consumption flows. Given that the goal is to elicit the discount rate on consumption, this could be problematic (Cohen et al. 2019, sec. 4B). The problems could be exacerbated if subjects derive utility from wealth. To assess this potential issue, suppose first (as most of the literature does) that monetary payments are consumed at the time of receipts, and that the utility function is locally linear. Under these two conditions, monetary experiments deliver estimates of the time discount rate—whether or not subjects derive utility from wealth (Cohen et al. 2019, sec. 4B). Then, all the experiments surveyed by Frederick, Loewenstein, and O'Donoghue and Andersen et al. indeed estimate a time discount rate.

On the other hand, if these conditions do not hold, it is more difficult to interpret the findings of the experiments. For instance, if subjects optimally smooth their consumption over time by borrowing and saving at some market interest rate, then experiments with financial flows would elicit only the interest rate faced by subjects; it would reveal nothing about how they discount time (Cohen et al. 2019, sec. 4B). In that case, we should rely on experiments using time-dated consumption rewards instead of monetary rewards. Such experiments directly deliver estimates of the time discount rate irrespective of whether wealth enters the utility function. Many such experiments have been conducted; a robust finding is that discount rates for consumption rewards are systematically higher than discount rates for monetary rewards (Cohen et al. 2019, sec. 3A). Hence, the estimates presented in the surveys by Frederick, Loewenstein, and O'Donoghue and by Andersen et al. are, if anything, lower bounds on actual time discount rates.

Overall, then, a time discount rate of $\delta = 0.35$ seems to be a midpoint estimate. Using the natural rate of interest $r^n = 2\%$, we obtain an estimate of the marginal rate of substitution between wealth and consumption: $u'(0)y^n = \delta - r^n = 0.35 - 0.02 = 0.33$. Although the financial return on wealth (r^n) is much lower than the time discount rate (δ), people are willing to hold wealth because they derive direct utility from it.

5.4. Assessment of the assumption

Combining our empirical estimates of r^n, δ, and $(\epsilon - 1)/\gamma$, we find that (30) easily holds: $\delta \times (\delta - r^n) = 0.35 \times (0.35 - 0.02) = 0.12 > 0.03 = (\epsilon - 1)/\gamma$. Hence the WUNK assumption holds comfortably in US data.[13] Of course there remains uncertainty about the exact values of δ and $(\epsilon - 1)/\gamma$; but the WUNK assumption holds for a broad range of estimates.

First, if we settle on the midrange estimate $\delta = 0.35$, condition (30) holds for any estimate of $(\epsilon - 1)/\gamma$ reported by Mavroeidis, Plagborg-Moller, and Stock (2014). Indeed, with $\delta = 0.35$, we have $\delta \times (\delta - r^n) = 0.12 > 0.08$, where 0.08 is the highest available estimate of $(\epsilon - 1)/\gamma$ in Mavroeidis, Plagborg-Moller, and Stock (2014, fig. 3).

Second, if we settle on the median estimate $(\epsilon - 1)/\gamma = 0.03$, condition (30) holds for any $\delta \geq 0.19$, since $0.19 \times (0.19 - 0.02) = 0.032 > 0.03$. If we use the estimate of $(\epsilon - 1)/\gamma$ obtained by Mavroeidis, Plagborg-Moller, and Stock in a modern vintage of data ($(\epsilon - 1)/\gamma = 0.018$), condition (30) even holds for any $\delta \geq 0.15$, since $0.15 \times (0.15 - 0.02) = 0.019 > 0.018$. Such values of δ are at the low end of available estimates: combining the estimates surveyed by Frederick, Loewenstein, and O'Donoghue (2002) and Andersen et al. (2014), we find that more than 70% of the estimates of δ are above 0.19.

6. Conclusion

This paper extends the textbook New Keynesian model by introducing relative wealth into the utility function. The marginal utility of wealth is assumed to be above a threshold that depends on price rigidity—an assumption that generally holds in the data. Although our model deviates only minimally from the New Keynesian model, it resolves all the New Keynesian anomalies at the ZLB: even when the ZLB is arbitrarily long-lasting, there is no collapse of inflation and output, and both forward guidance and government spending have limited, plausible effects. At the same

[13]The WUNK assumption also holds in the version of the model with finite Frisch elasticity of labor supply $1/\eta$. The assumption in that case is $\delta \times (\delta - r^n) > (1 + \eta)(\epsilon - 1)/\gamma$ (see equation (23)). Evidence suggests that $\eta = 1$ (Chetty et al. 2013, table 2). Since $\delta \times (\delta - r^n) = 0.12$ and $(\epsilon - 1)/\gamma = 0.03$, the assumption holds.

time, our model retains other properties of the New Keynesian model at the ZLB: paradox of thrift, paradox of toil, paradox of flexibility, and above-one government-spending multiplier.

Beyond the New Keynesian model, the wealth-in-the-utility assumption might be a simple way to model people's saving behavior better. First, it reconciles the single-digit interest rates observed on many markets with the double-digit time discount rates measured in most experimental studies. Relatedly, it explains why people appear to have a higher time discount rate for consumption rewards (such as food and beverage) than for monetary rewards. Second, it accommodates a broad range of steady-state real interest rates—including negative ones. Third, it reduces the effect of future interest rates on today's consumption. And fourth, it leads to a negative steady-state relationship between consumption and real interest rate, as in the old-fashioned IS curve. These properties contrast with the standard model, in which the steady-state real interest rate equals the time discount rate, and future interest rates have implausibly large effects on today's consumption.

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Appendix A. Derivations of Phillips curve and Euler equation

We derive the two differential equations that govern the equilibrium of our New Keynesian model: the Phillips curve, given by (7); and the Euler equation, given by (10).

Household saving and pricing

We begin by characterizing households' optimal saving and pricing. To solve household j's optimization problem, we write the current-value Hamiltonian:

$$\begin{aligned}\mathcal{H}_j = \frac{\epsilon}{\epsilon - 1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right) + u\left(\frac{b_j(t) - b(t)}{p(t)}\right) - \frac{\kappa}{a} y_j^d(p_j(t), t) - \frac{\gamma}{2}\pi_j(t)^2 \\ + \mathcal{A}_j(t)\left[i(t)b_j(t) + p_j(t)y_j^d(p_j(t), t) - \int_0^1 p_k(t)c_{jk}(t)\, dk - \tau(t)\right] + \mathcal{B}_j(t)\pi_j(t)p_j(t),\end{aligned}$$

with control variables $c_{jk}(t)$ for all $k \in [0, 1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian. To ease notation, we now drop the time index t.

The first optimality conditions are $\partial\mathcal{H}_j/\partial c_{jk} = 0$ for all $k \in [0, 1]$. The conditions yield

$$\frac{1}{c_j}\left(\frac{c_{jk}}{c_j}\right)^{-1/\epsilon} = \mathcal{A}_j p_k. \tag{A1}$$

Appropriately integrating (A1) over all $k \in [0, 1]$, and using (2) and (3), we find

$$\mathcal{A}_j = \frac{1}{pc_j}. \tag{A2}$$

Combining (A1) and (A2), we then obtain

$$c_{jk} = \left(\frac{p_k}{p}\right)^{-\epsilon} c_j. \tag{A3}$$

Integrating (A3) over all $j \in [0, 1]$, we get the usual demand for good k:

$$y_k^d(p_k) = \int_0^1 c_{jk}\, dj = \left(\frac{p_k}{p}\right)^{-\epsilon} c, \tag{A4}$$

where $c = \int_0^1 c_j\, dj$ measures aggregate consumption. We use this expression for $y_k^d(p_k)$ in household k's Hamiltonian. We also obtain $\int_0^1 p_k c_{jk}\, dk = pc_j$: the price of one unit of consumption index is indeed p.

The second optimality condition is $\partial \mathcal{H}_j / \partial b_j = \delta \mathcal{A}_j - \dot{\mathcal{A}}_j$, which gives

$$-\frac{\dot{\mathcal{A}}_j}{\mathcal{A}_j} = i + \frac{1}{p\mathcal{A}_j} \cdot u'\left(\frac{b_j - b}{p}\right) - \delta.$$

Using (A2), we obtain the household's Euler equation:

$$\frac{\dot{c}_j}{c_j} = i - \pi + c_j u'\left(\frac{b_j - b}{p}\right) - \delta. \tag{A5}$$

This Euler equation describes the optimal path of household j's consumption.

The third optimality condition is $\partial \mathcal{H}_j / \partial \pi_j = 0$, which yields

$$\mathcal{B}_j p_j = \gamma \pi_j. \tag{A6}$$

Differentiating (A6) with respect to time, we obtain

$$\frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j. \tag{A7}$$

The fourth optimality condition is $\partial \mathcal{H}_j / \partial p_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j$, which implies

$$\frac{\kappa}{a} \cdot \frac{\epsilon y_j}{p_j} - (\epsilon - 1)\mathcal{A}_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j.$$

Reshuffling the terms, we obtain

$$\pi_j - \frac{(\epsilon - 1) y_j \mathcal{A}_j}{\mathcal{B}_j p_j}\left(p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa}{a\mathcal{A}_j}\right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Then, using (A2), (A6), and (A7), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j}\left(\frac{p_j}{p} - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa c_j}{a}\right). \tag{A8}$$

This equation describes the optimal path of the price set by household j.

The previous four conditions are necessary for a maximum to the household's problem (Acemoglu 2009, theorem 7.9).

Equilibrium saving and pricing

We turn to saving and pricing in equilibrium. All households have the same initial wealth and initial price, so they all behave the same. We therefore omit the subscripts j and k.

Then, we simplify the household's Euler equation, given by (A5), and the household's Phillips curve, given by (A8), using two equilibrium conditions: relative wealth is zero; and production and consumption are equal ($y = c$). Accordingly, we simplify the household's Euler equation to

$$\frac{\dot{y}}{y} = r - \delta + u'(0)y,$$

where $r = i - \pi$. This is just the Euler equation given by (10). Since $y^n = (\epsilon - 1)a/(\epsilon\kappa)$, we also simplify the household's Phillips curve to

$$\dot{\pi} = \delta\pi - \frac{\epsilon\kappa}{\gamma a}(y - y^n).$$

This is the Phillips curve given by (7).

Appendix B. Phillips curve and Euler equation in discrete time

We recast the model of section 2 in discrete time and rederive the Phillips curve and Euler equation. This reformulation is helpful to compare our model to the textbook New Keynesian model, which is usually presented in discrete time. Moreover, the reformulation shows that introducing wealth in the utility function yields a discounted Euler equation.

In discrete time, households trade one-period government bonds. Bonds purchased in period t have a price $q(t)$ and pay one unit of money at maturity, in period $t+1$. The nominal interest rate between t and $t+1$ is defined as $i(t) = -\ln(q(t))$.

Household saving and pricing

We begin by characterizing saving and pricing by households. Household j chooses sequences $\left\{y_j(t), p_j(t), h_j(t), \left[c_{jk}(t)\right]_{k=0}^{1}, b_j(t)\right\}_{t=0}^{\infty}$ to maximize the discounted sum of instantaneous utilities

$$\sum_{t=0}^{\infty} \beta^t \left\{\frac{\epsilon}{\epsilon - 1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right) + u\left(\frac{b_j(t) - b(t)}{p(t)}\right) - \kappa h_j(t) - \frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)} - 1\right]^2\right\} dt,$$

where $\beta < 1$ is the time discount factor. The maximization is subject to three constraints. First, there is the production function (1). Second, there is the demand for good j, given by (6). The demand for good j remains the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time. And third, there is a budget constraint:

$$\int_0^1 p_k(t)c_{jk}(t)\, dk + q(t)b_j(t) + \tau(t) = p_j(t)y_j(t) + b_j(t-1).$$

Household j is also subject to a solvency constraint preventing Ponzi schemes: $\lim_{T\to\infty} b_j(T) \geq 0$. Finally, household j takes as given the initial conditions $b_j(-1)$ and $p_j(-1)$, as well as the sequences of aggregate variables $\{p(t), q(t), c(t)\}_{t=0}^{\infty}$. We assume that all households face the same initial conditions, so they all behave the same.

To solve household j's problem, we set up the Lagrangian:

$$\begin{aligned}\mathcal{L}_j = \sum_{t=0}^{\infty} \beta^t \Bigg\{ &\frac{\epsilon}{\epsilon-1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right) + u\left(\frac{b_j(t)-b(t)}{p(t)}\right) - \frac{\kappa}{a} y_j^d(p_j(t), t) - \frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)} - 1\right]^2 \\ &+ \mathcal{A}_j(t)\left[p_j(t)y_j^d(p_j(t), t) + b_j(t-1) - \int_0^1 p_k(t)c_{jk}(t)\, dk - q(t)b_j(t) - \tau(t)\right]\Bigg\}\end{aligned}$$

where $\mathcal{A}_j(t)$ is the Lagrange multiplier on the budget constraint in period t, and

$$y_j^d(p_j(t), t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t) \tag{A9}$$

is the demand for good j in period t. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Lagrangian.

We begin by computing the first-order conditions with respect to $c_{jk}(t)$ for all $k \in [0, 1]$. As in the continuous-time case, we obtain

$$\mathcal{A}_j(t) = \frac{1}{p(t)c_j(t)}. \tag{A10}$$

We then turn to the first-order condition with respect to $b_j(t)$. The condition is

$$q(t)\mathcal{A}_j(t) = \frac{1}{p(t)} u'\left(\frac{b_j(t)-b(t)}{p(t)}\right) + \beta\mathcal{A}_j(t+1).$$

Using (A10), we obtain the household's Euler equation for consumption:

$$q(t) = c_j(t)u'\left(\frac{b_j(t)-b(t)}{p(t)}\right) + \beta\frac{p(t)c_j(t)}{p(t+1)c_j(t+1)}. \tag{A11}$$

Finally, the first-order condition with respect to $p_j(t)$ is

$$0 = \frac{\kappa}{a} \cdot \frac{\epsilon y_j(t)}{p_j(t)} - \frac{\gamma}{p_j(t-1)} \left[\frac{p_j(t)}{p_j(t-1)} - 1 \right] + (1-\epsilon)\mathcal{A}_j(t) y_j(t) + \beta\gamma \frac{p_j(t+1)}{p_j(t)^2} \left[\frac{p_j(t+1)}{p_j(t)} - 1 \right].$$

Multiplying this equation by $p_j(t)/\gamma$ and using (A10), we obtain the household's Phillips curve:

$$\text{(A12)} \quad \frac{p_j(t)}{p_j(t-1)} \left[\frac{p_j(t)}{p_j(t-1)} - 1 \right] = \beta \frac{p_j(t+1)}{p_j(t)} \left[\frac{p_j(t+1)}{p_j(t)} - 1 \right] + \frac{\epsilon\kappa}{\gamma a} y_j(t) - \frac{\epsilon - 1}{\gamma} \cdot \frac{p_j(t) y_j(t)}{p(t) c_j(t)}.$$

Equilibrium saving and pricing

We turn to saving and pricing in equilibrium. Since all households behave the same in equilibrium, we drop the subscripts j and k. The fact that all households hold the same wealth implies that relative wealth is zero. As production and consumption are equal, we set $y(t) = c(t)$. Then, from (A11), we obtain the Euler equation:

$$\text{(A13)} \quad q(t) = u'(0) y(t) + \beta \frac{p(t) y(t)}{p(t+1) y(t+1)}.$$

Similarly, using (A12), we obtain the Phillips curve:

$$\text{(A14)} \quad \frac{p(t)}{p(t-1)} \left[\frac{p(t)}{p(t-1)} - 1 \right] = \beta \frac{p(t+1)}{p(t)} \left[\frac{p(t+1)}{p(t)} - 1 \right] + \frac{\epsilon - 1}{\gamma} \left[\frac{y(t)}{y^n} - 1 \right].$$

To obtain this equation we have used (8), which implies that $(\epsilon\kappa)/(\gamma a) = (\epsilon - 1)/(\gamma y^n)$.

Log-linearization

Last, to obtain standard expressions, we log-linearize the Euler equation and Phillips curve around the natural steady state—where $y = y^n$, $\pi = 0$, and $i = r^n$. We introduce the log-deviation of output from its steady-state level: $\hat{y}(t) = \ln(y(t)) - \ln(y^n)$. We also introduce the inflation rate between periods t and $t+1$: $\pi(t+1) = \ln(p(t+1)) - \ln(p(t))$.

We start by log-linearizing the Euler equation (A13). We first take the log of the left-hand side of (A13). Using the discrete-time definition of the interest rate $i(t)$, we obtain $\ln(q(t)) = -i(t)$.

Next we take the log of the right-hand side of (A13) and obtain $\Lambda \equiv \ln(\Lambda_1 + \Lambda_2)$, where

$$\Lambda_1 \equiv u'(0) y(t)$$
$$\Lambda_2 \equiv \beta \frac{p(t) y(t)}{p(t+1) y(t+1)}.$$

For future reference, we compute the values of Λ, Λ_1, and Λ_2 at the natural steady state. At the natural steady state, $i = r^n$, so the log of the left-hand side of (A13) equals $-r^n$, which implies that the log of the right-hand side of (A13) must also equal $-r^n$—thus $\Lambda = -r^n$. Moreover, at the natural steady state, $\Lambda_1 = u'(0)y^n$. And, since inflation is zero and output is constant at that steady state, $\Lambda_2 = \beta$.

Using the results, we obtain a first-order approximation of $\Lambda(\Lambda_1, \Lambda_2)$ around the natural steady state:

$$\Lambda = -r^n + \frac{\partial \Lambda}{\partial \Lambda_1}\left[\Lambda_1 - u'(0)y^n\right] + \frac{\partial \Lambda}{\partial \Lambda_2}\left[\Lambda_2 - \beta\right].$$

Factoring out $u'(0)y^n$ and β, and using the definitions of Λ_1 and Λ_2, we obtain

$$\text{(A15)} \qquad \Lambda = -r^n + u'(0)y^n \cdot \frac{\partial \Lambda}{\partial \Lambda_1} \cdot \left[\frac{y(t)}{y^n} - 1\right] + \beta \cdot \frac{\partial \Lambda}{\partial \Lambda_2} \cdot \left[\frac{p(t)y(t)}{p(t+1)y(t+1)} - 1\right].$$

Since $\Lambda = \ln(\Lambda_1 + \Lambda_2)$, we obviously have

$$\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{\Lambda_1 + \Lambda_2}.$$

In the first-order approximation, the partial derivatives are evaluated at the natural state, so their value is

$$\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{u'(0)y^n + \beta}.$$

Hence, (A15) becomes

$$\text{(A16)} \qquad \Lambda = -r^n + \frac{u'(0)y^n}{u'(0)y^n + \beta}\left[\frac{y(t)}{y^n} - 1\right] + \frac{\beta}{u'(0)y^n + \beta}\left[\frac{p(t)y(t)}{p(t+1)y(t+1)} - 1\right].$$

The last step is to note that the first-order approximation of $\ln(x)$ at $x = 1$ is $x - 1$, so that around the natural steady state, and up to second-order terms, we have

$$\text{(A17)} \qquad \frac{y(t)}{y^n} - 1 = \ln\left(\frac{y(t)}{y^n}\right) = \hat{y}(t)$$

and

$$\begin{aligned}\frac{p(t)y(t)}{p(t+1)y(t+1)} - 1 &= \ln\left(\frac{p(t)y(t)}{p(t+1)y(t+1)}\right) \\ &= \ln\left(\frac{y(t)}{y^n}\right) - \ln\left(\frac{y(t+1)}{y^n}\right) - \ln\left(\frac{p(t+1)}{p(t)}\right) \\ &= \hat{y}(t) - \hat{y}(t+1) - \pi(t+1).\end{aligned}$$

Hence, we rewrite (A16) as

$$\begin{aligned}\Lambda &= -r^n + \frac{u'(0)y^n}{u'(0)y^n + \beta}\hat{y}(t) + \frac{\beta}{u'(0)y^n + \beta}\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right] \\ &= -r^n + (1-\alpha)\hat{y}(t) + \alpha\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right]\end{aligned}$$

where

$$\alpha = \frac{\beta}{\beta + u'(0)y^n}.$$

Overall, then, taking the log of the Euler equation (A13) yields

$$-i(t) = -r^n + (1-\alpha)\hat{y}(t) + \alpha\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right]$$

This equation is valid up to terms that are second order around the natural steady state. Reshuffling the terms yields

$$\hat{y}(t) = \alpha\hat{y}(t+1) - \left[i(t) - r^n - \alpha\pi(t+1)\right]. \tag{A18}$$

Because the marginal utility of wealth is positive ($u'(0) > 0$), we have $\alpha < 1$. Thus the Euler equation is discounted: future output, $\hat{y}(t+1)$, appears discounted by the coefficient $\alpha < 1$. Such discounting occurs in a variety of models. For example, in McKay, Nakamura, and Steinsson (2017) it occurs because of income risk and borrowing constraints. As another example, in Gabaix (2016) it appears because of consumers' bounded rationality.

To make discounting more apparent, we solve the Euler equation forward:

$$\hat{y}(t) = -\sum_{k=0}^{+\infty}\alpha^k\left[i(t+k) - r^n - \alpha\pi(t+k+1)\right].$$

As noted by McKay, Nakamura, and Steinsson (2017, p. 821), in this type of Euler equations, the effect of interest rates k periods in the future on current output is discounted by $\alpha^k < 1$. Hence, discounting is stronger for interest rates further in the future.

Second, we log-linearize the Phillips curve (A14). We start with the left-hand side of (A14). Note that the first-order approximations of $x(x-1)$ and $\ln(x)$ at $x = 1$ are both $x - 1$. This means that up to second-order terms around $x = 1$, we have $x(x-1) = \ln(x)$. Hence, up to second-order terms around the natural steady state,

$$\frac{p(t)}{p(t-1)}\left[\frac{p(t)}{p(t-1)} - 1\right] = \ln\left(\frac{p(t)}{p(t-1)}\right) = \pi(t).$$

We turn to the right-hand side of (A14). Following the same logic, up to second-order terms around the natural steady state, we have

$$\beta\frac{p(t+1)}{p(t)}\left[\frac{p(t+1)}{p(t)}-1\right]=\beta\ln\left(\frac{p(t+1)}{p(t)}\right)=\beta\pi(t+1).$$

Furthermore, using (A17), we know that up to second-order terms around the natural steady state, we have

$$\frac{\epsilon-1}{\gamma}\left[\frac{y(t)}{y^n}-1\right]=\frac{\epsilon-1}{\gamma}\hat{y}(t)$$

Combining all these results, we find that the Phillips curve (A14) implies

$$\pi(t)=\beta\pi(t+1)+\frac{\epsilon-1}{\gamma}\hat{y}(t). \tag{A19}$$

This equation is valid up to terms that are second order around the natural steady state. Hence, despite wealth in the utility function, the Phillips curve remains the same as in a typical New Keynesian model: (A19) is the same as equation (21) in Gali (2008, p. 49).

Appendix C. Proofs of propositions 1 and 2

Proof of proposition 1

A steady state must satisfy the steady-state Phillips curve (9) and the steady-state Euler equation (12), where monetary policy imposes $r(\pi)=r^n+(\phi-1)\pi$. These equations form a linear system:

$$\pi=\frac{\epsilon\kappa}{\delta\gamma a}(y-y^n)$$
$$(\phi-1)\pi=-u'(0)(y-y^n).$$

As $[y=y^n,\pi=0]$ satisfies both equations, it is a steady state. Furthermore the two equations are non-parallel. In the NK model this is obvious since $u'(0)=0$. In the WUNK model the slope of the second equation is $-u'(0)/(\phi-1)$: if $\phi>1$, the slope is negative; if $\phi\in[0,1)$, the slope is positive and greater than $u'(0)$ and thus than $\epsilon\kappa/(\delta\gamma a)$, as (13) holds; in both cases the two equations have different slope. We conclude that $[y^n,0]$ is the unique steady state. The nominal interest rate at $[y^n,0]$ is given by $i=r^n+\phi\times 0=r^n>0$.

Dynamics are governed by the nonlinear dynamical system generated by the Phillips curve (7) and the Euler equation (10), where monetary policy imposes $r(\pi)=r^n+(\phi-1)\pi$. The Phillips curve can be written $\dot{\pi}(t)=P(y(t),\pi(t))$ where $P(y,\pi)=\delta\pi-\epsilon\kappa(y-y^n)/(\gamma a)$; the Euler equation (10) can be written $\dot{y}(t) = E(y(t), \pi(t))$ where $E(y, \pi) = y\left[(\phi - 1)\pi + u'(0)(y - y^n)\right]$. The dynamical system is linearized around the natural steady state as follows:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial E}{\partial y} & \frac{\partial E}{\partial \pi} \\ \frac{\partial P}{\partial y} & \frac{\partial P}{\partial \pi} \end{bmatrix} \begin{bmatrix} y(t) - y^n \\ \pi \end{bmatrix},$$

where the partial derivatives are evaluated at $[y = y^n, \pi = 0]$. This linearized system is just (15). Around the natural steady state, dynamics are described by this linearized system.

Next, we classify the linear dynamical system (15) following standard methodology. We denote by $\boldsymbol{M}$ the matrix in (15), and by $\lambda_1 \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C}$ the two eigenvalues of $\boldsymbol{M}$, assumed to be distinct and nonzero.

We begin with the first case: two real eigenvalues. Without loss of generality, we assume that $\lambda_1 < \lambda_2$. Then the solution to (15) takes the form

$$\begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix} = x_1 e^{\lambda_1 t} \boldsymbol{v}_1 + x_2 e^{\lambda_2 t} \boldsymbol{v}_2, \tag{A20}$$

where $\boldsymbol{v}_1 \in \mathbb{R}^2$ and $\boldsymbol{v}_2 \in \mathbb{R}^2$ are the linearly independent eigenvectors respectively associated with the eigenvalues λ_1 and λ_2, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, p. 35).

From (A20), we see that the system is a source around the steady state if $\lambda_1 > 0$ and $\lambda_2 > 0$. Moreover, the trajectories are tangent to $\boldsymbol{v}_1$ when $t \to -\infty$ and are parallel to $\boldsymbol{v}_2$ when $t \to +\infty$. The system is a saddle if $\lambda_1 < 0$ and $\lambda_2 > 0$; in that case, the vector $\boldsymbol{v}_1$ gives the direction of the stable line (saddle path) while the vector $\boldsymbol{v}_2$ gives the direction of the unstable line. Finally, if $\lambda_1 < 0$ and $\lambda_2 < 0$, the system is a sink. (See Hirsch, Smale, and Devaney 2013, pp. 40–44.)

Next, we consider the second case: two complex eigenvalues. The eigenvalues are complex conjugates, so we write them as $\lambda_1 = \theta + i\sigma$ and $\lambda_2 = \theta - i\sigma$. We also write the eigenvector associated with λ_1 as $\boldsymbol{v}_1 + i\boldsymbol{v}_2$, where the vectors $\boldsymbol{v}_1 \in \mathbb{R}^2$ and $\boldsymbol{v}_2 \in \mathbb{R}^2$ are linearly independent. Then the solution to (15) takes a more complicated form:

$$\begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix} = e^{\theta t}\left[\boldsymbol{v}_1, \boldsymbol{v}_2\right] \begin{bmatrix} \cos(\sigma t) & \sin(\sigma t) \\ -\sin(\sigma t) & \cos(\sigma t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $[\boldsymbol{v}_1, \boldsymbol{v}_2] \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix whose columns are respectively the real and imaginary components of an eigenvector of $\boldsymbol{M}$, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, pp. 44–55).

These solutions wind periodically around the steady state, either moving toward it ($\theta < 0$) or away from it ($\theta > 0$). Hence, the system is (spiral) source if the $\theta > 0$ and a (spiral) sink if $\theta < 0$. In the special case $\theta = 0$, the solutions just circle the steady state: the system is a center. (See Hirsch, Smale, and Devaney 2013, pp. 44–47.)

Given these results, we can use the trace and determinant of $\boldsymbol{M}$ to determine the type of a dynamical system (source, saddle, and so on)—using the well-known results that $\operatorname{tr}(\boldsymbol{M}) = \lambda_1 + \lambda_2$ and $\det(\boldsymbol{M}) = \lambda_1\lambda_2$ (Hirsch, Smale, and Devaney 2013, pp. 61–64). Here the trace and determinant of $\boldsymbol{M}$ are

$$\begin{aligned}\operatorname{tr}(\boldsymbol{M}) &= \delta + u'(0)y^n \\ \det(\boldsymbol{M}) &= \delta u'(0)y^n + (\phi - 1)\frac{\epsilon\kappa}{\gamma a}y^n.\end{aligned}$$

Let's start with the NK model. Since $u'(0) = 0$, $\det(\boldsymbol{M}) = (\phi - 1)y^n\epsilon\kappa/(\gamma a)$. We consider first the case $\phi < 1$. Then $\det(\boldsymbol{M}) < 0$, indicating that λ_1 and λ_2 are real and of opposite sign. (If they were real and of the same sign, $\det(\boldsymbol{M}) = \lambda_1\lambda_2 > 0$; if they were complex conjugates, $\det(\boldsymbol{M}) = \lambda_1\overline{\lambda_1} = \operatorname{Re}(\lambda_1)^2 + \operatorname{Im}(\lambda_1)^2 > 0$.) Thus, the system is a saddle if $\phi < 1$.

Second, we consider the case $\phi > 1$. Then $\det(\boldsymbol{M}) > 0$, indicating that λ_1 and λ_2 are either real and of the same sign, or complex conjugates. Since $u'(0) = 0$ in the NK model, we have $\operatorname{tr}(\boldsymbol{M}) = \delta > 0$, implying that λ_1 and λ_2 are either real and positive or complex with positive real part. (If they were real and negative, $\operatorname{tr}(\boldsymbol{M}) = \lambda_1 + \lambda_2 < 0$; if they were complex conjugates with negative real part, $\operatorname{tr}(\boldsymbol{M}) = \lambda_1 + \overline{\lambda_1} = 2\operatorname{Re}(\lambda_1) < 0$.) Either way, the system is a source.

We have just shown that the linear system (15) is a saddle when $\det(\boldsymbol{M}) < 0$ and a source when $\det(\boldsymbol{M}) > 0$ and $\operatorname{tr}(\boldsymbol{M}) > 0$. This classification is standard (for example, Hirsch, Smale, and Devaney 2013, fig. 4.1); we will use it repeatedly below.

Let's turn to the WUNK model. Using $\phi - 1 \geq -1$ and (13), we have

$$\det(\boldsymbol{M}) \geq \delta u'(0)y^n - \frac{\epsilon\kappa}{\gamma a}y^n = \delta y^n\left[u'(0) - \frac{\epsilon\kappa}{\delta\gamma a}\right] > 0.$$

Furthermore, $\operatorname{tr}(\boldsymbol{M}) > \delta > 0$. Since both $\det(\boldsymbol{M}) > 0$ and $\operatorname{tr}(\boldsymbol{M}) > 0$, using the same logic as above, we conclude that the system is a source.

Proof of proposition 2

A steady state must satisfy the steady-state Phillips curve (9) and the steady-state Euler equation (12), where monetary policy imposes $r(\pi) = -\pi$. These equations form a linear system:

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}(y - y^n) \tag{A21}$$

$$\pi = -r^n + u'(0)(y - y^n). \tag{A22}$$

A solution to this system with positive output is a steady state.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

$$\pi^z = -r^n \tag{A23}$$

$$y^z = y^n - \frac{\delta\gamma a}{\epsilon\kappa} r^n. \tag{A24}$$

Since $r^n < 0$, the solution satisfies $y^z > y^n > 0$: the solution has positive output so it is a steady state. Hence the NK model admits a unique steady state at the ZLB, where $\pi^z > 0$ (since $r^n < 0$) and $y^z > y^n$. Note that the expressions (16) and (17) reduce to (A24) and (A23) when $u'(0) = 0$.

In the WUNK model, since (13) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Substituting $y - y^n$ out of (A22) using (A21), we find that inflation in that solution is given by (17). Condition (13) implies that $u'(0)\delta\gamma a/(\epsilon\kappa) > 1$, so π^z has the sign of r^n; since $r^n < 0$, we infer that $\pi^z < 0$.

Next, using (A21) and the value of π given by (17), we find that output in the unique solution is given by (16). Since (13) holds and $r^n < 0$, we infer that $y^z < y^n$. The last step is to verify that $y^z > 0$. Using (16), we need

$$y^n > \frac{-r^n}{u'(0) - \epsilon\kappa/(\delta\gamma a)}.$$

Since $-r^n = u'(0)y^n - \delta$ and $u'(0) - \epsilon\kappa/(\delta\gamma a) > 0$ (from (11) and (13)), this is equivalent to

$$\left[u'(0) - \frac{\epsilon\kappa}{\delta\gamma a}\right] y^n > u'(0)y^n - \delta.$$

Eliminating $u'(0)y^n$ on both sides, we find that this is equivalent to

$$-\frac{\epsilon\kappa y^n}{\delta\gamma a} > -\delta,$$

or

$$\frac{\epsilon \kappa y^n}{\gamma a} < \delta^2.$$

Using (8), we have $(\epsilon \kappa y^n)/(\gamma a) = (\epsilon - 1)/\gamma$. So we need to verify that $\delta > \sqrt{(\epsilon - 1)/\gamma}$. But we have imposed $\delta > \sqrt{(\epsilon - 1)/\gamma}$ in the WUNK model, to ensure that the model accommodates positive natural rates of interest. Given this assumption, we conclude that $y^z > 0$: the solution to the system has positive output, so it is a steady state. In sum, the WUNK model admits a unique steady state at the ZLB, where $\pi^z < 0$ and $y^z < y^n$.

Next, dynamics around the ZLB steady state are described by the system resulting from the linearization of the dynamical system generated by the Phillips curve (7), the Euler equation (10), and the monetary policy rule $i(\pi) = 0$. This linear system is (18); the matrix derives from the matrix in (15), after setting $\phi = 0$ and replacing y^n by y^z.

We denote by $\boldsymbol{M}$ the matrix in (18). As in the proof of proposition 1, we classify the linear system (18) by computing the trace and determinant of $\boldsymbol{M}$. We have $\mathrm{tr}(\boldsymbol{M}) = \delta + u'(0)y^z > 0$ and

$$\det(\boldsymbol{M}) = \delta y^z \left[u'(0) - \frac{\epsilon \kappa}{\delta \gamma a} \right].$$

In the NK model $u'(0) = 0$ so $\det(\boldsymbol{M}) < 0$, implying that (18) is a saddle. In the WUNK model, (13) implies that $\det(\boldsymbol{M}) > 0$, implying that (18) is a source. In fact, the discriminant of the characteristic equation of the matrix $\boldsymbol{M}$ is strictly positive:

$$\mathrm{tr}(\boldsymbol{M})^2 - 4\det(\boldsymbol{M}) = \delta^2 + [u'(0)y^n]^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{\epsilon\kappa}{\gamma a}y^n = [\delta - u'(0)y^n]^2 + 4\frac{\epsilon\kappa}{\gamma a}y^n > 0.$$

Hence the eigenvalues of $\boldsymbol{M}$ are real, not complex: (18) is a nodal source, not a spiral source.

Appendix D. Model with government spending

We add government spending into the model of section 2. To obtain richer effects, we make the disutility of labor convex. We start by deriving the Phillips curve and Euler equation—thus obtaining equations (22) and (21). We then study the dynamics of the model.

Equilibrium

We begin by computing the government's demand for each good. At any time t the government chooses the amounts $g_j(t)$ of each good $j \in [0, 1]$ to minimize the expenditure

$$\int_0^1 p_j(t) g_j(t)\, dj$$

subject to the constraint of providing an amount of public consumption g:

$$\left[\int_0^1 g_j(t)^{(\epsilon-1)/\epsilon}\, dj\right]^{\epsilon/(\epsilon-1)} = g(t).$$

To solve the government's problem at time t, we set up a Lagrangian:

$$\mathcal{L} = \int_0^1 p_j(t) g_j(t)\, dj + C \cdot \left\{ g - \left[\int_0^1 g_j(t)^{(\epsilon-1)/\epsilon}\, dj\right]^{\epsilon/(\epsilon-1)} \right\},$$

where C is the Lagrange multiplier on the public-consumption constraint. We then follow the same steps as in the derivation of (A4). The first-order conditions with respect to $g_j(t)$ for all $j \in [0, 1]$ are $\partial\mathcal{L}/\partial g_j = 0$. These conditions imply

$$p_j(t) = C \cdot \left[\frac{g_j(t)}{g(t)}\right]^{-1/\epsilon}. \tag{A25}$$

Appropriately integrating (A25) over all $j \in [0, 1]$, and using (3) and (20), we find

$$C = p(t). \tag{A26}$$

Combining (A25) and (A26), we then obtain the government's demand for good j:

$$g_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} g(t). \tag{A27}$$

Next, we characterize households' optimal saving and pricing. To solve household j's optimization problem, we write the current-value Hamiltonian:

$$\mathcal{H}_j = \frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\,dk\right) + u\left(\frac{b_j(t)-b(t)}{p(t)}\right) - \frac{1}{1+\eta}\left[\frac{\kappa}{a}y_j^d(p_j(t),t)\right]^{1+\eta} - \frac{\gamma}{2}\pi_j(t)^2$$
$$+ \mathcal{A}_j(t)\left[i(t)b_j(t) + p_j(t)y_j^d(p_j(t),t) - \int_0^1 p_k(t)c_{jk}(t)\,dk - \tau(t)\right] + \mathcal{B}_j(t)\pi_j(t)p_j(t).$$

Compared to the baseline case of appendix A, the Hamiltonian's terms including the consumption levels $c_{jk}(t)$ have not changed. Thus the optimality conditions $\partial\mathcal{H}_j/\partial c_{jk} = 0$ remain the same, which implies that (A1), (A2), and (A3) remain valid. Adding the government's demand, given by (A27), to households' demand, given by (A3), we obtain the total demand for good j at time t:

$$y_j^d(p_j(t),t) = g_j(t) + \int_0^1 c_{jk}(t)\,dk = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} y(t),$$

where $y(t) \equiv g(t) + \int_0^1 c_j(t)\,dj$ measures total consumption. The expression for $y_j^d(p_j(t),t)$ enters into the Hamiltonian $\mathcal{H}_j$. The Hamiltonian's terms including bond holdings $b_j(t)$ have not changed either compared to the case of appendix A. Hence the optimality condition $\partial\mathcal{H}_j/\partial b_j = \delta\mathcal{A}_j - \dot{\mathcal{A}}_j$ remains the same, which implies that the Euler equation (A5) remains valid. Since the Hamiltonian's terms including inflation $\pi_j(t)$ have not changed compared to the case of appendix A, the optimality condition $\partial\mathcal{H}_j/\partial\pi_j = 0$ is unchanged. Equations (A6) and (A7) therefore remain valid. Last, because the disutility from labor is convex, the optimality condition $\partial\mathcal{H}_j/\partial p_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j$ is modified. Omitting the time index t, the condition now implies

$$\frac{\epsilon}{p_j}\left(\frac{\kappa}{a}y_j\right)^{1+\eta} + (1-\epsilon)\mathcal{A}_j y_j + \mathcal{B}_j\pi_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j,$$

which can be rewritten

$$\pi_j - \frac{(\epsilon-1)y_j\mathcal{A}_j}{\mathcal{B}_j p_j}\left[p_j - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta}\frac{y_j^\eta}{\mathcal{A}_j}\right] = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Then, using (A2), (A6), and (A7), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon-1)y_j}{\gamma c_j\pi_j}\left[\frac{p_j}{p} - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta} y_j^\eta c_j\right]. \tag{A28}$$

In equilibrium, we simplify the household's Euler equation, given by (A5), using the facts that all households behave the same (so $c_j = c$), and relative wealth is zero (so $b_j - b = 0$).

Accordingly, we obtain the following Euler equation:

$$\frac{\dot{c}}{c} = r - \delta + u'(0)c.$$

This is just the Euler equation given by (21). Unlike in appendix A, production and consumption are not equal (since $y = c + g$); therefore, we can no longer use y instead of c.

We also simplify the household's Phillips curve, given by (A28). As all households behave the same, $p_j = p$, $\pi_j = \pi$, $y_j = y$, and $c_j = c$. Moreover, output equals private plus public consumption: $y = c + g$. Hence, we obtain the following Phillips curve:

$$\dot{\pi} = \delta\pi + \frac{(\epsilon - 1)(c + g)}{\gamma c}\left[1 - \frac{\epsilon}{\epsilon - 1}\left(\frac{\kappa}{a}\right)^{1+\eta}(c + g)^{\eta}c\right].$$

This is just the Phillips curve given by (22).

Linearization

Since the disutility of labor is convex, the Phillips curve (22) is nonlinear. Because of the nonlinearity, additional steady states may appear in normal times and at the ZLB. Following the literature, we study dynamics around the steady state closest to the natural steady state. To that end, we linearize the Phillips curve and Euler equation around the natural steady state, and we analyze the dynamical system given by these two linear differential equations.

We begin by linearizing the Phillips curve (22) around the point $[c = c^n, \pi = 0, g = 0]$, where c^n is defined by

$$c^n = \left(\frac{\epsilon - 1}{\epsilon}\right)^{1/(1+\eta)}\frac{a}{\kappa}.$$

The Phillips curve can be written $\dot{\pi} = P(c, \pi, g)$ where

$$P(c, \pi, g) = \delta\pi + \frac{(\epsilon - 1)(c + g)}{\gamma c}\left[1 - \frac{\epsilon}{\epsilon - 1}\left(\frac{\kappa}{a}\right)^{1+\eta}(c + g)^{\eta}c\right].$$

The linearized version is

$$\dot{\pi} = P(c^n, 0, 0) + \frac{\partial P}{\partial c}(c - c^n) + \frac{\partial P}{\partial \pi}\pi + \frac{\partial P}{\partial g}g,$$

where the partial derivatives are evaluated at $[c = c^n, \pi = 0, g = 0]$. We have $P(c^n, 0, 0) = 0$ and

$$\begin{aligned}
\frac{\partial P}{\partial c} &= -\frac{\epsilon}{\gamma}\left(\frac{\kappa}{a}\right)^{1+\eta}(1+\eta)\left(c^n\right)^{\eta} = -(1+\eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)} \\
\frac{\partial P}{\partial \pi} &= \delta \\
\frac{\partial P}{\partial g} &= -\frac{\epsilon}{\gamma}\left(\frac{\kappa}{a}\right)^{1+\eta}\eta\left(c^n\right)^{\eta} = -\eta\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}.
\end{aligned}$$

Hence, the linearized Phillips curve is

$$\dot{\pi} = \delta\pi - \frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\left[(1+\eta)(c-c^n) + \eta g\right],$$

which yields (25).

Next we linearize the Euler equation (21) around the point $[c = c^n, \pi = 0]$. We consider two different monetary-policy rules. First, when monetary policy is normal, $r(\pi) = r^n + (\phi - 1)\pi$. Then the Euler equation is $\dot{c} = E(c, \pi)$, where

$$E(c, \pi) = c\left[(\phi - 1)\pi + u'(0)(c - c^n)\right].$$

The linearized version is

$$\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi}\pi,$$

where the partial derivatives are evaluated at $[c = c^n, \pi = 0]$. We have $E(c^n, 0) = 0$ and

$$\frac{\partial E}{\partial c} = c^n u'(0), \qquad \frac{\partial E}{\partial \pi} = c^n(\phi - 1).$$

So the linearized Euler equation with normal monetary policy is

$$\dot{c} = c^n\left[(\phi - 1)\pi + u'(0)(c - c^n)\right]. \tag{A29}$$

Second, when monetary policy is at the ZLB, $r(\pi) = -\pi$. Then the Euler equation can be written $\dot{c} = E(c, \pi)$ where

$$E(c, \pi) = c\left[-r^n - \pi + u'(0)(c - c^n)\right].$$

The linearized version is

$$\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi}\pi,$$

where the partial derivatives are evaluated at $[c = c^n, \pi = 0]$. We have $E(c^n, 0) = -c^n r^n$ and

$$\frac{\partial E}{\partial c} = c^n u'(0), \qquad \frac{\partial E}{\partial \pi} = -c^n.$$

So the linearized Euler equation at the ZLB is

$$\dot{c} = c^n \left[-r^n - \pi + u'(0)(c - c^n)\right],$$

which gives (24).

Dynamics

We start by studying the dynamics of the NK and WUNK models in normal times: the natural rate of interest is positive, monetary policy follows $i(\pi) = r^n + \phi\pi$, and government spending is zero. In the neighborhood of the point $[c = c^n, \pi = 0, g = 0]$, the Phillips curve (22) and Euler equation (21) are linearly approximated by (25) and (A29), respectively. Hence, we obtain the following proposition:

PROPOSITION A1. *Consider the NK and WUNK models with convex disutility of labor ($\eta > 0$) and no government spending ($g = 0$). Assume that the economy is in normal times. Then the models admit a steady state where inflation is zero ($\pi = 0$), consumption is at its natural level ($c = c^n$), and the ZLB is not binding ($i = r^n > 0$). Around this natural steady state, dynamics are governed by the linear dynamical system*

$$\text{(A30)} \qquad \begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)c^n & (\phi - 1)c^n \\ -(1+\eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} & \delta \end{bmatrix} \begin{bmatrix} c - c^n \\ \pi \end{bmatrix}.$$

In the NK model, the dynamical system is a source when monetary policy is active ($\phi > 1$) and a saddle when monetary policy is passive ($\phi < 1$). In the WUNK model, the dynamical system is a source whether monetary policy is active or passive.

Proof. The dynamics of the NK and WUNK models in the neighborhood of the point $[c = c^n, \pi = 0, g = 0]$ are well approximated by the linear dynamical system composed of (25) and (A29). This system is just (A30), whose matrix we denote $\boldsymbol{M}$.

As $[c = c^n, \pi = 0]$ satisfies (A30) with $\dot{\pi} = 0$ and $\dot{c} = 0$, it is a steady state of the models. The nominal interest rate at $[c^n, 0]$ is given by $i = r^n + \phi \times 0 = r^n > 0$.

As in the proofs of propositions 1 and 2, we classify the linear dynamical system (A30) following standard methodology. We begin by computing the trace and determinant of $\boldsymbol{M}$:

$$\begin{aligned}\operatorname{tr}(\boldsymbol{M}) &= \delta + u'(0)c^n \\ \det(\boldsymbol{M}) &= \delta c^n \left[u'(0) + (\phi - 1)(1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\right].\end{aligned}$$

Since $\delta > 0$ and $u'(0) \geq 0$, then $\operatorname{tr}(\boldsymbol{M}) > 0$. Next we determine the sign of the determinant.

In the NK model, $u'(0) = 0$ so the sign of $\det(\boldsymbol{M})$ is given by the sign of $\phi - 1$. Accordingly, when $\phi > 1$, $\det(\boldsymbol{M}) > 0$; since $\operatorname{tr}(\boldsymbol{M}) > 0$, the system (A30) is a source. In contrast, when $\phi < 1$, $\det(\boldsymbol{M}) < 0$, indicating that the system (A30) is a saddle.

In the WUNK model, since $\phi - 1 \geq -1$ for any $\phi \geq 0$, we have

$$\det(\boldsymbol{M}) \geq \delta c^n \left[u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\right]$$

Moreover, the WUNK assumption (23) says that the term in square brackets is positive. Hence, in the WUNK model, $\det(\boldsymbol{M}) > 0$. Since we also have $\operatorname{tr}(\boldsymbol{M}) > 0$, we conclude that the system (A30) is a source. ■

Proposition A1 extends proposition 1 to models with convex disutility of labor. It shows that in normal times, the dynamics of the NK and WUNK models around the natural steady state remain the same.

We turn to the dynamics of the models at the ZLB: the natural rate of interest is negative, and monetary policy simply sets $i = 0$. In the neighborhood of the point $[c = c^n, \pi = 0, g = 0]$, the Phillips curve (22) and Euler equation (21) are linearly approximated by (25) and (24), respectively. Hence, we obtain the following results:

PROPOSITION A2. *Consider the NK and WUNK models with convex disutility of labor ($\eta > 0$) and government spending ($g \geq 0$). Assume that the economy is at the ZLB. Then the models admit a steady state where private consumption and inflation are given by*

$$c^g = c^n + \frac{r^n + \frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\eta g}{u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}} \tag{A31}$$

$$\pi^g = \frac{(1+\eta)r^n + u'(0)\eta g}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)}. \tag{A32}$$

Around the steady state, dynamics are governed by the linear dynamical system

$$\text{(A33)} \qquad \begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)c^n & -c^n \\ -(1+\eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)} & \delta \end{bmatrix} \begin{bmatrix} c - c^g \\ \pi - \pi^g \end{bmatrix}.$$

In the NK model, the dynamical system is a saddle. In the WUNK model, the dynamical system is a source.

Proof. The dynamics of the NK and WUNK models around the point $[c = c^n, \pi = 0, g = 0]$ are well approximated by the linear dynamical system composed of (24) and (25).

A steady state needs to satisfy (24) and (25) with $\dot{\pi} = 0$ and $\dot{c} = 0$. These equations form the linear system

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} [(1+\eta)(c - c^n) + \eta g]$$
$$\pi = -r^n + u'(0)(c - c^n).$$

A solution to this system with positive consumption is a steady state.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

$$\text{(A34)} \qquad c = c^n - \frac{\eta}{1+\eta}g - \frac{\delta\gamma a}{\epsilon\kappa} \cdot \frac{1}{1+\eta}\left(\frac{\epsilon}{\epsilon - 1}\right)^{\eta/(1+\eta)} r^n$$

$$\text{(A35)} \qquad \pi = -r^n.$$

Since $r^n < 0$, for g not too large, the solution has positive consumption so it is a steady state. Hence the NK model admits a unique steady state at the ZLB. Note that (A31) and (A32) reduce to (A34) and (A35) when $u'(0) = 0$.

In the WUNK model, since (23) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Inflation and consumption in that unique solution are given by (A32) and (A31). Whether inflation is positive or negative, and whether consumption is above or below its natural level, depend on the amount of government spending, g. Furthermore, for g and r^n close enough to zero, c^g is close enough to c^n and positive. Then the solution to the system has positive consumption, implying that it is indeed a steady state.

The linear dynamical system (A33) is just the system composed of (25) and (A29) once it is rewritten in canonical form. Hence the dynamics of the NK and WUNK models are indeed governed by (A33).

Once again, we use standard methodology to classify the linear dynamical system (A33), whose matrix we denote $\boldsymbol{M}$. We start by computing the trace and determinant of $\boldsymbol{M}$:

$$\operatorname{tr}(\boldsymbol{M}) = \delta + u'(0)c^n$$

$$\det(\boldsymbol{M}) = \delta c^n \left[u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\right].$$

In the NK model, $u'(0) = 0$ so $\det(\boldsymbol{M}) < 0$, indicating that the system (A33) is a saddle. In the WUNK model, since we assume (23), $\det(\boldsymbol{M}) > 0$. Since we also have $\operatorname{tr}(\boldsymbol{M}) > 0$, we conclude that the system (A33) is a source. Using the same argument as at the end of the proof of proposition 2, we can also show that the system is a nodal source, not a spiral source. ■

Proposition A2 extends proposition 2 to models with convex disutility of labor. It shows that at the ZLB, the dynamics of the NK and WUNK models remain the same.

Appendix E. Proofs of propositions 5 and 9

Proof of proposition 5

In the NK model, the amount g^* of government spending is the amount that makes the unstable line of the dynamical system go through the natural steady state. With a bit less spending than g^* (panel B of figure 6), the natural steady state is on the left of the unstable line and is connected to trajectories coming from the southwest quadrant of the phase diagram. Hence, for $g < g^*$, $lim_{T\to\infty} c(0; T, g) = -\infty$. With a bit more spending than g^* (panel D of figure 6), the natural steady state is on the right of the unstable line and is connected to trajectories coming from the northeast quadrant of the phase diagram. Hence, for $g > g^*$, $lim_{T\to\infty} c(0; T, g) = +\infty$. Accordingly, for any $s > 0$, $\lim_{T\to\infty} \lambda(T, g^*, s) = +\infty$.

In the WUNK model, when the ZLB is infinitely long-lasting, the economy jumps to the ZLB steady state at time 0: $lim_{T\to\infty} c(0; T, g) = c^g(g)$, where $c^g(g)$ is given by (A31). The steady-state consumption $c^g(g)$ is linear in government spending g, with a coefficient in front of g of

$$\frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)}.$$

Accordingly, for any $s > 0$, we have

$$\begin{aligned}\lim_{T\to\infty}\lambda(T,g,s) &= 1+\frac{lim_{T\to\infty}c(0;T,g-s/2)-lim_{T\to\infty}c(0;T,g+s/2)}{s}\\ &= 1+\frac{c^g(g+s/2)-c^g(g-s/2)}{s}\\ &= 1+\frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)}-(1+\eta)},\end{aligned}$$

which yields (28).

Proof of proposition 9

Private consumption and inflation at the ZLB steady state are determined by (A31) and (A32). The coefficients in front of government spending g in these expressions are

$$\frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)}-(1+\eta)} \quad \text{and} \quad \frac{u'(0)\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)}-(1+\eta)}.$$

Since (23) holds, both coefficients are positive. Hence, an increase in g raises private consumption and inflation. Moreover, dc/dg is given by the first of these coefficient; this immediately gives us the expression for the multiplier $dy/dg = 1 + dc/dg$.