Optimal Public Expenditure with Inefficient Unemployment

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This paper proposes a theory of optimal public expenditure when unemployment is inefficient. The theory is based on a matching model. Optimal public expenditure deviates from the Samuelson rule to reduce the unemployment gap (the difference between the current and the efficient unemployment rate). Such optimal "stimulus spending" is described by a formula expressed with three sufficient statistics: the unemployment gap, the unemployment multiplier (the decrease in unemployment achieved by increasing public expenditure), and the elasticity of substitution between public and private consumption. When unemployment is inefficiently high and the multiplier is positive, the formula yields the following results. (a) Optimal stimulus spending is positive and increasing in the unemployment gap. (b) Optimal stimulus spending is zero for a zero multiplier, increasing in the multiplier for small multipliers, largest for a moderate multiplier, and decreasing in the multiplier beyond that. (c) Optimal stimulus spending is zero if extra public goods have no value, it becomes larger as the elasticity of substitution increases, and it completely fills the unemployment gap if extra public goods are as valuable as extra private goods.

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1. Introduction

The theory of optimal public expenditure developed by Samuelson (1954) is a cornerstone of public economics. This theory shows that public goods should be provided to the point where the marginal rate of substitution between public and private consumption equals their marginal rate of transformation. While the theory has been expanded in numerous directions since its inception, one question has not been answered: how is the theory modified in the presence of unemployment, especially inefficient unemployment? This question is relevant because public expenditure is one of the key tools used by governments to tackle high unemployment.[1]

In this paper, we expand Samuelson's theory to situations with inefficient unemployment. We begin in Section 2 by embedding Samuelson's framework into a matching model of the economy. This allows us to introduce unemployment into the analysis, especially inefficient unemployment. Indeed, in a matching model, there is always some unemployment: not all the labor services on offer are sold. Furthermore, productive efficiency usually fails: unemployment may be inefficiently high, when the price of labor services is too high, or inefficiently low, when the price of labor services is too low. When unemployment is inefficiently high, too many workers are idle; when unemployment is inefficiently low, too much labor is devoted to recruiting.

In Section 3, we find that when unemployment is efficient, the Samuelson rule remains valid; but when unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to bring unemployment closer to its efficient level. We denote the deviation of public expenditure from the Samuelson rule as "stimulus spending." We describe optimal stimulus spending with a formula expressed in terms of three sufficient statistics: (a) the unemployment gap, which is the difference between current and efficient unemployment rates; (b) the unemployment multiplier, which measures the reduction in unemployment achieved by increasing public expenditure; and (c) the elasticity of substitution between public and private consumption, which describes the value of additional public consumption.[2]

[1]See Kreiner and Verdelin (2012) for a survey of the public-economic literature on optimal public expenditure. A large literature in macroeconomics estimates or simulates the effect of public expenditure on output and employment, but only a handful of macroeconomic papers study optimal public expenditure. These papers (discussed in Section 3) feature productive inefficiency; but unlike in our study, the inefficiency does not arise from unemployment.

[2]See Chetty (2009) for an overview of the sufficient-statistic approach. Golosov, Tsyvinski, and Werquin (2014) and Badel and Huggett (2017) obtain formulas expressed in terms of estimable statistics for optimal income tax analysis in fully specified macroeconomic models. Badel and Huggett (2017) provide careful calibrations consistent with empirical estimates. Our paper emulates their approach for the problem of optimal stimulus spending.

Being expressed with sufficient statistics, the formula applies broadly, irrespective of the specification of the utility function, aggregate demand, and price mechanism. Furthermore, our formula addresses a common problem with optimal policy formulas expressed with sufficient statistics: the statistics usually are implicit functions of the policy, so the formulas cannot explicitly characterize the policy. We resolve this issue by expressing the relevant statistics as explicit functions of the policy and backing out optimal stimulus spending as a function of statistics that do not depend on the policy. This explicit formula yields several results about optimal stimulus spending. (Here we only discuss the case with positive unemployment multiplier and positive unemployment gap, but the paper considers all cases.)

The first result is that when the unemployment multiplier and unemployment gap are positive, optimal stimulus spending is positive. This result is simple to understand. At the Samuelson rule, the marginal utilities of public and private consumptions are equalized. Hence increasing public consumption has no first-order effect on welfare when we ignore the effect of public expenditure on unemployment. Now if public consumption reduces unemployment and unemployment is inefficiently high, increasing public consumption generates a positive first-order effect on welfare. It is therefore optimal to raise public expenditure above the Samuelson rule. Furthermore, we find that optimal stimulus spending is increasing in the unemployment gap.

The second result is that optimal stimulus spending is zero for a zero multiplier, increasing in the multiplier for small multipliers, maximized for a moderate multiplier, and decreasing in the multiplier for larger multipliers. When the unemployment multiplier is small, optimal stimulus spending is solely determined by how much public expenditure reduces the unemployment gap. A larger multiplier means a larger reduction, so it warrants more stimulus spending. When the multiplier is large, however, this logic breaks down: it becomes optimal to nearly entirely fill the unemployment gap. As less spending is required to fill the gap when the multiplier is larger, optimal stimulus spending is decreasing in the multiplier.

The third result is that optimal stimulus spending is increasing in the elasticity of substitution between public and private consumption in utility. This result is natural: a higher elasticity of substitution means that extra public goods are more valuable making stimulus spending more desirable. There are two interesting limit cases. With a zero elasticity, extra public goods are useless; as public consumption always crowds out private consumption, it is never optimal to provide public goods beyond the Samuelson rule. With an infinite elasticity, public and private goods are interchangeable, so it is optimal to maximize the sum of public and private consumption; this requires to provide enough public goods to completely fill the unemployment gap.

In addition, we establish that the formula for optimal stimulus spending remains the same whether the taxes used to finance public expenditure are distortionary or not. While it does not alter the formula, distortionary taxation does affect how stimulus spending should be designed. When taxes are nondistortionary, the unemployment multiplier and the output multiplier (the increase in output achieved by increasing public expenditure) are equal, so they can be used interchangeably in our formula. But with distortionary taxation, the output multiplier is no longer the same as the unemployment multiplier, so the output multiplier cannot be used in our formula anymore. When taxes are distortionary, higher taxes reduce labor supply, which reduces output but not unemployment. Hence, the output multiplier is smaller than the unemployment multiplier; with a strong labor-supply response, it is even possible to have a negative output multiplier and a positive unemployment multiplier. Accordingly, neither the size not the sign of the output multiplier are useful to design stimulus spending. This point is important because the output multiplier plays a prominent role in the stimulus debate.

Finally, since the statistics in the formula are estimable, we can use the formula to generate policy recommendations. As an illustration, in Section 4, we apply the formula to the Great Recession in the United States. The unemployment multiplier is estimated to be between 0.2 and 1 and according to research on state-dependent multipliers it could even be larger in bad times. (An unemployment multiplier of x means that raising public expenditure by 1% of GDP reduces unemployment by x percentage points). Estimates of the elasticity of substitution between public and private consumption are between 0.5 and 2. Given this uncertainty, we compute optimal stimulus spending for a range of multipliers and elasticities of substitution. For example, consider an elasticity of substitution of 1 (Cobb-Douglas utility). Then we find that optimal stimulus spending is quite large even with a small multiplier of 0.2: 2.8 percentage points of GDP; it is largest for a modest multiplier of 0.4: 3.7 points of GDP; it then decreases for larger multipliers; when the multiplier reaches 1.5, optimal stimulus spending has fallen to 1.9 points of GDP. Of course, optimal stimulus spending has a different impact on unemployment with small and large multipliers: it has virtually no effect on unemployment for a multiplier of 0.2 but almost completely fills the unemployment gap for a multiplier of 1.5.

Last, in Section 5, we calibrate and simulate a specific matching model. The simulation suggests that the matching model describes well the business cycle: in response to aggregate-demand shocks, the model generates countercyclical fluctuations in unemployment rate and unemployment multiplier. The simulation also confirms that the sufficient-statistic formula, obtained using first-order approximations, is accurate even for sizable business-cycle fluctuations.

2. A Matching Model of Inefficient Unemployment

We present the model used for the analysis. The model combines the public-expenditure framework of Samuelson (1954) with the general-equilibrium matching framework of Michaillat and Saez (2015). Because of the matching structure, the equilibrium features unemployment, and the rate of unemployment is generally inefficient.

2.1. Informal Description

The model is not standard, so to help readers understand the mechanisms, we begin by describing it informally. In the analysis the demand side of the model is completely generic; here for concreteness we describe a specific demand side (which we will also use as an example in the rest of the paper).

In our model, there are people and a government. People perform services for pay: they garden, cook, clean, educate children, cut hair, do administrative work, and so on. These people are like butlers who can do everything, very much like P.G. Wodehouse's Jeeves. Nobody can be their own butler, so people are butlers for others and use their income to hire their own butlers. This captures the fact that a modern economy is based on market exchanges rather than home production.

Beside spending their income to hire others, people also buy land, which provides utility and is also a savings vehicle. As land is in fixed supply, the choice between spending on services and saving determines the aggregate demand for services. The relevant price is the price of services in terms of land.

People are hired by other people and the government. People hired by other people produce private services (cleaning or cooking) while those hired by the government produce public services (tending public spaces or policing the streets). People value both public and private services. The government finances its hiring by taxing people.

Services are hired on a matching market. This means that while people are inelastically available to work for forty hours a week, they are not working the whole time. For simplicity, we assume that each person has the same number of idle hours each week. Since unemployment is equally spread over the population, everybody has the same consumption, and there are no insurance issues.

The matching market also means that people and the government need to post help-wanted ads to hire services. Posting ads involves a labor cost: workers have to spend time creating the ad, reading applications, and interviewing applicants. The time spent on recruiting by these human-resource workers depends on the number of positions to be filled and time spent filling each position. The services provided by human-resource workers do not directly provide utility; however, they are necessary to hire other workers whose services provide utility.

Once hired, everyone is paid the same price for their services. People continue to work for the same employer for a while, until the relationship exogenously stops. As services are sold by the hour, people usually work for several other people at the same time.

The state of the services market is described by a tightness variable—the ratio of help-wanted ads to unemployment. When tightness is high, it is easy for people to find work but hard to recruit new workers: the unemployment rate is low and employers devote a large share of their workforce to recruiting. When the tightness is low, it is hard to find work but easy to recruit: the unemployment rate is high and employers devote a small share of their workforce to recruiting. There is an efficient rate of unemployment: it maximizes the amount of services providing utility. When unemployment is inefficiently high, workers are idle for too many hours, so the amount of services consumed by people is too low. When unemployment is inefficiently low, too many hours are devoted to human-resource tasks, so the amount of services enjoyed by people is also too low.

In this economy, two variables—tightness and price—equilibrate demand and supply of services. If the price is high, demand for services is low (as land is relatively more attractive). If tightness were high, people would find work easily and the supply of services would be high. But then demand could not be equal to supply, so tightness must be low in equilibrium. If instead the price is low, demand is high, and tightness must be high. Effectively, for any price, tightness adjusts to equilibrate demand and supply. The price can be determined in many ways—bargained between employer and worker, fixed by a social norm, or set by government regulation—but once the price mechanism is specified, the equilibrium is unique. There is no guarantee, however, that the price ensures efficient tightness.

What happens then when the government hires more workers? In the simple situation where public expenditure does not affect private demand, public hiring raises tightness and reduces unemployment. If unemployment is already too low, reducing unemployment further reduces total consumption, which means that crowding out of private consumption by public consumption is more than one-for-one. If unemployment is efficient, reducing unemployment does not affect total consumption, and crowding out is exactly one-for-one. Finally, if unemployment is too high, reducing unemployment raises total consumption, so crowding out is less than one-for-one. This discussion illustrates why, when public expenditure raises tightness, public expenditure is more desirable in bad times than in good times.

2.2. Supply Side

We now formally describe the model. We start with the supply side.

The model is dynamic and set in continuous time. The economy consists of a government and a measure 1 of identical households. Households are self-employed: they produce services and sell them on a matching market.[3] There are two types of services: private services, purchased by households, and public services, purchased by the government. All services are bought on the same matching market at the same price p. Households value both the private services that they purchase and the public services provided by the government.

Each household has a fixed productive capacity $k > 0$; the capacity indicates the maximum amount of services that a household could sell at any point in time. (Here k is exogenous, but in Section 3.3, we will show that the results remain the same when k is chosen by households to maximize utility.) Since there is a measure 1 of households, the aggregate capacity in the economy is k. Because of the matching process, not all available services are sold at any point in time, so there is always some unemployment. At time t, households sell $C(t)$ services to other households and $G(t)$ services to the government; output $Y(t)$ is the sum of all sales:

$$Y(t) = C(t) + G(t).$$

As households are unable to sell their entire capacity, $Y(t) < k$. The unemployment rate is the share of aggregate capacity that is idle: $u(t) = (k - Y(t))/k$.

Services are sold through long-term relationships. Once a seller and a buyer have matched, the

[3]We abstract from firms for simplicity. Michaillat and Saez (2015) show how the model can be extended to include firms hiring workers on the labor market and selling their production on the product market.

seller serves the buyer at each instant until the relationship exogenously ends. The rate at which relationships separate is $s > 0$. Since $Y(t)$ services are committed to existing relationships at time t, the amount of services available for purchase at time t is $k - Y(t)$.

To buy new services, households and the government advertise a total of $v(t)$ vacancies. (In Section 2.3, we will explain how households and government form their demand for services.) A matching function taking as arguments the aggregate number of available services and the aggregate number of vacancies determines the rate $h(t)$ at which new long-term relationships are formed. For convenience, we use a standard Cobb-Douglas specification:

$$h(t) = \omega \cdot v(t)^{1-\eta} \cdot (k - Y(t))^{\eta},$$

where $\eta \in (0, 1)$ is the matching elasticity, and $\omega > 0$ is the matching efficacy.

With constant returns to scale in matching, the rates at which sellers and buyers form new long-term relationships is determined by the market tightness. The market tightness $x(t)$ is the ratio of the two arguments in the matching function: $x(t) \equiv v(t)/(k - Y(t))$. Each of the $k - Y(t)$ available services is sold at rate $f(x(t)) = h(t)/(k - Y(t)) = \omega x(t)^{1-\eta}$ and each of the $v(t)$ vacancies is filled at rate $q(x(t)) = h(t)/v(t) = \omega x(t)^{-\eta}$. The selling rate $f(x)$ is increasing in x and the buying rate $q(x)$ is decreasing in x. Hence, when tightness is higher, it is easier to sell services but harder to buy them.

In such a model, output follows the law of motion $\dot{Y}(t) = f(x(t))(k - Y(t)) - sY(t)$. The term $f(x(t))(k - Y(t))$ is the number of new relationships forming at time t; the term $sY(t)$ is the number of existing relationships separating at time t. If $f(x)$ and s are constant over time, output converges to the steady-state level

$$Y(x, k) = \frac{f(x)}{f(x) + s} k. \tag{1}$$

The function $Y(x, k)$ is positive and increasing in x and k, and its elasticity with respect to x is $(1 - \eta)u(x)$. The unemployment rate is directly related to output: $u = 1 - Y/k$; hence, the steady-state unemployment rate is

$$u(x) = \frac{s}{s + f(x)}. \tag{2}$$

The function $u(x)$ is positive and decreasing in x. Its elasticity with respect to x is $-(1-\eta)(1-u(x))$. When tightness is high, output is high and unemployment low because it is easy to sell services.

In US data, unemployment reaches this steady-state level quickly because labor market flows are large. In fact, Hall (2005, Figure 1) shows that the unemployment rate obtained from (2) and the actual employment rate are indistinguishable. Thus, as Hall does, we ignore the transitional dynamics of output and unemployment and assume that the two variables depend on tightness according to (1) and (2). To simplify the analysis further, we abstract from transitional dynamics and randomness at the household level: we assume that when tightness is x, all households sell a share $1 - u(x)$ of their capacity k; the remaining share $u(x)$ is idle.

Posting a vacancy costs $\rho > 0$ services per unit of time. These services represent the resources devoted by households and the government to matching with appropriate providers of services. These matching services do not provide utility to households, so we distinguish services that are purchased from services that provide utility. Households purchases $C(t)$ services and the government purchases $G(t)$ services. We refer to $C(t)$ as *private expenditure* and $G(t)$ as *public expenditure*. But households only derive utility from $c(t) < C(t)$ private services and $g(t) < G(t)$ public services; $c(t)$ and $g(t)$ are computed by subtracting matching services used by households and the government from $C(t)$ and $G(t)$. We refer to $c(t)$ as *private consumption*, $g(t)$ as *public consumption*, and $y(t) = c(t) + g(t)$ as *total consumption*.

The wedge between expenditure and consumption is determined by tightness. As we did with sellers of services, we abstract from transitional dynamics and randomness with buyers of services (households and the government). This means that by posting v_0 vacancies, a buyer establishes exactly $v_0 q(x)$ new matches at any point in time. It also means that a buyer is always in a situation where the same number of relationships form and separate. So if a buyer wants to continuously purchase Y_0 services, sY_0 new matches must be continuously created to replace the matches that have separated. This requires $v_0 = sY_0/q(x)$ vacancies and $\rho v_0 = \rho s Y_0/q(x)$ services spent on filling vacancies. Hence, only $y_0 = Y_0 - \rho s Y_0/q(x)$ services actually provide utility. We can rewrite this relationship as $Y_0 = [1 + \tau(x)]\, y_0$, where

$$\tau(x) \equiv \frac{\rho s}{q(x) - \rho s}. \tag{3}$$

is the wedge between consumption and expenditure. This logic holds for any consumption level

y_0. Thus, if a household or the government desire to consume one service, they need to purchase $1+\tau(x)$ services—one service for consumption plus $\tau(x)$ services for matching. Thus, private consumption is related to private expenditure by $c(t) = C(t)/[1+\tau(x(t))]$ and public consumption to public expenditure by $g(t) = G(t)/[1+\tau(x(t))]$. The matching wedge $\tau(x)$ is positive and increasing for $x \in [0, x^m)$, where $x^m \in (0, +\infty)$ is defined by $q(x^m) = \rho s$ and $\lim_{x\to x^m} \tau(x) = +\infty$, and the elasticity of $\tau(x)$ with respect to x is $(1+\tau(x))\eta$. When tightness is higher, the matching wedge is higher because it is more difficult to fill a vacancy.

We write total consumption as a function of tightness and capacity:

$$y(x,k) = \frac{1-u(x)}{1+\tau(x)}k. \tag{4}$$

This function $y(x,k)$ plays a central role in the analysis because it gives the amount of services that can be consumed for a given tightness. We refer to $y(x,k)$ as the *aggregate supply*. Equation (4) shows that consumption is below the capacity k because some services are not sold ($u(x) > 0$) and some services are used for matching instead of consumption ($\tau(x) > 0$). The function $y(x,k)$ is positive for $x \in [0, x^m)$ and $k > 0$, increasing in k, and with an elasticity with respect to x of $(1-\eta)u(x) - \eta\tau(x)$.

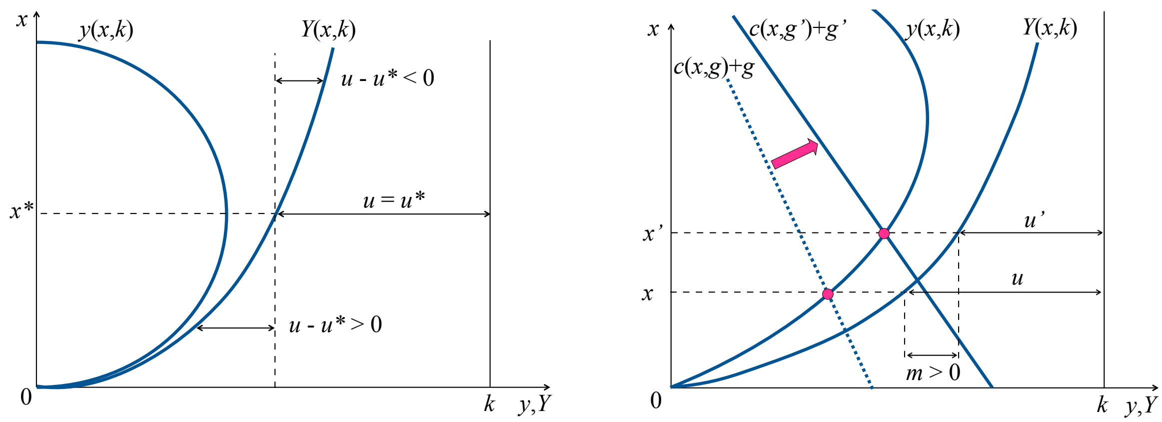
In such a matching model there always is unemployment, and the level of unemployment is generally inefficient—because prices generally fail to maintain productive efficiency (Michaillat and Saez 2015, p. 525, p. 528). The formal definition of efficiency is the following:

DEFINITION 1. *Tightness and unemployment rate are* efficient *if they maximize total consumption for a given productive capacity. The efficient tightness is denoted by x^* and the efficient unemployment rate by u^*. We refer to $u - u^*$ as the* unemployment gap.

We have seen that the elasticity of $y(x,k)$ with respect to x is $(1-\eta)u(x) - \eta\tau(x)$. This elasticity is $1-\eta > 0$ for $x = 0$, strictly decreasing in x, and $-\infty$ at $x = x^m$. Thus, there is a unique x^* where the elasticity is zero. Furthermore, the derivative of $y(x,k)$ with respect to x is positive for $x < x^*$, zero at x^*, and negative for x^*: the tightness x^* maximizes $y(x,k)$ for a given k. Efficient tightness and unemployment are therefore characterized as follows:

LEMMA 1. *The efficient tightness is defined by*

$$(1-\eta)u(x^*) - \eta\tau(x^*) = 0. \tag{5}$$



A. Unemployment gap ($u - u^*$)   B. Unemployment multiplier (m)

Figure 1. Aggregate Demand and Supply, and Sufficient Statistics

The efficient unemployment rate is given by $u^* = u(x^*)$.

An increase in tightness has two opposite effects on consumption: on the one hand, it increases consumption by reducing the amount of unsold services; on the other hand, it decreases consumption by raising the amount of services devoted to matching. When the efficiency condition (5) is satisfied, an increase in tightness reduces unsold services as much as it increases matching services, indicating that consumption is maximized. Hence, determining whether unemployment is inefficiently high or low requires to compare the unemployment rate u to the matching wedge τ (labor devoted to recruiting efforts). For instance, when the unemployment rate is high relative to the matching wedge, such that $u/\tau > \eta/(1-\eta)$, then unemployment is inefficiently high.

Panel A in Figure 1 summarizes the supply side of the model. It depicts how total consumption and output depend on tightness. It also depicts the efficient tightness and unemployment rate, situations in which tightness is inefficiently high and unemployment is inefficiently low, and situations in which tightness is inefficiently low and unemployment is inefficiently high.

2.3. Demand Side and Equilibrium: General Case

We turn to the demand side and equilibrium of the model. Specifying the supply side is necessary to compute social welfare and study optimal policy. But the sufficient-statistic approach makes it unnecessary to specify demand side and equilibrium: we therefore keep them generic and find sufficient statistics that summarize their relevant features. Then we will express the formula for optimal public expenditure with these statistics.

The representative household derives instantaneous utility $\mathcal{U}(c, g)$, where the function $\mathcal{U}$ is strictly increasing in c and g and concave. The marginal rate of substitution between public and private consumption is

$$MRS_{gc} \equiv \frac{\partial \mathcal{U}/\partial g}{\partial \mathcal{U}/\partial c} > 0.$$

We assume that $\mathcal{U}$ is such that MRS_{gc} is a decreasing function of g/c; for example, $\mathcal{U}$ could be a constant-elasticity-of-substitution utility function. For convenience we also assume that $MRS_{gc}(0) > 1$. The elasticity of substitution between public and private consumption measures how the marginal rate of substitution varies with g/c:

DEFINITION 2. *The* elasticity of substitution between public and private consumption*, denoted ϵ, is given by*

$$\frac{1}{\epsilon} = -\frac{d\ln(MRS_{gc})}{d\ln(g/c)}.$$

The elasticity of substitution is positive because MRS_{gc} is decreasing in g/c. When $\epsilon < 1$, public and private services are gross complements. When $\epsilon = 1$, public and private services are independent.[4] And when $\epsilon > 1$, public and private services are gross substitutes.

The elasticity of substitution has two interesting limits. When $\epsilon \to 0$, public and private consumption are perfect complements. This means that a certain number of public services are needed for a given level of private consumption, but beyond that, additional public services have zero value and the marginal rate of substitution falls to zero. At this point, public workers dig and fill holes.[5] When $\epsilon \to +\infty$, the public and private consumption are perfect substitutes. This means that the marginal rate of substitution is constant at 1, such that households are equally happy to consume one private or one public service.[6]

To generate a demand for services, households need to have the choice between spending on services and something else. We assume that households save what they do not spend. They choose how much to spend and save to maximize utility. We assume that the asset used for saving is in fixed supply; thus there are no aggregate state variables, and the equilibrium immediately converges to its steady-state position.

[4] The Cobb-Douglas function $\mathcal{U}(c, g) = c^{1-\gamma} g^{\gamma}$ has $\epsilon = 1$.
[5] The Leontief function $\mathcal{U}(c, g) = \min\{c, g\}$ has $\epsilon = 0$.
[6] The linear function $\mathcal{U}(c, g) = c + g$ has $\epsilon \to +\infty$.

In steady state, the household demands a quantity $c(x, p, g)$ of consumption. This demand depends negatively on tightness x because a higher tightness makes purchasing services more costly. For obvious reasons, the demand depends negatively on the price of services p. Finally, the demand depends on public consumption g because g may affect the marginal utility of private consumption. As we saw, to consume $c(x, p, g)$ services, the household purchases a total of $C(x, p, g) = [1 + \tau(x)]\, c(x, p, g)$ services; the extra $\tau(x)c(x, p, g)$ services are used to fill vacancies.[7]

Besides, the government demands an amount g of consumption. This requires the purchase of $G = [1 + \tau(x]\, g$ services.[8] The government balances its budget at all time with a lump-sum tax $T = G$. The total demand for consumption then is $g + c(x, p, g)$. We refer to $c(x, p, g)$ as *private demand* and $g + c(x, p, g)$ as *aggregate demand*.[9]

Finally we specify a mechanism for the price of services: $p = p(x, g)$. The price of services is a function of tightness x and public consumption g; but since x and g determine all other variables in a feasible allocation, the price could be any function of any variable—it is as generic as possible. The price mechanism generally fails to maintain unemployment at its efficient level; hence, policies correcting prices could be useful to bring unemployment closer to its efficient level.[10] We assume that the price schedule $p(x, g)$ embeds all such price policies. If the price policies ensure that unemployment is always efficient, our analysis trivially applies. We focus on the more interesting situation where price policies are unable to keep unemployment at its efficient level. Then we explore how public expenditure can improve welfare, taking all price policies as given.

Given the price mechanism and public expenditure, tightness adjusts to equalize aggregate supply and demand:

$$y(x, k) = c(x, p(x, g), g) + g. \tag{6}$$

This equation implicitly defines equilibrium tightness as a function $x(g)$ of public consumption. Panel B in Figure 1 illustrates how equilibrium tightness $x(g)$ is given by the intersection of the aggregate-demand and aggregate-supply curves.

For the welfare analysis, the relevant information about the tightness function $x(g)$ is summa-

[7]To purchase $C(x, p, g)$ services, households post $sC(x, p, g)/q(x)$ vacancies.

[8]To purchase G services, the government posts $sG/q(x)$ vacancies.

[9]We express demand in terms of consumption because consumption matters for welfare and aggregate supply (4) is expressed with consumption. But we could equivalently describe demand in terms of expenditure.

[10]In some contexts, monetary policy could be such a policy. See Appendix B.

rized by the following sufficient statistic:

DEFINITION 3. *The* unemployment multiplier *is given by*

$$m = -y \cdot \frac{du}{dg}.$$

The unemployment multiplier measures the decrease of the unemployment rate, measured in percentage points, when public consumption increases by 1 percent of total consumption.

As unemployment is determined by tightness (through (2)), the unemployment multiplier is determined by the response of tightness to public consumption. As showed in Figure 1, Panel B, public consumption affects tightness by shifting the aggregate-demand curve. This shift occurs through a mechanical channel, as public consumption directly contributes to aggregate demand; a private-demand channel, as public consumption may affect private demand in various ways (for instance, by altering the marginal utility from private consumption); and a price channel, as public consumption may affect the price of services and thus private demand. Through these channels, the multiplier can take a broad range of values: negative, positive, below 1, or above 1.

2.4. Demand Side and Equilibrium: An Example with Land

To provide an example of demand side, we describe a model in which households save using land, as in Iacoviello (2005) and Liu, Wang, and Zha (2013). Appendix A contains the derivations and Appendix B provides other examples. This example illustrates the type of models covered by the analysis, and shows how demand-side parameters influence the values of the sufficient statistics.

The representative household purchases a quantity $l(t)$ of land. Land is traded on a perfectly competitive market and is in fixed supply, l_0. In equilibrium, the land market clears so $l(t) = l_0$.

The household derives utility from holding land, for instance from the housing services provided by land. The instantaneous utility function is $\mathcal{U}(c(t), g(t)) + \mathcal{V}(l(t))$, where $\mathcal{V}$ is strictly increasing and concave. We use a constant-elasticity-of-substitution specification for $\mathcal{U}$:

$$\mathcal{U}(c, g) = \left[(1-\gamma)^{\frac{1}{\epsilon}} c^{\frac{\epsilon-1}{\epsilon}} + \gamma^{\frac{1}{\epsilon}} g^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}.$$

The parameter $\gamma \in (0, 1)$ indicates the value of public services relative to private services, and the parameter $\epsilon > 0$ gives the elasticity of substitution between public and private consumption. The household's utility at time 0 is

$$\int_0^{+\infty} e^{-\delta t} \left[\mathcal{U}\left(c(t), g(t)\right) + \mathcal{V}(l(t))\right] dt, \tag{7}$$

where $\delta > 0$ is the subjective discount rate. The law of motion of the household's land holding is

$$\dot{l}(t) = p(t) \left[1 - u(x(t))\right] k - p(t) \left[1 + \tau(x(t))\right] c(t) - T(t). \tag{8}$$

In the law of motion, $p(t) \left[1 - u(x(t))\right] k$ is the household's labor income, $p(t) \left[1 + \tau(x(t))\right] c(t)$ is its spending on services, and $T(t)$ is the lump-sum tax used to finance public expenditure.

The household takes $l(0)$ and the paths of $x(t)$, $g(t)$, $p(t)$, and $T(t)$ as given. It chooses the paths of $c(t)$ and $l(t)$ to maximize (7) subject to (8). To solve this maximization problem, we set up the usual Hamiltonian and obtain the following first-order conditions:

$$\frac{\partial \mathcal{U}}{\partial c}(c(t), g(t)) = \lambda(t)p(t) \left[1 + \tau(x(t))\right], \tag{9}$$

$$\mathcal{V}'(l(t)) = \delta\lambda(t) - \dot{\lambda}(t), \tag{10}$$

where $\lambda(t)$ is the costate variable associated with land.

For a given public consumption g, an equilibrium consists of paths for $[x(t),\ c(t),\ l(t),\ p(t),\ \lambda(t)]_{t=0}^{\infty}$ that satisfy supply = demand on the services market: $y(x(t)) = c(t) + g$; supply = demand on the land market: $l_0 = l(t)$; price mechanism on the services market: $p(t) = p(x(t), g)$; and the first-order conditions (9) and (10) of the household's problem. Since all the variables can be recovered from the costate variable $\lambda(t)$, the equilibrium boils down to a dynamical system of dimension 1 with variable $\lambda(t)$. As $\lambda(t)$ is nonpredetermined and the dynamical system is a source, the equilibrium jumps to its steady-state position from any initial condition. As the economy is always in steady state, the welfare associated with an equilibrium simply is $\mathcal{U}(c, g) + \mathcal{V}(l_0)$.

To describe the steady state, we compute the private demand, $c(x, p, g)$. First, we combine (9) and (10):

$$\frac{\partial \mathcal{U}}{\partial c}(c, g) = (1 + \tau(x))p\frac{\mathcal{V}'(l_0)}{\delta}.$$

The equation says that at the optimum the household is indifferent between purchasing one private service, which costs $(1 + \tau(x))p$ units of land and yields utility $\partial\mathcal{U}/\partial c$, and purchasing $(1 + \tau(x))p$

units of land, which costs the same amount and yields utility $\mathcal{V}'(l_0)/\delta$ over a lifetime. Using the expression for $\partial\mathcal{U}/\partial c$, we find that the private demand is implicitly defined by

$$\left[(1-\gamma)+\gamma^{\frac{1}{\epsilon}}(1-\gamma)^{\frac{\epsilon-1}{\epsilon}}\left(\frac{g}{c}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}=(1+\tau(x))p\frac{\mathcal{V}'(l_0)}{\delta}. \tag{11}$$

Examples of aggregate-demand shocks are shocks to the marginal utility of land, $\mathcal{V}'(l_0)$, or to the time discount rate, δ. With a higher marginal utility of land or lower time discount rate, households desire to save more and consume less, which depresses private and aggregate demands. With a rigid price, such a negative shock leads to lower tightness and higher unemployment.

To complete the description of the steady state, we specify the price of services. We choose a price mechanism that allows for price rigidity (required to obtain unemployment fluctuations) and yields a simple expression for the multiplier:

$$p(g)=p_0\left[(1-\gamma)+\gamma^{\frac{1}{\epsilon}}\left((1-\gamma)\frac{g}{y^*-g}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1-r}{\epsilon-1}}. \tag{12}$$

The parameter $p_0 > 0$ governs the price level. The parameter r determines the effect of public consumption on prices: if $r < 1$, the price is increasing in g; if $r = 1$, the price is fixed; and if $r > 1$, the price is decreasing in g (this seems less realistic).

The parameter r is a key determinant of the unemployment multiplier:[11]

$$m=\frac{(1-u^*)r}{(1-\gamma)\epsilon}. \tag{13}$$

The multiplier is usually positive, except if $r < 0$.[12] Besides, the multiplier depends on ϵ and γ, because these parameters affect the shape of the aggregate-demand curve. In particular, the multiplier is larger when public and private services are stronger complements (lower ϵ), smaller when they are stronger substitutes (higher ϵ), and zero when goods are perfect substitutes ($\epsilon \to \infty$). What happens when public and private services are stronger substitutes is that an increase in public consumption reduces the demand for private consumption more, because public consumption

[11] This expression is valid when unemployment is efficient and public expenditure optimal; otherwise the multiplier admits another expression, slightly more complicated but with the same properties.

[12] In this case $r < 0$, an increase in public consumption raises the price of services so much that it reduces the demand for private consumption more than one-for-one.

replaces private consumption more easily.

2.5. Comparison with the Diamond-Mortensen-Pissarides (DMP) Model

Our matching model shares many features with the standard matching model—the DMP model. Such features include the matching function, random search, long-term relationships, hiring through vacancies, a fixed productive capacity, and the central role of market tightness. But it also differs from the DMP model on several aspects. Here we present these differences and explain how they make the model more suited to the analysis of optimal public expenditure. We use as reference the textbook version of the DMP model, developed by Pissarides (2000).

We begin with the conceptual differences. The first conceptual difference is that our model is more general than the DMP model. The generalizations make the model more suited to the sufficient-statistic approach. First, the price mechanism is general and not restricted to Nash bargaining. This generalization allows for a broader range of multipliers and unemployment gaps. Second, we introduce functional forms that are more general than in the DMP model, thus obtaining a downward-sloping demand curve in the (y, x) plan. Because changes in public expenditure shift the demand curve, they affect tightness, and crowding-out of private consumption by public consumption is generally not one-for-one. In contrast, in the DMP model, changes in public expenditure do not shift the demand curve because that curve is horizontal in the (y, x) plan. Therefore public consumption crowds out private consumption one-for-one (Michaillat 2014).

The second conceptual difference rests in our formulation of the efficiency condition. In the DMP model, the Hosios (1990) condition gives workers' bargaining power such that the bargained wage yields the efficient rate of unemployment. Our efficiency condition (5) is mathematically equivalent to the Hosios condition, but it is more general because it is not tied to Nash bargaining: it applies to any price mechanism. Instead of describing the bargaining power leading to efficient unemployment, our condition describes how observable variables (unemployment and matching wedge) are related when unemployment is efficient.

Next, there are several cosmetic differences that make our matching model closer to the Walrasian model—the workhorse model in public economics. These changes make it easier to use public-economic tools and compare our results to standard public-economic results.

First, we model a service economy instead of a labor market. This means that the traded goods are services instead of labor, the price of the goods is the price of services instead of the real wage, the buyers are households (and the government) instead of firms, and the sellers are self-employed workers instead of unemployed workers.

Second, the Beveridge curve is recast as an aggregate-supply curve and the job-creation condition as an aggregate-demand curve.[13] The aggregate-supply curve is mathematically equivalent to the Beveridge curve, and the aggregate-demand curve to the job-creation condition, but our curves are closer to the Walrasian supply and demand concepts.

Third, the condition determining equilibrium tightness is recast as a supply = demand condition. In fact, it is useful to think of tightness as another price: both actual price and tightness ensure that supply = demand (Michaillat and Saez 2015, pp. 526–529). As both price and tightness are determined in equilibrium, the matching framework can be seen as a generalization of the Walrasian framework—where only the price is determined in equilibrium.[14] This generalization implies that equilibria in the matching model may be inefficient; unlike in the Walrasian equilibrium, where productive efficiency is respected whenever supply equals demand. Because it allows for inefficiency, the matching model is useful to think about inefficient unemployment. Since we use the supply-demand formalism, the graphical representation of the equilibrium is different. In the DMP model, the equilibrium is the intersection of the Beveridge and job-creation curves in an (unemployment, vacancy) plan. In our model, the equilibrium is the intersection of the aggregate-supply and aggregate-demand curves in a (output, tightness) plan.[15]

Fourth, the recruiting cost takes a different form. In the DMP model, the vacancy-posting cost is measured in terms of final good, so there are effectively two goods in the model: labor and final good. This complicates the welfare analysis. Here the cost is measured in terms of services (the traded good), so there is a single good in the model. This simplifies the welfare analysis, because once consumption is defined as output net of recruiting services, welfare solely depends on consumption.

Fifth, while the DMP model focuses on atomistic workers and jobs, our model studies households selling and buying many services. This brings the model closer to the Walrasian framework, in which agents buy and sell many goods. Furthermore, since households buy and sell many services,

[13]In Pissarides (2000), the Beveridge curve is equation (1.5) and the job-creation condition is equation (1.9). In this paper, the aggregate-supply curve is (4) and in the example with land the aggregate-demand curve is (11).

[14]In the Walrasian framework, the implicit tightness is fixed so sellers always sell anything they bring to the market.

[15]In Pissarides (2000), the equilibrium is depicted in Figure 1.2. Here, the equilibrium is depicted in Figure 1.

we can avoid heterogeneity across households, which would complicate the welfare analysis with insurance problems (as an atomistic worker can only be employed or unemployed).

3. A Sufficient-Statistic Formula for Optimal Public Expenditure

We derive a sufficient-statistic formula for optimal public expenditure and explore its theoretical implications. The main implication is that whenever unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to reduce the unemployment gap.

3.1. Derivation

We obtain the formula by finding the public consumption g that maximizes welfare $\mathcal{U}(c, g)$. In equilibrium, $c = y(x, k) - g$ and $x = x(g)$. Thus, the optimal g maximizes $\mathcal{U}\left(y(x(g), k) - g, g\right)$. The first-order condition of the maximization is

$$0 = \frac{\partial \mathcal{U}}{\partial g} - \frac{\partial \mathcal{U}}{\partial c} + \frac{\partial \mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}. \tag{14}$$

We assume that the maximization problem is well-behaved: $g \mapsto \mathcal{U}(y(x(g), k) - g, g)$ admits a unique extremum, and the extremum is an interior maximum. Under this assumption, (14) is a necessary and sufficient condition for optimality.

Equation (14) shows that an increase in public expenditure affects welfare through three channels: it raises public consumption (first term in the right-hand side); for a given level of total consumption, it reduces private consumption one-for-one (second term in the right-hand side); and it affects the level of total consumption and thus private consumption (third term in the right-hand side). Dividing equation (14) by $\partial \mathcal{U}/\partial c$, we obtain the following lemma:

LEMMA 2. *Optimal public expenditure satisfies*

$$\underbrace{1 = MRS_{gc}}_{\textit{Samuelson rule}} + \underbrace{\frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\textit{correction}}. \tag{15}$$

Equation (15) shows that in a matching model the Samuelson rule needs to be corrected. The correction term is the product of the effect of public consumption on tightness, dx/dg, and the effect of tightness on total consumption, $\partial y/\partial x$. That is, the correction term measures the effect of public consumption on total consumption, dy/dg; it is positive whenever an increase in public consumption leads to an increase in total consumption.[16] Hence, an insight from (15) is that at the optimum, public consumption must be crowding out private consumption ($dc/dg < 0$).[17] Our theory allows for either crowding in or crowding out of private consumption by public consumption, but if there is crowding in ($dc/dg > 0$), the government should increase public consumption until it starts crowding out private consumption. Crowding out necessarily happens at some point because once unemployment is efficient, total consumption is maximized and crowding out is one-for-one.

When the rate of unemployment is efficient, consumption is maximized ($\partial y/\partial x = 0$) and the correction term is zero. Thus the Samuelson rule, which was originally derived in a neoclassical model, remains valid in a model with unemployment as long as unemployment is efficient. When the rate of unemployment is inefficient, consumption is below its maximum ($\partial y/\partial x \neq 0$), and the correction term may not be zero. To describe deviations from the Samuelson rule, we decompose public spending into two components:

DEFINITION 4. Samuelson spending $(g/c)^*$ *is given by the Samuelson rule:* $MRS_{gc}((g/c)^*) = 1$. Stimulus spending *is given by* $g/c - (g/c)^*$.

Since $MRS_{gc}(0) > 1$ and MRS_{gc} is decreasing in g/c, Samuelson spending is always well-defined. Next, we express the elements of (15) with our three key sufficient statistics: the elasticity of substitution between public and private consumption ϵ, the unemployment gap $u - u^*$, and the unemployment multiplier m.

LEMMA 3. *The term* $1 - MRS_{gc}$ *can be approximated as follows:*

$$1 - MRS_{gc} \approx \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*}, \tag{16}$$

where ϵ *is evaluated at* g/c. *The approximation is valid up to a remainder that is* $O\left([g/c - (g/c)^*]^2\right)$.

[16]Formula (15) is closely related to the optimal unemployment insurance formula in Landais, Michaillat, and Saez (2016, equation (23)): the two formulas show that in matching models standard optimal-policy formulas needs to be corrected with a term that is positive whenever the policy improves welfare through tightness.

[17]Formally: since $MRS_{gc} > 0$, equation (15) implies that $dy/dg < 1$ and thus $dc/dg = dy/dg - 1 < 0$.

The term $\partial y / \partial x$ can be approximated as follows:

$$\frac{x}{y} \cdot \frac{\partial y}{\partial x} \approx \frac{u - u^*}{1 - u^*}. \tag{17}$$

The approximation is valid up to a remainder that is $O\left([u - u^*]^2\right)$. Last, the term dx/dg satisfies

$$\frac{y}{x} \cdot \frac{dx}{dg} = \frac{m}{(1-\eta)(1-u)u}. \tag{18}$$

Equation (16) immediately follows from the definition of the elasticity of substitution ϵ, and equation (18) from the definition of the unemployment multiplier m. But the derivation of equation (17) is more complex, so the proof of the lemma is relegated to Appendix C.

Using Lemma 3, we prove in Appendix C that (15) can be rewritten as follows:

LEMMA 4. *Optimal stimulus spending satisfies*

$$\frac{g/c - (g/c)^*}{(g/c)^*} \approx z_0 \cdot \epsilon \cdot m \cdot \frac{u - u^*}{u^*}, \tag{19}$$

where ϵ and m are evaluated at $[g/c, u]$ and $z_0 = 1 / \left[(1-\eta)(1-u^*)^2\right]$. The approximation is valid up to a remainder that is $O\left([u - u^*]^2 + [g/c - (g/c)^*]^2\right)$.

Formula (19) characterizes optimal public expenditure in terms of estimable statistics. But the formula is implicit because the statistics in the right-hand side of the formula (especially u) are endogenous to the policy. Hence, we cannot plug the current values of the statistics in the right-hand side of the formula to compute optimal stimulus spending. We cannot use the formula to perform comparative statics either. This is a typical limitation of the sufficient-statistic approach (Chetty 2009). To address this limitation, we develop a new sufficient-statistic formula.

We assume that public expenditure is at the Samuelson level $(g/c)^*$ and unemployment is at an inefficient rate $u_0 \neq u^*$. We have in mind the following scenario. Initially everything is going well: unemployment is efficient and accordingly public expenditure satisfies the Samuelson rule. Then a shock occurs that pushes unemployment away from its efficient level. The shock could be anything: aggregate-demand shock, aggregate-supply shock, shock to the price of services, shock to the matching function, or shock to the separation rate. Given the initial unemployment gap $u_0 - u^*$, we determine optimal stimulus spending $g/c - (g/c)^*$. As g/c deviates from $(g/c)^*$, unemployment responds, so we cannot directly plug $u_0 - u^*$ into (19). But, taking the response of unemployment to g/c into account, we can transform (19) into an explicit formula, expressed in terms of sufficient statistics independent of the policy.

PROPOSITION 1. *Suppose that the economy is initially at an equilibrium* $[(g/c)^*, u_0]$. *Then optimal stimulus spending satisfies*

$$\frac{g/c - (g/c)^*}{(g/c)^*} \approx \frac{z_0 \cdot \epsilon \cdot m}{1 + z_1 z_0 \epsilon m^2} \cdot \frac{u_0 - u^*}{u^*}, \tag{20}$$

where ϵ *and* m *are evaluated at* $[(g/c)^*, u_0]$, $z_0 = 1/\left[(1-\eta)(1-u^*)^2\right]$, *and* $z_1 = (g/y)^*(c/y)^*/u^*$. *Under the optimal policy, the unemployment rate is*

$$u \approx u^* + \frac{1}{1 + z_1 z_0 \epsilon m^2} \cdot (u_0 - u^*)\,. \tag{21}$$

The approximations (20) *and* (21) *are valid up to a remainder that is* $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$.

The formal proof, presented in Appendix C, builds on a simple argument: since the unemployment multiplier m is proportional to du/dg, a first-order Taylor expansion of u at u_0 yields

$$u \approx u_0 - \text{constant} \cdot m \cdot \frac{g/c - (g/c)^*}{(g/c)^*}.$$

Substituting u by this expression in (19) leads to (20).

Formula (20) links optimal stimulus spending $g/c - (g/c)^*$ to three main sufficient statistics: the elasticity of substitution between public and private consumption (ϵ), the unemployment multiplier (m), and the unemployment gap ($u_0 - u^*$). Furthermore, the formula involves Samuelson spending ($(g/c)^*$), the efficient unemployment rate (u^*), and constants $z_0 > 0$ and $z_1 > 0$, which depend on $(g/c)^*$, u^*, and the matching elasticity η. Formula (21) links the unemployment rate under optimal public expenditure to the same statistics. The advantage of (20) over (19) is that the statistics in the right-hand side are not endogenous to the policy. Thus, the formula allows us to compute optimal stimulus spending using the current values of the statistics.

The policy debate on stimulus spending often revolves around the sizes of public-spending multipliers and output or unemployment gaps (for example, Romer and Bernstein 2009). Formula (20) confirms that optimal stimulus spending can be expressed as a function of a public-spending multiplier—the unemployment multiplier—and the unemployment gap. Yet these statistics are not sufficient to measure the effect of public expenditure on welfare because an increase in public expenditure also modifies the composition of households' consumption. Consequently, optimal stimulus spending also depends on the elasticity of substitution between public and private consumption.

While policy discussions focus on the output multiplier, formula (20) is based on the unemployment multiplier. Yet, unemployment and output multipliers are closely related, so we can translate the results in terms of output multiplier. We start by introducing a new unemployment multiplier:

DEFINITION 5. *The* empirical unemployment multiplier *is given by*

$$M = -\frac{Y}{1-u} \cdot \frac{du}{dG}.$$

The empirical unemployment multiplier measures the percent increase of the employment rate, $1-u$, when public expenditure increases by 1 percent of GDP. In practice $1-u \approx 1$ so the multiplier approximately measures the decrease of the unemployment rate when public expenditure increases by 1 percent of GDP.

This multiplier plays an important role in the empirical applications of Sections 4 and 5. Indeed the theoretical unemployment multiplier m, which enters our formulas, is difficult to estimate because it measures the response of unemployment to a change in public consumption, which is not directly observable. The issues is that public expenditure on matching resources—which are unobserved—must be subtracted from total public expenditure to obtain public consumption. The empirical unemployment multiplier M is much easier to estimate because it measures the response of unemployment to a change in public expenditure, which is reported in national accounts. As m and M are closely related, we will use empirical estimates of M to calibrate m in the formulas.[18]

Additionally, the empirical unemployment multiplier acts as a bridge between the theoretical unemployment multiplier in our formulas and the output multiplier:

LEMMA 5. *Theoretical and empirical unemployment multipliers are related by*

$$m = \frac{(1-u) \cdot M}{1 - \frac{G}{Y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot M}. \tag{22}$$

[18] Although public consumption g and private consumption c are not observable, the consumption ratio g/c can be measured from national accounts, because $g/c = G/C$ and both public expenditure G and private expenditure C are observable in national accounts. Thus, our formula is usable once m is replaced by M.

Furthermore, empirical unemployment multiplier and output multiplier are equal: $M = dY/dG$.

The proof of the lemma is relegated to Appendix C. However, it is easy to see why empirical unemployment multiplier and output multiplier are directly related: because public expenditure must necessarily reduce the unemployment rate to raise output.

The lemma implies that m and M always have the same sign, and m is increasing in M. Moreover, the lemma shows that M and the output multiplier are equal. Thus, our formula can be expressed with the output multiplier by replacing m by the function of M given by (22) and then replacing M by dY/dG. After this manipulation, it appears that the output multiplier affects optimal stimulus spending in the same way as the unemployment multiplier m.

A caveat is that the output multiplier is only useful when taxation is nondistortionary. Section 3.3 shows that when taxation is distortionary, the link between unemployment and output multipliers breaks down, and it is necessary to use the unemployment multiplier to design stimulus spending.

3.2. Implications

Using our sufficient-statistic formula (20), we explore how the sign and amplitude of optimal stimulus spending depend on the unemployment gap, unemployment multiplier, and elasticity of substitution between public and private consumption. We also use (21) to describe the properties of the unemployment gap under optimal stimulus spending.

Sign of Optimal Stimulus Spending. Formula (20) gives the sign of optimal stimulus spending in various situations:

PROPOSITION 2. *If the unemployment multiplier is zero ($m = 0$), or the unemployment gap is zero ($u_0 = u^*$), optimal stimulus spending is zero ($g/c = (g/c)^*$). In all other situations, optimal public expenditure deviates from the Samuelson rule to partially fill the initial unemployment gap. Consider first a positive unemployment multiplier ($m > 0$). If the unemployment gap is positive ($u_0 > u^*$), optimal stimulus spending is positive ($g/c > (g/c)^*$) but does not completely fill the unemployment gap ($u > u^*$). If the unemployment gap is negative ($u_0 < u^*$), optimal stimulus spending is negative ($g/c < (g/c)^*$) but does not completely eliminate the unemployment gap ($u < u^*$). If the unemployment multiplier is negative ($m < 0$), the sign of optimal stimulus spending is the opposite.*

It is only if the unemployment multiplier is always zero or the unemployment gap is always zero that public expenditure should always be at the Samuelson level. In all other situations, optimal public expenditure deviates from the Samuelson rule.

The general pattern is that public expenditure should deviate from the Samuelson level to partially fill the initial unemployment gap. To understand these results, imagine that public expenditure is at the Samuelson level, the unemployment multiplier is positive, and unemployment is inefficiently high. Keeping total consumption constant, increasing public consumption by 1 service reduces private consumption by 1 service. So far, since the marginal utilities of public and private consumption are equalized at the Samuelson rule, the increase in public expenditure has no first-order effect on welfare. Yet, since the unemployment multiplier is positive, increasing public consumption lowers unemployment; and since unemployment is inefficiently high, reducing unemployment raises total consumption. Hence, after accounting for the effect of public expenditure on unemployment, the increase in public expenditure raises welfare. Thus, it is optimal to increase public expenditure above the Samuelson level, which reduces the unemployment gap.

Why is it not optimal to completely fill the unemployment gap? If the government did that, we would reach a situation where one unit of public consumption costs one unit of private consumption (since crowding out is one-for-one when the unemployment gap is zero) but is less valuable than one unit of private consumption (since public spending is above Samuelson spending). This situation is suboptimal: welfare can be increased by reducing public consumption.

These results have implications for the optimal cyclicality of public expenditure. Under the presumptions that the unemployment gap is positive in slumps but negative in booms, and that the unemployment multiplier is nonzero with a constant sign, then optimal stimulus spending changes sign over the business cycle. Thus optimal public expenditure fluctuates over the business cycle around the Samuelson level.

Role of the Unemployment Multiplier. We determine how optimal stimulus spending depends on the unemployment multiplier. From (20) and (21), we obtain the following proposition:

PROPOSITION 3. *Assume that the initial unemployment gap is positive ($u_0 > u^*$). Let $m^\dagger = 1/\sqrt{z_1 z_0 \epsilon}$. Then optimal stimulus spending is a hump-shaped function of the unemployment multiplier: it is 0 when $m = 0$, increasing in m for $m \in [0, m^\dagger]$, maximized at $m = m^\dagger$, decreasing in m for $m \in [m^\dagger, +\infty)$, and 0 for $m \to +\infty$. The maximum optimal stimulus spending (reached at*

$m = m^\dagger$) is

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1}{2} \cdot \sqrt{\frac{z_0 \epsilon}{z_1}} \cdot \frac{u_0 - u^*}{u^*}.$$

Furthermore, the unemployment gap under optimal stimulus spending is a decreasing function of the unemployment multiplier: it falls from $u_0 - u^*$ when $m = 0$ to 0 when $m \to +\infty$.

The proposition shows that for a given unemployment gap, optimal stimulus spending is a hump-shaped function of the unemployment multiplier: it is increasing in the multiplier until the multiplier reaches $1 \big/ \sqrt{z_1 z_0 \epsilon}$, and decreasing after that. The proposition also gives the maximum optimal stimulus spending. For concreteness this proposition and the next only consider positive unemployment multipliers and unemployment gaps, but we could of course derive corresponding results with negative multipliers and gaps.

What is the intuition behind the hump shape? When public expenditure is optimal, the marginal social cost from consuming too many public services and too few private services equals the marginal social value from reducing unemployment. This marginal social value is determined by two factors: the current unemployment multiplier, which measures how much unemployment can be reduced by additional expenditure, and the current unemployment gap, which measures the social value from lower unemployment. For a given amount of stimulus spending and a given initial unemployment gap, a larger initial multiplier has conflicting effects on the two factors: it means a larger current multiplier (a higher marginal social value) but a smaller current unemployment gap (a lower marginal social value). The first effect advocates for more spending but the second for less spending. It turns out that the first effect dominates for small multipliers, so optimal stimulus spending is increasing in the multiplier; but the second effect dominates for large multipliers, so optimal stimulus spending is decreasing in the multiplier. In fact, for large multipliers, it becomes optimal to nearly entirely fill the unemployment gap; since less spending is required to fill the gap when the multiplier is larger, optimal stimulus spending is decreasing in the multiplier.

Our results qualify the view that larger multipliers entail larger stimulus spending—the "bang-for-the-buck" logic often used in policy discussions (see Mankiw and Weinzierl 2011, p. 212). Stimulus skeptics usually believe in small multipliers and infer that stimulus spending should be small or zero in slumps; similarly, stimulus advocates usually believe in large multipliers and infer that stimulus spending should be large in slumps. Our theory shows that the bang-for-the-buck logic does hold for small multipliers, but not for large ones, such that large multipliers may not justify a large stimulus. Instead, the relationship between unemployment multiplier and optimal stimulus spending is hump-shaped, such that optimal stimulus spending may be quite similar for some small and large multipliers.

Role of the Elasticity of Substitution Between Public and Private Consumption. We determine how optimal stimulus spending depends on the elasticity of substitution between public and private consumption. From (20), we immediately obtain the following results:

PROPOSITION 4. *Assume that the unemployment multiplier and initial unemployment gap are positive ($m > 0$ and $u_0 - u^* > 0$). Then optimal stimulus spending is an increasing function of the elasticity of substitution between public and private consumption: it rises from 0 when $\epsilon = 0$ to*

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1}{z_1 m} \cdot \frac{u_0 - u^*}{u^*}$$

when $\epsilon \to +\infty$. The unemployment gap under the optimal policy is a decreasing function of the elasticity of substitution: it falls from $u_0 - u^*$ when $\epsilon \to 0$ to 0 when $\epsilon \to +\infty$.

The proposition shows that both optimal stimulus spending and the share of the unemployment gap filled under the optimal policy are increasing in the elasticity of substitution between public and private consumption with two interesting polar cases.

The first is $\epsilon \to 0$. In this case, additional public services have zero value: additional public workers "dig and fill holes in the ground", and optimal stimulus spending is zero, irrespective of the unemployment rate and multiplier. Intuitively, public consumption beyond the Samuelson level is useless. Since public consumption crowds out private consumption in the vicinity of u^*, it is never optimal to provide more public consumption than in the Samuelson rule.[19]

The second special case is $\epsilon \to +\infty$. Then, the public services provided by the government perfectly substitute for the private services purchased by households. In this case, optimal stimulus

[19]The results in Proposition 4 are based on (20), which is a first-order approximation around $[u^*, (g/c)^*]$. When u_0 is relatively far from u^* and g/c from $(g/c)^*$, the equation may not be accurate and results may change. In particular, when $\epsilon \to 0$ the first-order approximation of MRS_{gc} works well only very close to $(g/c)^*$, so (20) works well only if the optimal g/c is close to $(g/c)^*$. However, when public consumption crowds in private consumption ($dc/dg > 0$), the optimal g/c is not close to $(g/c)^*$ and (20) does not work well. (Note that having $dc/dg > 0$ requires a large deviation from u^* because at u^* total consumption is maximized so $dc/dg = -1$.) Equation (20) suggests that stimulus spending should be zero in this situation. But going back to (15)—which can be written $0 = MRS_{gc} + dc/dg$—we see that optimal stimulus spending is positive when $dc/dg > 0$, even if $\epsilon \to 0$. Indeed then $MRS_{gc}(g/c) = 0$ as soon as $g/c > g/c^*$; nevertheless, since $dc/dg > 0$ at $(g/c)^*$, it is optimal to increase spending above $(g/c)^*$ and continue spending until dc/dg falls to 0, which necessarily occurs when unemployment is close enough to u^*.

spending completely fills the unemployment gap such that $u = u^*$. This result holds even if the multiplier is very small and public expenditure severely crowds out private consumption. Intuitively, public and private consumption are interchangeable, so it is optimal to maximize total consumption, irrespective of its composition. Accordingly, it is optimal to completely fill the unemployment gap.

Overall, Proposition 4 clarifies the link between usefulness of public expenditure and optimal stimulus spending. A concern of stimulus skeptics is that additional public expenditure could be wasteful. Our theory develops this argument. It is true that when the elasticity of substitution between public and private consumption is zero, public expenditure should remain at the Samuelson level. But in the more realistic case where the elasticity of substitution is positive, some stimulus spending is indeed desirable in slumps.

3.3. Distortionary Taxation

So far labor supply has been fixed and taxation nondistortionary. We now introduce endogenous labor supply: households supply a productive capacity k at a utility cost $\mathcal{W}(k)$, where the function $\mathcal{W}$ is strictly increasing and convex. We examine how distortionary taxation affects optimal public expenditure. Here we present a summary; complete results and derivations are in Appendix D.

We begin by considering the traditional approach to taxation, which consists in using a linear income tax τ^L to finance public expenditure. With the linear income tax, the household's labor income becomes $(1-\tau^L)Y(x,k) = (1-\tau^L)(1-u(x))k$. To finance public expenditure G, the tax rate must be $\tau^L = G/Y = g/y$.

The household chooses k to maximize utility. The labor supply decision is distorted by the income tax: a higher tax reduces the returns to supplying labor and thus reduces the capacity k supplied by the household through substitution effects. Because of this distortion, the formula for optimal public expenditure is modified as follows:

$$\underbrace{1 - \frac{d\ln(k)}{d\ln(g)} = MRS_{gc}}_{\text{modified Samuelson rule}} + \underbrace{\frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}.$$

This optimality condition differs from (15), but the two have the same structure once the Samuelson rule is modified to account for distortionary taxation.[20] The statistic $1 - d\ln(k)/d\ln(g) > 1$ is the

[20]A modified Samuelson rule was developed by Stiglitz and Dasgupta (1971), Diamond and Mirrlees (1971), and

marginal cost of funds. It is more than one because the linear income tax distorts the labor supply. Because the marginal cost of funds is greater than one, the modified Samuelson rule recommends lower public expenditure than the regular Samuelson rule.

Although Samuelson spending is lower with a linear income tax, the correction to the Samuelson rule remains the same, and our sufficient-statistic formula spending remains the same. If the economy is initially at an equilibrium $[(g/c)^*, u_0]$, then with a linear income tax, optimal stimulus spending satisfies (20) and the unemployment rate under optimal stimulus spending satisfies (21), where the statistic z_1 is generalized to allow for supply-side responses.

Section 3.1 shows that when taxes are nondistortionary, unemployment and output multipliers are equal, so all our results can be reformulated with the output multiplier. Things are different when taxes are distortionary. Higher taxes reduce labor supply, which reduces output but not unemployment. The output multiplier becomes smaller than the unemployment multiplier. Therefore the output multiplier cannot be used to design optimal public expenditure—only the unemployment multiplier can be used.

Stimulus skeptics are concerned that output is already too low in slumps and that increasing taxes to fund stimulus spending would further reduce output through supply-side responses.[21] Nonetheless, our theory shows that as long as the unemployment multiplier is positive, stimulus spending should be positive in slumps—even if the output multiplier is negative. How can such a policy be optimal? Starting from the modified Samuelson rule, a small increase in public expenditure reduces unemployment, reduces labor supply, and increases public consumption, which are all good for welfare; but it reduces output and thus private consumption, which is bad for welfare. At the modified Samuelson rule, the cost of lower private consumption offsets the benefits of higher public consumption and lower labor supply; the only remaining effect on welfare is the positive effect from lower unemployment. Therefore increasing public expenditure above the modified Samuelson rule raises welfare, so it is indeed desirable.[22]

Atkinson and Stern (1974) to describe optimal public expenditure with a linear income tax. A large literature has built on these papers (see Ballard and Fullerton 1992; Kreiner and Verdelin 2012).

[21]Barro and Redlick (2011) find in US data that the deficit-financed output multiplier is positive (around 0.5) but because taxation significantly depresses output, the balanced-budget output multiplier is negative (around −0.6).

[22]Here we have considered the standard approach of funding public goods with distortionary taxation. In Appendix D, we also consider the "modern approach" that follows the benefit principle. The benefit principle, which was introduced by Hylland and Zeckhauser (1979) and fully developed by Kaplow (1996, 1998), is an important result in modern public-economic theory: it states that the optimal provision of public expenditure can be disconnected from distortionary taxation. Extra public expenditure is financed by a tax change leaving all individual utilities unchanged and thus not altering further labor supply. This is done by changing the nonlinear income tax schedule to absorb the

3.4. Comparison with a Keynesian, Fixprice Model

We have showed that the Samuelson rule breaks down when productive efficiency fails, which happens when the price of services is not at the efficient level. We treat the failure of productive efficiency using a matching model, but in macroeconomics failures of productive efficiency are usually studied using models in which prices are fixed at some inefficient level, in the tradition of Barro and Grossman (1971) and Bénassy (1993).[23] Here we apply our methodology to such a fixprice model. We discuss advantages and disadvantages of each approach.

The economy has the same structure as in the model of Section 2, except that services are traded on a perfectly competitive market instead of a matching market. The price of services is fixed at a level p, which may not be the market-clearing level. The private demand for services is given by a function $c(p, g)$, with $\partial c/\partial p < 0$. The aggregate demand for services is $y(p, g) = c(p, g) + g$. The aggregate supply of services is fixed at k. Since there is no wedge between output and consumption, y, c, and g are both output and consumption of services. Importantly this fixprice model can be seen as the limit case of our matching model when matching costs become vanishingly small: $\rho \to 0$. Hence all the equations from the matching model apply once we set $\tau(x) = 0$.[24]

Since $c = y - g$, the optimal g maximizes $\mathcal{U}(y - g, g)$. The first-order condition of the maximization is

$$1 = MRS_{gc} + \frac{dy}{dg}. \tag{23}$$

This is the same condition as (15) in the matching model. The implications of the optimality condition are different than in the matching model because of the values taken by the output multiplier dy/dg in the fixprice model.

In equilibrium, the price of services clears the market for services, so aggregate demand = aggregate supply. In that case, $y = k$ so $dy/dg = 0$ and $MRS_{gc} = 1$: the Samuelson rule holds.

What happens in disequilibrium, when the price of services is fixed at a level that does not

benefits derived from extra public expenditure, individual by individual. In this case, the original Samuelson rule holds (when unemployment is efficient) and all our formulas for optimal stimulus spending that we derived with fixed capacity carry over to the case with endogenous capacity.

[23]New Keynesian models build upon this tradition, but replace fixed prices by prices that slowly adjust from their initial level to the flexible-price level. Slow price adjustments make the equilibrium dynamics more interesting but the theoretical analysis more difficult. We focus on fixed prices for tractability and consistency with our earlier analysis.

[24]Michaillat and Saez (2015, pp. 539–540) discuss the link between the two models in more detail.

clear the market for services? When there is excess demand, $y(p, g) > k$. The demand-side of the market has to be rationed, such that output is determined by the supply side: $y = k$. Thus $dy/dg = 0$ and $MRS_{gc} = 1$: the Samuelson rule also holds. However, when there is excess supply, the supply-side of the market has to be rationed, and output is determined by the demand side: $y = y(p, g) < k$. The output multiplier can be 1, above 1, or below 1, depending on the effect of public consumption on the marginal utility from private consumption (that is, depending on whether c and g are complements or substitutes in the Edgeworth-Pareto sense).[25]

If the output multiplier is 1 or greater than 1, then for any public expenditure, $MRS_{gc} + dy/dg > 1$. Thus, it is optimal for the government to spend until the output gap is filled, irrespective of the usefulness of additional public expenditure. Intuitively, with such large multipliers, there is either no effect of public consumption on private consumption, or crowding in of private consumption by public consumption, so increasing public consumption raises all inputs into the welfare function, until the output gap is filled. Clearly, public consumption should fill the output gap. As showed in Appendix E, this implies that up to a second-order remainder, optimal stimulus spending is

$$\frac{g/c - (g/c)^*}{(g/c)^*} \approx z_2 \cdot \frac{1 - (g/y)^*(dy/dg)}{dy/dg} \cdot \frac{k - y_0}{y_0}, \tag{24}$$

where y_0 is the initial level of output, dy/dg is evaluated at $[(g/c)^*, y_0]$, and $z_2 \equiv 1 \big/ [(c/y)^*(g/y)^*]$.

The results when the output multiplier is 1 or greater than 1 are consistent with results obtained by others using other fixprice models. For instance, using a fixprice model generating a multiplier of 1, Mankiw and Weinzierl (2011, pp. 232–234) find that it is optimal to completely fill the output gap. This result has three implications. First, optimal stimulus spending grows in proportion to the output gap. Second, optimal stimulus spending is decreasing in the output multiplier: with a larger multiplier, less spending is required to fill the output gap. Finally, the value of additional public spending (measured by the elasticity of substitution between public and private consumption) is irrelevant: whether additional public workers provide services that substitute perfectly for private services or they dig and fill holes, optimal stimulus spending is the same.

[25] To see this, consider the demand side with land of Section 2.4. The output multiplier is $dy/dg = 1 + dc/dg$, so we need to determine the sign of dc/dg. Since there is excess supply, private consumption is determined by private demand, so we study the response of private demand to g. Private demand $c(p, g)$ is defined by $\frac{\partial \mathcal{U}}{\partial c}(y - g, g) = p\mathcal{V}'(l_0)/\delta$; thus $dy/dg = 1 - (\partial^2\mathcal{U}/\partial c \partial g)/(\partial^2\mathcal{U}/\partial c^2)$. The multiplier is 1 if $\partial^2\mathcal{U}/\partial c\partial g = 0$; that is, c and g are unrelated in the Edgeworth-Pareto sense. The multiplier is above 1 if $\partial^2\mathcal{U}/\partial c\partial g > 0$; that is, c and g are complements in the Edgeworth-Pareto sense. And the multiplier is below 1 if $\partial^2\mathcal{U}/\partial c\partial g < 0$; that is, c and g are substitutes in the Edgeworth-Pareto sense.

If the output multiplier is lower than 1, it is not necessarily optimal to fill the output gap. At the Samuelson level of spending, $MRS_{gc} = 1$ so $MRS_{gc} + dy/dg > 1$: it is optimal to increase public expenditure to reduce the output gap. As public expenditure increases, MRS_{gc} decreases. If $MRS_{gc} + dy/dg$ is above 1 once the output gap is filled, then it is optimal to completely fill the output gap. On the other hand, if $MRS_{gc} + dy/dg$ reaches 1 before the output gap is filled, then optimal public expenditure does not completely fill the output gap. Then, optimal stimulus spending satisfies

$$\frac{g/c - (g/c)^*}{(g/c)^*} \approx \epsilon \cdot (dy/dg). \tag{25}$$

This equation applies only if optimal stimulus spending is small enough that it does not completely fill the output gap—so only if multiplier and elasticity of substitution are small enough.

Equation (25) implies that as long as public consumption is valuable at the margin, stimulus spending should be positive for any positive output multiplier. Additionally, optimal stimulus spending grows in proportion to the elasticity of substitution between public and private consumption and the output multiplier. The size of the output gap does not influence the size of optimal stimulus spending, however.

Overall, when the economy is slack, the fixprice model leads to similar qualitative insights as the matching model. This is good news: irrespective of how productive inefficiency is modeled, in slumps stimulus spending obeys similar principles. While the fixprice model has the advantage of simplicity relative to the matching model, it also has two limitations.

First, while the fixprice model can describe an excessively slack economy, it cannot describe an excessively tight economy. In the matching model there is a welfare cost associated with excessively high tightness (excessive resources devoted to recruiting). But in the fixprice model welfare is the same in the excess-demand and market-clearing regimes. Indeed, when the price of services is below the market-clearing price, output is given by k, which is independent of the price. Hence welfare is independent of the price: it remains the same as when the market clears. This explains why the recommendations differ when the economy is tight: the matching model recommends to reduce public spending to reduce the (negative) unemployment gap; the fixprice model recommends to keep public spending at the Samuelson level.

Second, the fixprice model is harder to apply. The model offers starkly different policy recommendations depending on the value of the output multiplier. If the multiplier is large enough, public expenditure should completely fill the output gap, irrespective of the utility derived from public services. When the multiplier is sufficiently small, on the other hand, optimal stimulus spending only partially fills the output gap; then optimal stimulus spending is larger when the multiplier is larger and when public consumption substitute better for private consumption. Given the uncertainty surrounding the value of the multiplier, these precepts are difficult to apply.[26]

4. Application to the Great Recession in the United States

We now complement our theoretical results with a numerical application. We calibrate our sufficient-statistic formula and compute optimal stimulus spending at the onset of the Great Recession in the United States. This exercise illustrates how much optimal public expenditure may deviate from the Samuelson rule, and how the deviation depends on the values of the sufficient statistics. Since the formula is valid whether taxes are distortionary or not, the numerical results apply in both cases.

Our starting point is 2008:Q3 in the United States: the unemployment rate is $u = 6\%$ and public expenditure is $G/C = 19.7\%$. For simplicity, we assume that in 2008:Q3 the unemployment rate is efficient and public expenditure satisfies the Samuelson rule: $u^* = 6\%$ and $(G/C)^* = 19.7\%$. These assumptions seem reasonable as unemployment and public expenditure in 2008:Q3 are close to their 25-year averages, and there is a presumption, going back at least to Okun (1963), that the economy is efficient on average.[27]

In 2008, an adverse shock hits the US economy and unemployment starts rising toward an inefficient level $u_0 > u^*$.[28] In our model unemployment immediately reaches the higher level u_0,

[26]Endogenizing productive capacity (k) in the fixprice model makes optimal stimulus spending a smooth function of the sufficient statistics. This makes it easier to apply. The fixprice model with endogenous capacity also offers a symmetric treatment of the regimes with insufficient and excessive activity. The analysis of the fixprice model with endogenous capacity is presented in Appendix E (it parallels our analysis of the matching model with endogenous capacity in Appendix D.)

[27]We set u to the seasonally adjusted unemployment rate constructed by the Bureau of Labor Statistics from the Current Population Survey. To construct G/C, we set G to the seasonally adjusted employment level in the government industry and C to the seasonally adjusted employment level in the private industry. Both series constructed are by the Bureau of Labor Statistics from the Current Employment Statistics survey. Over the 1990–2014 period, average unemployment rate is $u = 6.1\%$ and average public expenditure is $G/C = 19.7\%$.

[28]Our formula accommodates any type of shock. But since we calibrate u^* and $(G/C)^*$ using preshock observations, the shock should not affect u^* and $(G/C)^*$. So it could be a shock to aggregate demand, to aggregate supply, or to prices, but not to the matching process (matching function, s, or ρ).

but in reality unemployment slowly rises to u_0. The challenge for policymakers is to forecast u_0 in advance. In the winter 2008–2009, when the US government designed the stimulus package, they forecast $u_0 = 9\%$, so we use $u_0 = 9\%$ (Romer and Bernstein 2009, Figure 1). Then, to apply formula (20), we collect estimates of the two main statistics: the elasticity of substitution between public and private consumption (ϵ) and the unemployment multiplier (m).

The elasticity of substitution describes how well public goods substitute for private goods. A literature attempts to provide empirical estimates. The general idea is to exploit variations in the ratio of public-consumption price to private-consumption price and assess its impact on the ratio of public consumption to private consumption. For example, if the ratio of public to private consumption stays constant in spite of secular variations in the price ratio, then the elasticity should be one.[29] The identification is challenging because the empirical analysis has to use time series at the country level and has to assume that public spending is set optimally, at least on average. The modern literature has used the co-integration approach developed by Ogaki (1992). With US data, Amano and Wirjanto (1997, 1998) estimate elasticities of 0.9 and 1.56. Kwan (2007, p. 52) provides estimates for nine East Asian countries ranging from 0.57 to 1.05, depending on the specification and sample period.[30] These estimates are somewhat sensitive to the specification and time period chosen. However, virtually all estimates fall in the range 0.5–2, with $\epsilon = 1$ as a plausible midrange estimate. Accordingly, we consider three values spanning the range to available estimates: $\epsilon = 0.5$, $\epsilon = 1$, and $\epsilon = 2$.

Next, we determine plausible values for the unemployment multiplier m. Since m is not directly observable, we report estimates for the multiplier M and then translate M into m. Using (22), the values above for G/C and u, and the calibration of η and τ in Landais, Michaillat, and Saez (2017), we find $m = 0.91 \times M/(1 - 0.046 \times M)$. Hence m is almost identical to M.[31]

The unemployment multiplier M is estimated by measuring the response of the unemployment rate (in percentage points) when public expenditure increases by 1% of GDP. Monacelli, Perotti, and Trigari (2010, pp. 533–536) estimate a structural vector autoregression (SVAR) on US data

[29]This is plausible in light of the fairly stable ratio of government consumption to GDP in OECD countries since 1980 (see https://data.worldbank.org/indicator/).

[30]Earlier work finds estimates of 1.1 for Taiwan (Chiu 2001) and 1.39 for Japan (Okubo 2003, pp. 79–80).

[31]We calibrate (22) as follows. We set G/Y and u to their values after the shock but before the stimulus: $G/Y = (G/C)/(1 + G/C) = 0.197/(1 + 0.197) = 16.5\%$ and $u = 9\%$. Landais, Michaillat, and Saez (2017, Figure 1) measure the resources devoted to matching; when the unemployment rate is 9%, as in 2009:Q2, they find that $\tau = 1.7\%$; we use this value. Last, following Landais, Michaillat, and Saez (2017, Online Appendix D), we set $\eta = 0.6$.

and find multipliers between 0.2 and 0.6. Ramey (2013, pp. 40–42) estimates SVARs on US data with various identification schemes and sample periods. She finds multipliers between 0.2 and 0.5, except in one specification where the multiplier is 1.[32]

Overall, the average unemployment multiplier is estimated to be in the 0.2–1 range. If multipliers are larger when unemployment is higher, as suggested by recent research on state-dependent multipliers, the multiplier entering our formula could even be larger. For instance, using regime-switching SVARs on US data, Auerbach and Gorodnichenko (2012, Table 1, rows 1–3) find that while the output multiplier is 0.6 in expansions and 1 on average, it is as high as 2.5 in recessions. To account for the uncertainty about the exact value of the multiplier, we compute optimal stimulus spending for M between 0 and 2.

The results are displayed in Figure 2. Panel A displays optimal stimulus spending as a share of GPD ($G/Y - (G/Y)^*$), constructed using (20). Panel B displays the unemployment rate under optimal stimulus spending, constructed using (21).[33] Several observations stand out.

First, even with a small multiplier of 0.2, optimal stimulus spending is significant. With $\epsilon = 0.5$, optimal stimulus spending is 1.6 percentage points of GDP. With $\epsilon = 1$, it is 2.8 points of GDP. And with $\epsilon = 2$, it is 4.7 points of GDP.

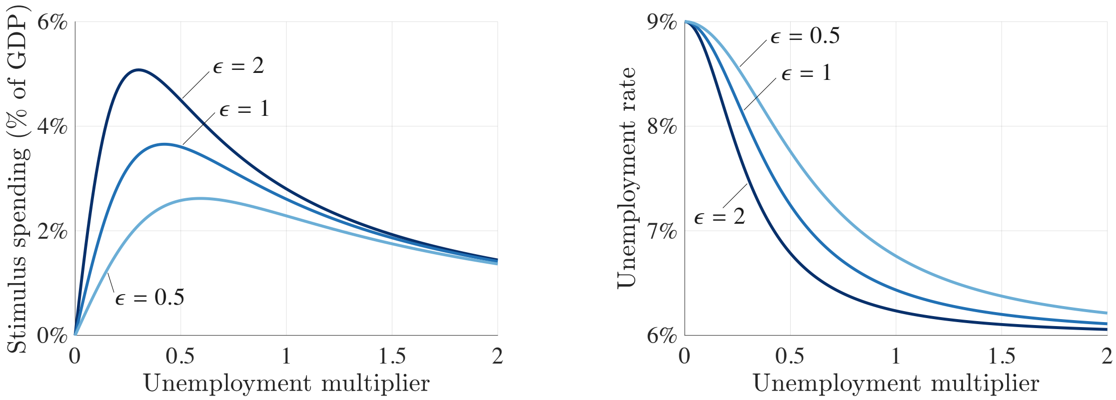
Second, the multiplier warranting the largest stimulus is fairly modest. With $\epsilon = 0.5$ the largest stimulus (2.6 points of GDP) occurs with a multiplier of 0.6. With $\epsilon = 1$ the largest stimulus (3.7 points of GDP) occurs with a multiplier of 0.4. And with $\epsilon = 2$ the largest increase (5.1 points of GDP) occurs with a multiplier of 0.3.

Third, optimal stimulus spending is the same for small and large multipliers. For instance, fix $\epsilon = 1$: optimal stimulus spending is the same for multipliers of 0.12 and 1.5 (1.9 points of GDP). Of course the resulting unemployment rates are very different.

Fourth, for small multipliers, unemployment barely falls below its initial level of 9% although optimal stimulus spending is large. With a multiplier of 0.2, unemployment only falls to 8.7% with

[32]One potential issue is that the multipliers are estimated using deficit-financed changes in public expenditure instead of balanced-budget changes (Barro and Redlick 2011). This issue is unimportant here: unemployment multipliers are larger once distortionary taxation is taken into account. This is because an increase in distortionary taxes reduces labor supply, which raises tightness and lowers the unemployment rate, thus complementing the increase in public expenditure (Michaillat and Saez 2015, pp. 530–532). Hence, these estimates are lower bounds on the estimates of balanced-budget multipliers.

[33]We calibrate z_0 and z_1 in (20) and (21) as we have calibrated (22). We set $(g/y)^* = 19.7\%$, $(g/y)^* = 16.5\%$, $(c/y)^* = 1 - (g/y)^* = 83.5\%$, $u^* = 6\%$, and $\eta = 0.6$. This values imply $z_0 = 2.83$ and $z_1 = 2.30$. We also translate the ratio g/c given by (20) into a ratio G/Y using the identity $G/Y = (g/c)/(1 + g/c)$.



A. Optimal stimulus spending

B. Unemployment rate under optimal policy

Figure 2. Optimal Stimulus Spending During the Great Recession in the United States

Notes: In the United States in 2008:Q3 unemployment was 6% and public expenditure was 16.5% of GDP. (We assume that in 2008:Q3 unemployment and public expenditure were efficient.) The Great Recession shock hit the US economy at that time, raising unemployment to a projected level of 9%. Panel A displays optimal stimulus spending as a share of GPD for various values of the empirical unemployment multiplier (M) and elasticity of substitution between public and private consumption (ϵ). Panel B displays the corresponding unemployment rate. Panel A is computed using (20), and Panel B is computed using (21).

$\epsilon = 0.5$, 8.5% with $\epsilon = 1$, and 8.1% with $\epsilon = 2$. This is because public expenditure has little effect on unemployment when the multiplier is small.

Fifth, with a multiplier above 1, optimal stimulus spending almost brings back unemployment to its efficient level of 6%. With a multiplier of 1, the unemployment rate falls below 6.8%, so the remaining unemployment gap is less than 0.8 percentage points. With a multiplier of 2, the remaining unemployment gap is less than 0.2 percentage points.

Sixth, the elasticity of substitution between public and private consumption plays a significant role for small to medium multipliers, but not for large multipliers. Consider first a multiplier of 0.4. With $\epsilon = 0.5$, optimal stimulus spending is 2.4 percentage points of GDP; with $\epsilon = 1$, public expenditure should increase by an additional 1.2 points of GDP; and with $\epsilon = 2$, public expenditure should increase by another 1.2 points of GDP. Hence, ϵ significantly influences optimal stimulus spending. In contrast, for a multiplier above 1, optimal stimulus spending with $\epsilon = 0.5$, $\epsilon = 1$, or $\epsilon = 2$ is nearly indistinguishable. This is because for large multipliers, the optimal policy is to fill the unemployment gap, so it is not influenced by the elasticity of substitution.

Finally, we calculate optimal stimulus spending at the onset of the Great Recession using midrange values for the unemployment multiplier and elasticity of substitution: $M = 0.5$ and

$\epsilon = 1$. Under this calibration, optimal stimulus spending is 3.6 points of GDP; since US GDP in 2008 is \$14,700 billion, optimal stimulus spending is \$530 billion per year. How does this optimal stimulus package compare to the actual stimulus package? According to the Congressional Budget Office (CBO), the American Recovery and Reinvestment Act (ARRA), enacted into law in February 2009, is estimated to cost \$840 billion over ten years, with half of that amount spent in 2010.[34] So at the peak of the Great Recession in 2010, stimulus spending was \$420 billion. This is below but of the same order of magnitude as our optimal stimulus package of \$530 billion. Yet, evaluating the adequacy of ARRA would be more complicated than comparing these two numbers. Our model focuses on one stabilization policy: government expenditure on goods and services. ARRA was more complex; it was a blend of three policies: increase in government expenditure, increase in government transfers, and increase in government deficit. According to the CBO, only 30% of ARRA was devoted to government expenditure, so about $0.3 \times \$420$ billion = \$130 billion.[35] Since government expenditure on goods and services was combined with other stabilization policies, optimal stimulus spending on goods and services is less than \$530 billion. Determining whether it is above or below \$130 billion would require a more sophisticated model describing jointly the effect of government transfers, government deficit, and government expenditure on goods and services.

5. Simulations

This section simulates a fully specified, structural matching model. The simulations show that the matching model provides a good description of the business cycle: in response to aggregate-demand shocks the model generates realistic, countercyclical fluctuations in the unemployment rate and unemployment multiplier. This realism suggests that the matching framework is adapted to study optimal policy over the business cycle. The simulations also show that our sufficient-statistic formula, obtained with first-order approximations, is accurate even for large business-cycle fluctuations. Indeed, in our matching model, the sufficient-statistic formula and the exact optimality condition deliver almost identical policies.

[34] See https://www.cbo.gov/sites/default/files/114th-congress-2015-2016/reports/49958-ARRA.pdf.

[35] See https://web.archive.org/web/20150905195457/http://www.recovery.gov/arra/Transparency/fundingoverview/Pages/fundingbreakdown.aspx for a breakdown of ARRA's funding.

5.1. Quantitative Properties of the Matching Model

We simulate the matching model with land described in Section 2.4. The model is calibrated to US data using the empirical evidence from Section 4. (The calibration is relegated to Appendix A.) We represent the business cycle as a succession of unexpected permanent aggregate-demand shocks. We focus on these shocks because they generate inefficient fluctuations in unemployment as well as negative comovements between tightness and unemployment.[36]

We parameterize aggregate demand with $\alpha = \delta/\mathcal{V}'(l_0)$. Since the economy jumps to its new steady-state equilibrium in response to a shock, we only need to compute a collection of steady states parameterized by $\alpha \in [0.97, 1.03]$. We run two simulations: one in which G/Y is constant at 16.5%, its average value in the United States for 1990–2014, and one in which G/Y is at its optimal level, given by (15).

Figure 3 illustrates the simulations. The unemployment rate is countercyclical: when $G/Y =$ 16.5%, it rises from 4.4% when aggregate demand is highest ($\alpha = 1.03$), to 6.1% (the average unemployment rate in the United States for 1990–2014) when aggregate demand is average ($\alpha = 1$), and to 11.0% when aggregate demand is lowest ($\alpha = 0.97$). Unemployment fluctuates in response to aggregate-demand shocks because the price of services is rigid: when α goes up, the price does not adjust, which stimulates the aggregate-demand curve (11) and reduces unemployment.

The unemployment multiplier is also countercyclical: it increases from 0.2 when unemployment is 4.4%, to 0.5 (the midrange of US estimates) when unemployment is 6.1%, to 1.4 when unemployment is 11.0%. This countercyclicality is consistent with evidence suggesting that in the United States, multipliers are higher when unemployment is higher or output is lower (Auerbach and Gorodnichenko 2012; Candelon and Lieb 2013; Fazzari, Morley, and Panovska 2015). The mechanism behind this countercyclicality is described in Michaillat (2014). When unemployment is high, there is a lot of idle capacity so the matching process is congested by sellers of services. Hence, an increase in public expenditure has very little effect on other buyers of services. Crowding out of private expenditure by public expenditure is therefore weak, and the multiplier is large. When unemployment is low, the opposite occurs: matching is congested by buyers of services, crowding out of private expenditure by public expenditure is sharp, and the multiplier is small.

We can also compute optimal public expenditure over the business cycle. We find that optimal

[36]Other shocks generate unrealistic comovements between tightness and unemployment (Michaillat and Saez 2015).

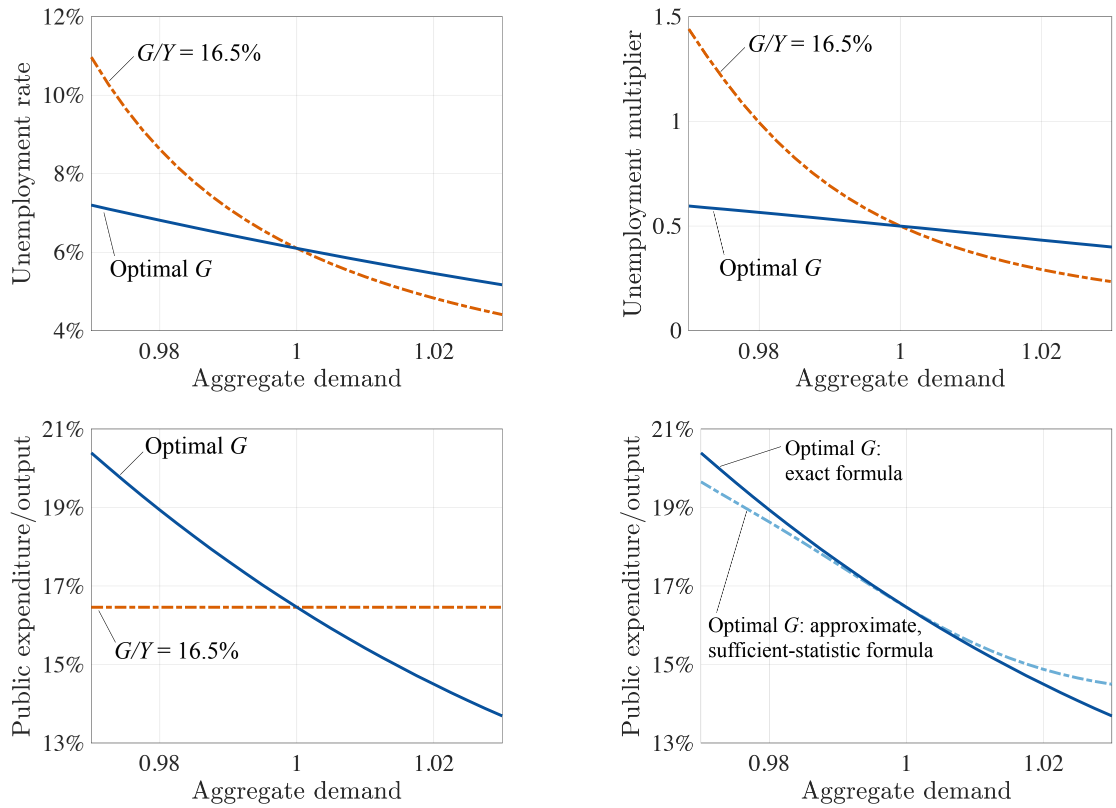


Figure 3. Business-Cycle Simulation in the Matching Model with Land

Notes: The figure displays business cycles both when public expenditure is $G/Y = 16.5\%$ and when it is set using the optimality condition (15). The business cycles are generated by aggregate-demand shocks. The multiplier in the top-right panel is the empirical unemployment multiplier, also equal to the output multiplier. The bottom-right panel compares the results of the sufficient-statistic formula (20) and the exact optimality condition (15). The results are obtained by simulating the matching model with land from Section 2.4 under the calibration in Appendix A.

public spending is markedly countercyclical, decreasing from $G/Y = 20.4\%$ to $G/Y = 13.7\%$ when α increases from 0.97 to 1.03. This is as expected. The unemployment rate is efficient when $\alpha = 1$, inefficiently high when $\alpha < 1$, and inefficiently low when $\alpha > 1$. Furthermore, the unemployment multiplier is positive. Hence, public spending should be above Samuelson spending when $\alpha < 1$ and below it when $\alpha > 1$.

Finally, unemployment responds when public expenditure is adjusted from $G/Y = 16.5\%$ to its optimal level. When aggregate demand is low, optimal public expenditure is higher than $G/Y = 16.5\%$ so unemployment falls below its original level. For instance, at $\alpha = 0.97$ the unemployment rate falls from 11.0% to 7.2%. When aggregate demand is high, optimal public expenditure is below $G/Y = 16.5\%$ so unemployment rises above its original level. For instance, at $\alpha = 1.03$ the unemployment rate increases from 4.4% to 5.2%. The unemployment multiplier heavily depends on the unemployment rate, so it adjusts accordingly.

5.2. Accuracy of the Sufficient-Statistic Formula

The sufficient-statistic formula is valid up to a second-order remainder, but since unemployment fluctuations are large, the remainder could be large and the approximation given by the formula inaccurate. In our simulations, however, this does not happen: Figure 3 shows that the sufficient-statistic formula is quite accurate. The figure compares the level of public expenditure given by our sufficient-statistic formula, which is approximate and given by (20), to the level given by the exact optimality condition (15). When aggregate demand α departs from 1 (where the two formulas give the same results by construction), the deviation between the results remains below one percentage point: at $\alpha = 1.03$, the exact condition gives $G/Y = 13.7\%$ while our formula gives $G/Y = 14.5\%$; at $\alpha = 0.97$, the exact condition gives $G/Y = 20.4\%$ while our formula gives $G/Y = 19.7\%$.

6. Conclusion

This paper has presented a theory of optimal public expenditure in the presence of unemployment. The theory shows that when unemployment is efficient, the Samuelson rule remains valid; but when unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to bring unemployment closer to its efficient level.

In the past few decades, monetary policy has been governments' preferred stabilization policy. Yet it has become clear that because of the zero lower bound on nominal interest rates, governments cannot rely on monetary policy alone to stabilize the economy—after the Great Recession, the zero lower bound was binding is Japan, the United States, and the eurozone. Our theory suggests that public expenditure could contribute to stabilization whenever monetary policy is constrained.

In addition, public expenditure could be helpful to members of monetary unions, such as countries in the eurozone or US states. These governments have no control over monetary policy, so they cannot use it to tackle unemployment—but they can adjust public expenditure. Furthermore, our theory focuses on budget-balanced spending, so it is appropriate for US states, which cannot run budget deficits, and to eurozone countries, which face strict constraints on their budget deficits.

In this paper we have limited ourselves to static considerations. It would be useful to enrich our analysis with dynamic elements. Several such elements seem important: the political process associated with the design of stimulus packages (Battaglini and Coate 2016); the dynamic effects of public spending in a liquidity trap (Woodford 2011; Werning 2011); the use of government debt (Barro 1979); the distinction between temporary and permanent changes in public spending (Barro 1981); and public investment in infrastructure (Baxter and King 1993).

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Appendix A. The Model with Land

We derive several results that are useful in the analysis and simulation of the matching model with land presented in Section 2.4. We also calibrate the model to US data.

Utility Function

We compute the derivatives of the constant-elasticity-of-substitution utility function:

$$\begin{aligned}
\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} &= (1-\gamma)^{\frac{1}{\epsilon}}\left(\frac{c}{\mathcal{U}}\right)^{\frac{\epsilon-1}{\epsilon}}, \quad \mathcal{U}_c \equiv \frac{\partial \mathcal{U}}{\partial c} = \left(\frac{(1-\gamma)\mathcal{U}}{c}\right)^{\frac{1}{\epsilon}} \\
\frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} &= \gamma^{\frac{1}{\epsilon}}\left(\frac{g}{\mathcal{U}}\right)^{\frac{\epsilon-1}{\epsilon}}, \quad \mathcal{U}_g \equiv \frac{\partial \mathcal{U}}{\partial g} = \left(\frac{\gamma\mathcal{U}}{g}\right)^{\frac{1}{\epsilon}} \\
\frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} &= \frac{1}{\epsilon}\cdot\left(\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} - 1\right) \\
\frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} &= \frac{1}{\epsilon}\cdot\frac{\partial \ln(\mathcal{U})}{\partial \ln(g)}.
\end{aligned}$$

When the Samuelson rule holds, we have $MRS_{gc} = \mathcal{U}_g/\mathcal{U}_c = 1$, so

$$(g/c)^* = \frac{\gamma}{1-\gamma}, \qquad (g/y)^* = \gamma, \qquad (c/y)^* = 1-\gamma,$$

and the derivatives of the utility function simplify to

$$\begin{aligned}
&\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} = 1-\gamma, \quad \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} = \gamma \\
&\mathcal{U}_c = 1, \quad \mathcal{U}_g = 1 \\
&\frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} = -\frac{\gamma}{\epsilon}, \quad \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} = \frac{\gamma}{\epsilon}.
\end{aligned}$$

Household's Problem and Equilibrium

We solve the household's utility-maximization problem and analyze equilibrium dynamics.

The current-value Hamiltonian of the household's problem is

$$\mathcal{H}(t, c(t), l(t)) = \mathcal{U}\left(c(t), g(t)\right) + \mathcal{V}(l(t)) + \lambda(t)\left\{p(t)\left[1-u(x(t))\right]k - p(t)\left[1+\tau(x(t))\right]c(t) - T(t)\right\}.$$

It has control variable $c(t)$, state variable $l(t)$, and current-value costate variable $\lambda(t)$. The first-order conditions for an interior solution to the maximization problem are $\partial\mathcal{H}/\partial c = 0$, $\partial\mathcal{H}/\partial l = \delta\lambda(t) - \dot{\lambda}(t)$, and the appropriate transversality condition. Since $\mathcal{U}$ and $\mathcal{V}$ are concave, these first-order conditions are not only necessary but also sufficient. These conditions yield (9) and (10).

Since all the equilibrium variables can be recovered from the costate variable $\lambda(t)$, the equilibrium can be represented as a dynamical system of dimension 1 with variable $\lambda(t)$. The variable $\lambda(t)$ satisfies the differential equation $\dot{\lambda}(t) = \delta\lambda(t) - \mathcal{V}'(l_0)$. The steady-state value of $\lambda(t)$ is $\lambda = \mathcal{V}'(l_0)/\delta > 0$. Since $\delta > 0$, the dynamical system is a source. As $\lambda(t)$ is a nonpredetermined variable, the system jumps to the steady state from any initial condition.

Unemployment Multiplier

We compute the unemployment multiplier. We first express dx/dg as a function of the derivatives of the utility function. Then, we compute the theoretical unemployment multiplier (m) from dx/dg and the empirical unemployment multiplier (M) from m.

The price schedule is $p(g) = p_0\mathcal{U}_c(y^* - g, g)^{1-r}$. Its elasticity is

$$\frac{d\ln(p)}{d\ln(g)} = (1-r)\left[\frac{\partial\ln(\mathcal{U}_c)}{\partial\ln(g)} - \frac{g}{y^*-g}\cdot\frac{\partial\ln(\mathcal{U}_c)}{\partial\ln(c)}\right],$$

and the value of the elasticity at $g^* \equiv \gamma y^*$ is

$$\frac{d\ln(p)}{d\ln(g)} = (1-r)\cdot\frac{1}{\epsilon}\cdot\frac{\gamma}{1-\gamma}.$$

The private demand $c(x, g)$ is defined by $\mathcal{U}_c(c, g) = p(g)(1+\tau(x))\mathcal{V}'(l_0)/\delta$. The elasticities of demand are

$$\frac{\partial\ln(c)}{\partial\ln(x)} = \frac{\eta\tau(x)}{\partial\ln(\mathcal{U}_c)/\partial\ln(c)}$$
$$\frac{\partial\ln(c)}{\partial\ln(g)} = -\frac{\partial\ln(\mathcal{U}_c)/\partial\ln(g) - \partial\ln(p)/\partial\ln(g)}{\partial\ln(\mathcal{U}_c)/\partial\ln(c)}.$$

We calibrate the price level such that unemployment is efficient when public expenditure is at the Samuelson level, or equivalently $x(g^*) = x^*$. This means that $c(x^*, g^*) = c^* \equiv (1-\gamma)y^*$ and

$\eta\tau(x^*) = (1-\eta)u^*$. Thus, the elasticities of demand at x^* and g^* are

$$\frac{\partial \ln(c)}{\partial \ln(x)} = -(1-\eta)u^* \frac{\epsilon}{\gamma} \quad \text{and} \quad \frac{\partial \ln(c)}{\partial \ln(g)} = \frac{r-\gamma}{1-\gamma}.$$

The equilibrium condition determining tightness $x(g)$ is $y(x,k) = g + c(x,g)$. In the simulations, k is fixed. Differentiating this equation with respect to g, we obtain the elasticity of $x(g)$ with respect to g:

$$\frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} = \frac{g}{y} + \frac{c}{y} \left[\frac{\partial \ln(c)}{\partial \ln(g)} + \frac{\partial \ln(c)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} \right]$$

so that

$$\frac{d \ln(x)}{d \ln(g)} = \frac{(g/y) + (c/y)(\partial \ln(c)/\partial \ln(g))}{\partial \ln(y)/\partial \ln(x) - (c/y)(\partial \ln(c)/\partial \ln(x))}.$$

As discussed above, we calibrate the model such that $x(g^*) = x^*$ and $c(x^*, g^*) = c^*$. In addition, $(c/y)^* = 1-\gamma$ and $(g/y)^* = \gamma$. Hence, the value of the elasticity at g^* is

$$\frac{d \ln(x)}{d \ln(g)} = \frac{1}{(1-\eta)u^*} \cdot \frac{r}{\epsilon} \cdot \frac{\gamma}{1-\gamma}.$$

From the expression for $d\ln(x)/d\ln(g)$, we obtain m and M using (18) and (22):

$$m = (1-\eta)(1-u)u \cdot \frac{y}{g} \cdot \frac{d \ln(x)}{d \ln(g)} \quad \text{and} \quad M = \frac{m}{1 - u + \frac{g}{y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot m}.$$

Since $(g/y)^* = \gamma$, the values of m and M at g^* are

$$m = \frac{(1-u^*)r}{(1-\gamma)\epsilon} \quad \text{and} \quad M = \frac{r}{\gamma r + (1-\gamma)\epsilon}.$$

Calibration

We calibrate the matching model with land using evidence from the United States. The calibration is summarized in Table A1. We use the calibration in the simulations presented in Section 5.

We begin by calibrating the utility function. We arbitrarily set the elasticity of substitution between public and private consumption to $\epsilon = 1$. As showed above, the parameter γ determines Samuelson spending: $(G/C)^* = \gamma/(1-\gamma)$. We assume that Samuelson spending is the average level of public expenditure in the United States for 1990–2014: $(G/C)^* = 19.7\%$ (Section 4). We

Table A1. Parameter Values in Simulations

	Description	Source
$\epsilon = 1$	Elasticity of substitution between g and c	–
$\gamma = 0.16$	Parameter of utility function	Matches $(G/C)^* = 19.7\%$
$s = 2.8\%$	Monthly separation rate	Landais, Michaillat, and Saez (2017)
$\eta = 0.6$	Matching elasticity	Landais, Michaillat, and Saez (2017)
$\omega = 0.60$	Matching efficacy	Landais, Michaillat, and Saez (2017)
$\rho = 1.4$	Matching cost	Matches $u^* = 6.1\%$
$r = 0.46$	Price rigidity	Matches $M = 0.5$ at $\alpha = 1$
$p_0 = 0.78$	Price level	Matches $u = u^*$ at $\alpha = 1$

therefore set $\gamma = 0.16$.

Then we calibrate matching parameters. Landais, Michaillat, and Saez (2017) use evidence from the US labor market for 1990–2014 and find a separation rate of $s = 2.8\%$ (Online Appendix B), a matching elasticity of $\eta = 0.6$ (Online Appendix D), and a matching efficacy of $\omega = 0.60$ (Online Appendix G). We use these values. They also find that the average US unemployment rate and tightness for 1990–2014 are $u = 6.1\%$ and $x = 0.43$ (Online Appendix G). We assume that these averages are efficient: $u^* = 6.1\%$ and $x^* = 0.43$. Then we set the matching cost from the relationship $\tau = \rho s/[\omega x^{-\eta} - \rho s]$, which implies $\rho = \omega x^{-\eta}\tau/[(1+\tau)s]$. This relation holds for any τ and x, in particular when tightness is efficient. But when tightness is efficient, $\tau^* = (1-\eta)u^*/\eta$ so $\tau^* = 4.1\%$. Plugging $x^* = 0.43$ and $\tau^* = 4.1\%$ in the expression for ρ yields $\rho = 1.4$.

Last we calibrate the price mechanism. The empirical evidence suggest that on average in the United States the unemployment multiplier is $M = 0.5$ (Section 4). Relying on (13), we set $r = 0.46$ to match $M = 0.5$. We calibrate the price level such that $u = u^* = 6.1\%$ when the demand parameter $\alpha \equiv \delta/\mathcal{V}'(l_0) = 1$ and $G/C = (G/C)^* = 19.7\%$. Using (12), we find that $p_0 = 0.78$.

Appendix B. Demand Side and Equilibrium: Other Examples

In Section 2.4 we describe a demand side and equilibrium with land. Here we present two other demand sides: one with money and another one with bonds. We find that they yield equilibria with the same properties as the land equilibrium.

Money in the Utility

We replace land by money and assume that households derive utility from holding real money balances. Introducing money in the utility function is a classical way to generate an aggregate demand: following Sidrauski (1967), a large number of business-cycle models with money in the utility function have been developed (for example, Barro and Grossman 1971; Blanchard and Kiyotaki 1987). Money is introduced in the utility function to capture the fact that money helps conducting transactions.

A household holds $M(t)$ units of money and the supply of money is fixed at M_0. In equilibrium, the money market clears and $M(t) = M_0$. The price of services in terms of money is $p(t)$. We specify a general price mechanism that determines the price of services: $p(t) = p(x(t), g(t))$. The household's instantaneous utility function is $\mathcal{U}(c(t), g(t)) + \mathcal{V}(M(t)/p(t))$. The law of motion of the household's real money balances $m(t) \equiv M(t)/p(t)$ is

$$\dot{m}(t) = [1 - u(x(t))]\, k - [1 + \tau(x(t))]\, c(t) - \pi(t)m(t) - T(t),$$

where $\pi(t) \equiv \dot{p}(t)/p(t)$ is the inflation rate. In steady state, g and thus p are fixed so inflation is zero. The equilibrium immediately converges to steady state. In steady state the desired private consumption $c(x, p, g)$ is given by

$$\frac{\partial \mathcal{U}}{\partial c} = (1 + \tau(x))\frac{\mathcal{V}'(M_0/p)}{\delta}.$$

Equilibrium tightness $x(g)$ is implicitly defined by

$$c(x, p(x, g), g) + g = y(x, k).$$

Bonds in the Utility

Here we replace land by nominal bonds and assume that households derive utility from holding real bonds. Introducing bonds in the utility function is a simple way to generate an aggregate demand in a dynamic cashless economy. Furthermore, the presence of wealth in the utility function captures the fact that people care about wealth for its own sake, not only as future consumption.

Wealth could be valued for several reasons: it provides high social status; it provides political power; people value frugality and dignify the accumulation of wealth; or people value bequests. The assumption of wealth in the utility function has been used in growth models (Kurz 1968; Zou 1994), microeconomic models (Robson 1992; Cole, Mailath, and Postlewaite 1995), life-cycle models (Carroll 2000; Francis 2009), asset-pricing models (Bakshi and Chen 1996; Gong and Zou 2002), business-cycle models (Michaillat and Saez 2014; Ono and Yamada 2014), and public-economic models (Saez and Stantcheva 2016).

Bonds are issued and purchased by households, and they have a price of 1 in terms of money. Money only plays the role of a unit of account. A household holds $B(t)$ bonds and bonds are in zero net supply. In equilibrium, the bond market clears and $B(t) = 0$. The rate of return on bonds is the nominal interest rate $i(t)$. The nominal interest rate is determined by the central bank, which sets an interest rate $i(t) = i(x(t), g(t))$. Since the interest rate depends on tightness public consumption, the central bank potentially responds to both economic activity and fiscal policy.

The price of services in terms of money is $p(t)$. The inflation rate is $\pi(t) \equiv \dot{p}(t)/p(t)$. In the economy there are two goods—services and bonds—and hence one relative price (public and private services have the same price). The price of bonds relative to services is determined by the real interest rate, $i(t) - \pi(t)$. Since the nominal interest rate is determined by the central bank, it is the inflation rate that determines the real interest rate. The inflation rate is determined by a general price mechanism: $\pi(t) = \pi(x(t), g(t))$. Given the inflation rate, the price of services moves according to $\dot{p}(t) = \pi(t)p(t)$. The initial price $p(0)$ is given. Given the inflation rate and nominal interest rate, tightness adjusts such that supply equals demand on the market for services.

The household's instantaneous utility function is $\mathcal{U}(c(t), g(t)) + \mathcal{V}(B(t)/p(t))$. The law of motion of the household's real wealth $b(t) \equiv B(t)/p(t)$ is

$$\dot{b}(t) = [1 - u(x(t))]\, k - [1 + \tau(x(t))]\, c(t) + [i(t) - \pi(t)]\, b(t) - T(t).$$

As earlier, the equilibrium immediately converges to steady state. In steady state, the desired amount of private consumption $c(x, i, \pi, g)$ is given by

$$\frac{\partial \mathcal{U}}{\partial c} = (1 + \tau(x)) \frac{\mathcal{V}'(0)}{\delta - i + \pi}.$$

This equation is the usual consumption Euler equation modified by the utility of wealth and evaluated in steady state. The demand for saving arises in part from intertemporal consumption-smoothing considerations and in part from the utility provided by wealth. The equation implies that at the margin, the household is indifferent between consuming and holding real wealth. Equilibrium tightness $x(g)$ is implicitly defined by

$$c(x, i(x, g), \pi(x, g), g) + g = y(x, k).$$

Appendix C. Long Proofs

In this appendix we provide the proofs that are relatively long. We incorporate the shorter proofs directly in the main text.

Proof of Lemma 3

Since MRS_{gc} is a function of g/c, the first-order Taylor expansion of $MRS_{gc}(g/c)$ at $(g/c)^*$ is

$$MRS_{gc}(g/c) = MRS_{gc}((g/c)^*) + \frac{dMRS_{gc}}{dg/c} \cdot (g/c - (g/c)^*) + O\left([g/c - (g/c)^*]^2\right).$$

In addition, $MRS_{gc}((g/c)^*) = 1$ and $dMRS_{gc}/d(g/c) = -1/[\epsilon \cdot (g/c)^*]$. Hence,

$$\text{(A1)} \qquad 1 - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right).$$

The $1/\epsilon$ in the Taylor expansion is evaluated at $(g/c)^*$. But we can replace it by the $1/\epsilon$ evaluated at g/c because the difference between the two is proportional to $g/c - (g/c)^*$. So once the difference is multiplied by $g/c - (g/c)^*$ in (A1), it is absorbed in $O\left([g/c - (g/c)^*]^2\right)$. Thus, equation (A1) yields equation (16).

Next, we write $\partial \ln(y)/\partial \ln(x)$ as a function of u:

$$\frac{\partial \ln(y)}{\partial \ln(x)} = (1 - \eta)u - \eta\tau(u).$$

The function $\tau(u)$ is defined by $\tau(u) = \tau(x(u))$, where $\tau(x)$ is given by (3) and $x(u) = u^{-1}(u)$ is the inverse of the function $u(x)$ given by (2). We have

$$\tau'(u) = \tau'(x) \cdot x'(u) = \frac{\tau'(x)}{u'(x(u))} = \frac{(1+\tau)\eta\tau/x}{-(1-\eta)(1-u)u/x} = -\frac{(1+\tau)\eta\tau}{(1-\eta)(1-u)u}.$$

Since $(1-\eta)u^* = \eta\tau(u^*)$, we have $\tau'(u^*) = -(1+\tau(u^*))/(1-u^*)$ and

$$-\eta\tau'(u^*) = \frac{\eta + \eta\tau(u^*)}{1-u^*} = \frac{\eta + (1-\eta)u^*}{1-u^*} = \eta + \frac{u^*}{1-u^*}.$$

Hence, the derivative of $\partial \ln(y)/\partial \ln(x)$ with respect to u at u^* is $(1-\eta) - \eta\tau'(u^*) = 1/(1-u^*)$. Furthermore, $\partial \ln(y)/\partial \ln(x) = 0$ at u^*. Thus, a first-order Taylor expansion of $\partial \ln(y)/\partial \ln(x)$ at u^* yields (17).

Finally, since the elasticity of $u(x)$ with respect to x is $-(1-\eta)(1-u)$, we find that

$$m = -y \cdot \frac{u}{g} \cdot \frac{d\ln(u)}{d\ln(g)} = \frac{y}{g} \cdot (1-\eta) \cdot u \cdot (1-u) \cdot \frac{d\ln(x)}{d\ln(g)} = \frac{y}{x} \cdot (1-\eta) \cdot u \cdot (1-u) \cdot \frac{dx}{dg}.$$

We obtain (18) by rearranging this equation.

Proof of Lemma 4

We start from (15). First, we approximate $1 - MRS_{gc}$ with (16). Next, we rewrite dx/dg with (18) and approximate $\partial y/\partial x$ with (17). This yields

$$\text{(A2)} \qquad \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1-\eta} \cdot \frac{(u-u^*)}{u \cdot (1-u) \cdot (1-u^*)} + 0([g/c - (g/c)^*]^2 + [u-u^*]^2).$$

We can rewrite this as

$$\frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1-\eta} \cdot \frac{(u-u^*)}{u^* \cdot (1-u^*)^2} + 0([g/c - (g/c)^*]^2 + [u-u^*]^2).$$

This is because the difference between $1/[u \cdot (1-u) \cdot (1-u^*)]$ and $1/\left[u^* \cdot (1-u^*)^2\right]$ is $O\left(u-u^*\right)$. Once this difference is multiplied by $u - u^*$ in (A2), it is absorbed by the term $0([g/c - (g/c)^*]^2 + [u-u^*]^2)$. We obtain (19) from this last equation.

Proof of Proposition 1

The economy starts at an equilibrium $[(g/c)^*, u_0]$, where the unemployment rate u_0 is inefficient. Since $u_0 \neq u^*$, the optimal g/c departs from $(g/c)^*$. In (19), the multiplier m and unemployment rate u are functions of g/c, so they respond as g/c moves away from $(g/c)^*$, and we cannot read the optimal g/c off the formula. In this proof, we derive a formula giving the optimal g/c as a function of fixed quantities.

First, we express the equilibrium values of all variables as functions of $[u, g/c]$. The proof of Lemma 3 showed that x and τ can be written as functions of u. Since $y = (1-u)\cdot k/(1+\tau)$, we can also write y as a function of u. Since $g = y \cdot (g/c)/[1 + g/c]$, g can be written as a function of u and g/c. As $c = y - g$, c can also be written as a function of u and g/c. Last, since $C = c \cdot (1+\tau)$, $C = c \cdot (1+\tau)$, and $C = c \cdot (1+\tau)$, we can write C, G, and Y as functions of u and g/c.

Among all pairs $[u, g/c]$, the only pairs describing an equilibrium are those consistent with the equilibrium condition $u = u(x(g))$, where g is the function of u and g/c described above, $x(g)$ is the function defined by (6), and $u(x)$ is the function defined by (2). This equilibrium condition defines the unemployment rate as an implicit function of g/c, denoted $u(g/c)$. Then, the pairs $[u(g/c), g/c]$ for all $g/c > 0$ are the equilibria for all possible levels of public expenditure.

We start by linking u to u_0 and g/c. We write a first-order Taylor expansion of $u(g/c)$ around $u((g/c)^*) = u_0$, subtract u^* on both sides, and divide both sides by u^*:

$$\text{(A3)} \qquad \frac{u-u^*}{u^*} = \frac{u_0-u^*}{u^*} + \frac{1}{u^*} \cdot \frac{du}{d\ln(g/c)} \cdot \frac{g/c-(g/c)^*}{(g/c)^*} + O\left(\left[g/c-(g/c)^*\right]^2\right).$$

To compute $du/d\ln(g/c)$ at $[u_0, (g/c)^*]$, we decompose the derivative:

$$\frac{du}{d\ln(g/c)} = \frac{du}{d\ln(g)} \cdot \frac{d\ln(g)}{d\ln(g/c)}.$$

First, the definition of the unemployment multiplier implies that

$$\frac{du}{d\ln(g)} = -m \cdot (g/y)^*,$$

where m is evaluated at $[u_0, (g/c)^*]$. Second, we compute $d\ln(g)/d\ln(g/c)$. We have

$$\ln(g/c) = \ln(g) - \ln(y(x(g/c), k) - g).$$

Differentiating with respect to $\ln(g/c)$ yields

$$1 = \frac{d\ln(g)}{d\ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(g/c)} + \frac{g}{c} \cdot \frac{d\ln(g)}{d\ln(g/c)}. \tag{A4}$$

Reshuffling the terms, we obtain

$$\frac{d\ln(g)}{d\ln(g/c)} = \frac{c}{y} + \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(g/c)}.$$

At u^*, $\partial \ln(y)/\partial \ln(x) = 0$, so at u_0, $\partial \ln(y)/\partial \ln(x)$ is $O(u_0 - u^*)$. Once this term is multiplied by $g/c - (g/c)^*$ in (A3), it creates a term that is $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$. Thus, we omit the term $(\partial \ln(y)/\partial \ln(x)) \cdot (d\ln(x)/d\ln(g/c))$ and set

$$\frac{d\ln(g)}{d\ln(g/c)} = (c/y)^*.$$

So far, we have showed that

$$\frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} - m z_1 \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right), \tag{A5}$$

where

$$z_1 \equiv \frac{(g/y)^*(c/y)^*}{u^*}.$$

Equation (19) includes a remainder that is $O\left([u - u^*]^2 + [g/c - (g/c)^*]^2\right)$. Equation (A5) implies that $(u - u^*)^2$ is $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$. Thus the remainder in formula (19) is $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$. Combining (19) and (A5), we therefore obtain

$$\frac{g/c - (g/c)^*}{(g/c)^*} = z_0 \epsilon m \left[\frac{u_0 - u^*}{u^*} - m z_1 \frac{g/c - (g/c)^*}{(g/c)^*}\right] + O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right),$$

where ϵ and m in the first bracket are evaluated at $[u, g/c]$. But instead we can use the values of ϵ and m evaluated at $[u_0, (g/c)^*]$ because the difference between the two values of each statistic is

$O\left([u-u_0]+[g/c-(g/c)^*]\right)$. So once the differences are multiplied by $g/c-(g/c)^*$ and u_0-u^* in the above equation, they are absorbed by the term $O\left([u_0-u^*]^2+[g/c-(g/c)^*]^2\right)$. Thus, this last equation yields equation (20).

To finish the proof, we derive (21). With the arguments that we have just used, (19) can be written

$$\frac{g/c-(g/c)^*}{(g/c)^*}=z_0\epsilon m\cdot\frac{u-u^*}{u^*}+O\left([u_0-u^*]^2+[g/c-(g/c)^*]^2\right),$$

where ϵ and m are evaluated at $[u_0,(g/c)^*]$. Replacing the left-hand side of the last equation by the right-hand side in (20), and dividing everything by $z_0\epsilon m$, we obtain (21).

Proof of Lemma 5

As $G=[1+\tau(x(g))]\,g$ and the elasticity of $1+\tau(x)$ with respect to x is $\eta\tau$, we have

$$\text{(A6)}\qquad \frac{d\ln(G)}{d\ln(g)}=1+\eta\tau\frac{d\ln(x)}{d\ln(g)}=1+\frac{g}{y}\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u\cdot(1-u)}\cdot m,$$

where the last equality is obtained using (18). Furthermore, the definitions of m and M imply

$$m=-y\frac{du}{dg}=-\frac{Y}{1+\tau(x)}\cdot\frac{du}{dG}\cdot\frac{dG}{dg}=\frac{g}{G}\,(1-u)M\frac{dG}{dg}=(1-u)M\frac{d\ln(G)}{d\ln(g)}.$$

We now plug into this equation the expression for $d\ln(G)/d\ln(g)$ obtained in (A6):

$$m=(1-u)\cdot M+\frac{g}{y}\cdot\frac{\eta}{1-\eta}\cdot\frac{\tau}{u}\cdot M\cdot m.$$

We obtain (22) by rearranging this equation.

Next, consider a change in public expenditure dG. This change leads to a change du in unemployment and, since $Y=(1-u)k$, to a change $dY=-du\cdot k$ in output. Hence,

$$\frac{dY}{dG}=-k\frac{du}{dG}=-\frac{Y}{1-u}\cdot\frac{du}{dG}=M.$$

Appendix D. Distortionary Taxation

We introduce an endogenous labor supply and distortionary income tax to study how distortionary taxation affects optimal public expenditure. We compare two approaches to taxation: the traditional approach in public economics and macroeconomics, which consists in using a linear income tax; and the modern approach in public economics, which consists in using a nonlinear income tax implemented following the benefit principle. We find with both approaches that the formula for optimal stimulus spending remains the same as when taxes are nondistortionary.

Traditional Approach

In the traditional approach to taxation, the government uses a linear income tax τ^L to finance public expenditure. With the linear income tax, the household's labor income becomes $(1-\tau^L)Y(x,k) = (1-\tau^L)(1-u(x))k$. To finance public expenditure G, the tax rate must be $\tau^L = G/Y = g/y$.

The household chooses k to maximize utility. Let $MRS_{kc} \equiv \mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$ be the marginal rate of substitution between labor and private consumption. As usual, the household supplies labor until the marginal rate of substitution between labor and consumption equals the post-tax real wage:

$$MRS_{kc} = (1-\tau^L)\frac{1-u(x)}{1+\tau(x)}. \tag{A7}$$

Indeed, one unit of labor is only sold with probability $1-u(x)$. When it is sold, it only yields $1/(1+\tau(x))$ units of consumption. Hence, the effective real wage is $(1-u(x))/(1+\tau(x))$, and the post-tax real wage is $(1-\tau^L)(1-u(x))/(1+\tau(x))$.[37]

The supply decision is distorted by the income tax: a higher τ^L implies a lower k. In fact, (A7) implicitly defines a function $k(g)$ describing how productive capacity responds to a change in public expenditure and the associated tax change. As the income tax is distortionary, the function $k(g)$ is decreasing in g.

The welfare of an equilibrium is $\mathcal{U}(c,g) - \mathcal{W}(k)$. Given a tightness function $x(g)$ and a capacity function $k(g)$, the government chooses g to maximize $\mathcal{U}(y(x(g),k(g)) - g, g) - \mathcal{W}(k(g))$. The

[37]Formally, for all the models in Section 2 and Appendix B, the first-order condition with respect to k is $\mathcal{W}'(k) = (1-\tau^L)(1-u(x))\lambda$, where λ is the costate variable associated with real wealth in the household's Hamiltonian. We combine this equation and the first-order condition with respect to c, given by (9), and obtain (A7).

first-order condition of the government's problem is

$$0=\frac{\partial\mathcal{U}}{\partial g}-\frac{\partial\mathcal{U}}{\partial c}-\mathcal{W}'(k)\frac{dk}{dg}+\frac{\partial\mathcal{U}}{\partial c}\cdot\frac{\partial y}{\partial k}\cdot\frac{dk}{dg}+\frac{\partial\mathcal{U}}{\partial c}\cdot\frac{\partial y}{\partial x}\cdot\frac{dx}{dg}.$$

Dividing the condition by $\partial\mathcal{U}/\partial c$, we obtain

$$1=MRS_{gc}-\left(MRS_{kc}-\frac{\partial y}{\partial k}\right)\cdot\frac{dk}{dg}+\frac{\partial y}{\partial x}\cdot\frac{dx}{dg}.$$

Households' optimal labor supply, given by (A7), implies that $MRS_{kc}=(1-\tau^L)(\partial y/\partial k)$. The government's budget constraint implies that $\tau^L=g/y$. Last, from equation (4), $\partial y/\partial k=y/k$. Hence, $-(MRS_{kc}-\partial y/\partial k)=\tau_L y/k=g/k$ and we have proved the following:

LEMMA A1. *With a linear income tax, optimal public expenditure satisfies*

$$\underbrace{1-\frac{d\ln(k)}{d\ln(g)}=MRS_{gc}}_{\textit{modified Samuelson rule}}+\underbrace{\frac{\partial y}{\partial x}\cdot\frac{dx}{dg}}_{\textit{correction}}.$$

This optimality condition differs from (15), but the two have the same structure once the Samuelson rule is modified to account for distortionary taxation. Indeed, this condition can be written as the modified Samuelson rule plus a correction equal to $(\partial y/\partial x)\cdot(dx/dg)$. The statistic $1-d\ln(k)/d\ln(g)>1$ in the modified Samuelson rule is the marginal cost of funds. It is more than one because the linear income tax distorts the labor supply. Because the marginal cost of funds is greater than one, the modified Samuelson rule recommends a lower level of public expenditure than the regular rule.

We generalize the definition of Samuelson spending with distortionary taxation:

DEFINITION 6. *With a linear income tax,* Samuelson spending $(g/c)^*$ *is given by the modified Samuelson rule:*

$$MRS_{gc}((g/c)^*)=1-\frac{d\ln(k)}{d\ln(g)}.$$

The elasticity $d\ln(k)/d\ln(g)<0$ *is evaluated at optimal public expenditure.*

Although Samuelson spending is lower with a linear income tax, the correction to the Samuelson rule is the same, so our sufficient-statistic formula for optimal stimulus spending remains the same:

PROPOSITION A1. *Suppose that the economy is initially at an equilibrium* $[(g/c)^*, u_0]$. *Then, with a linear income tax, optimal stimulus spending satisfies* (20) *and the unemployment rate under the optimal policy satisfies* (21), *where the statistic* z_1 *is generalized to allow for supply-side responses:*

$$z_1 = \frac{(g/y)^*(c/y)^*}{u^*} \cdot \frac{1}{1 - d\ln(k)/d\ln(g)}.$$

The elasticity $d\ln(k)/d\ln(g)$ *is evaluated at* $[(g/c)^*, u^*]$.

Proof. With a linear income tax, Samuelson spending satisfies

$$MRS_{gc}(g/c^*) = 1 - \frac{d\ln(k)}{d\ln(g)},$$

so Lemma A2 implies that optimal public expenditure satisfies

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

As in Lemma 3, we have

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*}.$$

Moreover, (17) and (18) remain valid. Combining these results, we obtain (19).

Since formula (19) remains valid, the proof follows the same steps as the proof of Proposition 1. The only difference occurs once we reach equation (A4). With a supply-side response to taxation, the equation becomes

$$1 = \frac{d\ln(g)}{d\ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d\ln(x)}{d\ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(k)} \cdot \frac{d\ln(k)}{d\ln(g)} \cdot \frac{d\ln(g)}{d\ln(g/c)} + \frac{g}{c} \cdot \frac{d\ln(g)}{d\ln(g/c)}.$$

Using the same argument as in the proof of Proposition 1, we can omit the term containing the factor $\partial\ln(y)/\partial\ln(x)$. Since $\partial\ln(y)/\partial\ln(k) = 1$, we therefore obtain $d\ln(g)/d\ln(g/c) = (c/y)^*/(1 - d\ln(k)/d\ln(g))$. Using the new expression for $d\ln(g)/d\ln(g/c)$, we conclude the proof like the proof of Proposition 1. □

The unemployment multiplier in formulas (20) and (21) is a policy elasticity, in the sense of Hendren (2016). It measures the change in unemployment for a change in public expenditure accompanied by the change in taxes maintaining a balanced government budget. In Section 3 taxes are not distortionary, so the unemployment multiplier should be measured using a policy reform in which taxes are nondistortionary. Here taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the tax change distorts the labor supply.

When taxation is nondistortionary, equation (22) shows that the unemployment multiplier m in our sufficient-statistic formula is closely related to the empirical unemployment multiplier M. Furthermore, the output multiplier is equal to M, so all our results remain the same if we reformulate them with the output multiplier instead of m. But when taxation is distortionary, things are different, and the output multiplier cannot be used to design optimal public expenditure. With distortionary taxation, equation (22) remains valid, but the link between the output multiplier and M break down. Indeed, output is $Y = (1-u)k$ so

$$\frac{dY}{dG} = -k\frac{du}{dG} + (1-u)\frac{dk}{dG} = -\frac{Y}{1-u}\cdot\frac{du}{dG} + \frac{Y}{k}\cdot\frac{dk}{dG} = M + \frac{Y}{k}\cdot\frac{dk}{dG}.$$

Since taxes are distortionary, $dk/dG < 0$ and

$$M = \frac{dY}{dG} - \frac{Y}{k}\cdot\frac{dk}{dG} > \frac{dY}{dG}.$$

Thus, when a change in taxes distort the capacity supplied by households, the unemployment multiplier M is the output multiplier net of the supply-side response $(Y/k)(dk/dG)$. The supply-side response measures the percentage change in labor supply when public expenditure increases by 1 percent of GDP. As taxation is distortionary, the supply-side response is negative and the unemployment multiplier is larger than the output multiplier. The unemployment multiplier is the correct sufficient statistic whether taxation is distortionary or not. With distortionary taxation, there is a wedge between unemployment and output multipliers equal to the supply-side responses, so the output multiplier is not useful to calibrate optimal stimulus spending.

Intuitively, an increase in public expenditure affects unemployment and the associated increase in taxes reduces labor supply. The negative effect on labor supply determines the marginal cost of fund and Samuelson spending but has nothing to do with the correction to the Samuelson rule and thus stimulus spending. The effect on unemployment, on the other hand, determines the correction to the Samuelson rule and thus stimulus spending. Since the unemployment multiplier measures the effect of public spending on unemployment, it governs optimal stimulus spending. Since the output multiplier conveys information about the effect of public spending on labor supply, it is not directly relevant to stimulus spending.

Modern Approach

We turn to the modern approach to taxation in public economics, which consists in using a nonlinear income tax implemented according to the benefit principle. The benefit principle, which was introduced by Hylland and Zeckhauser (1979) and fully developed by Kaplow (1996, 1998), is an important result in modern public-economic theory: it states that the optimal provision of public expenditure should be disconnected from distortionary taxation.[38] Hence, extra public expenditure should be financed by a change in the nonlinear tax schedule leaving all individual utilities unchanged—thus not altering further labor supply.

We assume that the government finances any increase in public expenditure by an increase in nonlinear income tax following the benefit principle: the tax schedule is changed to offset the extra benefit received by any individual from the extra public expenditure. Thus, the change in public expenditure leaves all individual utilities unchanged and does not further alter labor supply. In this case, although taxation is distortionary, we obtain the same results as with a fixed labor supply.

More precisely, we assume that households choose capacity k to maximize utility, and that public expenditure is funded by a distortionary, nonlinear income tax $T(k)$. We start from an equilibrium $[c, g, x, k]$. To ease notation, we introduce $\phi(x) \equiv (1-u(x))/(1+\tau(x))$. With the income tax, the household's disposable income becomes $(1-u(x))(k-T(k))$. In equilibrium, households' disposable income equals their expenses: $(1-u(x))(k-T(k)) = (1+\tau(x))c$ so $c = \phi(x)(k-T(k))$.

We implement a small change in public expenditure dg funded by a small tax change $dT(k)$ that satisfies the benefit principle. This change triggers a small change dx in tightness. By the benefit principle, the tax change $dT(k)$ is designed to keep the household's utility constant for any choice of k. For all k, $dT(k)$ satisfies

$$\text{(A8)} \qquad \mathcal{U}\left(\phi(x)\cdot(k-T(k)), g\right) = \mathcal{U}\left(\phi(x+dx)\cdot(k-T(k)-dT(k)), g+dg\right).$$

The left-hand and right-hand sides of this equation define two identical functions of k. This implies

[38]See Kaplow (2004) and Kreiner and Verdelin (2012) for a survey of the benefit-principle approach.

that the household does not change his choice of k after the reform: the labor supply is unaffected by a change dg funded by the benefit principle.

Taking a first-order expansion of the right-hand side of (A8) and subtracting the left-hand side from the right-hand side, we obtain

$$\frac{\partial \mathcal{U}}{\partial c} \cdot [\phi'(x) \cdot (k - T(k)) \cdot dx - \phi(x) \cdot dT(k)] + \frac{\partial \mathcal{U}}{\partial g} \cdot dg = 0.$$

Dividing by $\partial \mathcal{U}/\partial c$ and re-arranging yields

$$T(k) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k) = MRS_{gc} \cdot dg + \phi'(x) \cdot k \cdot dx.$$

Accordingly, the effect of the reform on the government budget balance $R = \phi(x)T(k) - g$ is

$$dR = T(k)\cdot\phi'(x)\cdot dx+\phi(x)\cdot dT(k)-dg = \left(MRS_{gc} - 1\right)\cdot dg+\phi'(x)\cdot k\cdot dx = \left(MRS_{gc} - 1\right)\cdot dg+\frac{\partial y}{\partial x}\cdot dx.$$

(We used $dk = 0$ and $\phi'(x)k = \partial y/\partial x$.) At the optimum, $dR = 0$, so we have proved the following:

LEMMA A2. *Under the benefit principle, optimal public expenditure satisfies* (15).

Under the benefit principle, (15) remains valid and capacity k is not affected by changes in public expenditure. Thus, our sufficient-statistic formula remains valid:

PROPOSITION A2. *Suppose that the economy is initially at an equilibrium* $[(g/c)^*, u_0]$. *Then, under the benefit principle, optimal stimulus spending satisfies* (20) *and the unemployment rate under the optimal policy satisfies* (21).

Finally, under the benefit principle, there are no labor-supply distortions for a marginal increase in public expenditure; therefore, output and unemployment multipliers are equal, and the output multiplier can be used to design optimal stimulus spending.

Appendix E. Fixprice Model

Optimal public expenditure in the fixprice model presented in Section 3.4 is straightforward. Only one result is cumbersome to obtain: the amount of stimulus spending required to completely fill the output gap, given by equation (24). We derive this result here. In addition, we present an extension of the fixprice model in which productive capacity is endogenous, not fixed, and we derive a sufficient-statistic formula for optimal public expenditure in that model. Compared to the fixprice model with fixed capacity, three differences arise: (a) the model offers a symmetric treatment of excessive production and insufficient production; (b) it is never optimal to completely fill the output gap; and (c) optimal stimulus spending is a smooth function of the sufficient statistics.

Stimulus Spending Required to Fill the Output Gap

We derive (24). The economy starts at an equilibrium $[(g/c)^*, y_0]$, where output $y_0 < k$ is inefficiently low. We compute the stimulus spending $g/c - (g/c)^*$ required to fill the output gap $k - y_0$. To that end, we link y to g/c. We write a first-order Taylor expansion of $y(g/c)$ around $y((g/c)^*) = y_0$, evaluate it at $y(g/c) = k$, and divide it by y_0:

$$\frac{k - y_0}{y_0} = \frac{d\ln(y)}{d\ln(\ln(g/c))} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right). \tag{A9}$$

Next we compute $d\ln(y)/d\ln(g/c)$ using the following decomposition:

$$\frac{d\ln(y)}{d\ln(g/c)} = \frac{d\ln(y)}{d\ln(g)} \cdot \frac{d\ln(g)}{d\ln(g/c)} = (g/y)^* \cdot \frac{dy}{dg} \cdot \frac{d\ln(g)}{d\ln(g/c)}, \tag{A10}$$

where dy/dg is evaluated at $[(g/c)^*, y_0]$. The last step is to compute $d\ln(g)/d\ln(g/c)$. We have $\ln(g/c) = \ln(g) - \ln(y - g)$. Differentiating this equation with respect to $\ln(g/c)$ yields

$$1 = \frac{d\ln(g)}{d\ln(g/c)} - (y/c)^* \frac{d\ln(y)}{d\ln(g/c)} + (g/c)^* \frac{d\ln(g)}{d\ln(g/c)}.$$

Using (A10) and reshuffling the terms, we obtain

$$\frac{d\ln(g)}{d\ln(g/c)} = \frac{1}{1 + (g/c)^* - (g/c)^*(dy/dg)}.$$

And using (A10) again, we find

$$\frac{d\ln(y)}{d\ln(g/c)} = \frac{(g/y)^*(dy/dg)}{1 + (g/c)^* - (g/c)^*(dy/dg)} = \frac{(c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)}. \tag{A11}$$

Combining (A11) with (A9), we finally obtain

$$\frac{g/c-(g/c)^*}{(g/c)^*}=\frac{1-(g/y)^*(dy/dg)}{(c/y)^*(g/y)^*(dy/dg)}\cdot\frac{k-y_0}{y_0}+O\left([g/c-(g/c)^*]^2\right),$$

where the output multiplier dy/dg is evaluated at $[(g/c)^*, y_0]$. This equation yields equation (24).

Endogenous Productive Capacity

We extend the fixprice model by introducing endogenous productive capacity, and we describe optimal public expenditure in that model. We could introduce endogenous capacity by assuming that households are price-takers: they supply capacity k to maximize utility given the price of services. But proceeding this way has a downside: it introduces an internal inconsistency in the model when there is excess supply. Indeed, aggregate supply would describe how much households desire to work for a given price, assuming that they can sell all the services that they supply to the market. But in fact they would not be able to sell all the services because there is excess supply. To be consistent, the model should allow households to revise their supply decision given that in fact the probability to sell a given service is less than one.[39]

Here we address this issue as in the New Keynesian literature. We assume that households are price-setters: they set the price of services to maximize profits and supply the amount of services demanded at the profit-maximizing price. When the price is fixed, households simply supply as many services as required to satisfy demand (for example, Nakamura and Steinsson 2014, p. 773). Let y be aggregate output of services, which is demand-determined. Since households supply exactly the amount of services required by demand, aggregate supply of services is $k = y$.

The government now chooses g to maximize $\mathcal{U}(y-g,g)-\mathcal{W}(y)$. The first-order condition of the maximization is

$$1 = MRS_{gc}+\frac{dy}{dg}\cdot(1-MRS_{kc}), \tag{A12}$$

where $MRS_{kc}\equiv\mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$ is the marginal rate of substitution between labor and private consumption. This equation is the same as equation (23), except that the output multiplier is

[39]The matching model addresses this issue by introducing a matching function that gives the probabilities to sell services, and by letting households take these probabilities into account when they make their supply decisions.

multiplied by the labor wedge $1 - MRS_{kc}$.[40] This equation is also the same as equation (45) in Woodford (2011)—this is not surprising since our fixprice model has all the key ingredients of the New Keynesian model considered by Woodford.

The economy can be in three possible regimes, depending on the labor wedge : efficient production when $1 - MRS_{kc} = 0$, insufficient production when $1 - MRS_{kc} > 0$ (a slump), and excessive production $1 - MRS_{kc} < 0$ (a boom). When there is efficient production, $MRS_{kc} = 1$ and the Samuelson rule remains valid. When there is excessive or insufficient production, things change: $MRS_{kc} \neq 1$ so the correction to the Samuelson rule is nonzero.

We assume that the economy starts at $[(g/c)^*, y_0]$, with a marginal rate of substitution $(MRS_{kc})_0 \neq 1$. Following the procedure developed in the matching model, we obtain a formula expressed as a function of fixed (not endogenous) sufficient statistics:

PROPOSITION A3. *Suppose that the economy is initially at an equilibrium* $[(g/c)^*, y_0]$. *Then optimal stimulus spending satisfies*

$$\text{(A13)} \qquad \frac{g/c - (g/c)^*}{(g/c)^*} \approx \frac{\epsilon \cdot (dy/dg)}{1 + z_3 \cdot \epsilon \cdot \frac{(dy/dg)^2}{1-(g/y)^*(dy/dg)}} \left[1 - (MRS_{kc})_0\right].$$

The statistics ϵ *and* dy/dg *are evaluated at* $[(g/c)^*, y_0]$, *and* $z_3 \equiv (MRS_{kc})_0\,(c/y)^*(g/y)^*/\kappa$, *where* $\kappa \equiv 1/[d\ln(\mathcal{W}'(k))/d\ln(k)]$ *is the Frisch elasticity of labor supply. Under the optimal policy, the labor wedge is*

$$\text{(A14)} \qquad 1 - MRS_{kc} \approx \frac{1}{1 + z_3 \cdot \epsilon \cdot \frac{(dy/dg)^2}{1-(g/y)^*(dy/dg)}} \left[1 - (MRS_{kc})_0\right].$$

The approximations (20) *and* (21) *are valid up to a remainder that is* $O\left([g/c - (g/c)^*]^2\right)$.

Proof. Optimal stimulus spending satisfies (A12), which can be rewritten using (16):

$$\text{(A15)} \qquad \frac{g/c - (g/c)^*}{(g/c)^*} = \epsilon \cdot (dy/dg) \cdot (1 - MRS_{kc}) + O\left([g/c - (g/c)^*]^2\right).$$

As in the matching model, MRS_{kc} responds to g/c when it deviates from $(g/c)^*$, so we cannot use (A15) to compute optimal stimulus spending. We follow the procedure developed in the

[40]The labor wedge plays an important role in macroeconomics. See Shimer (2009) for a discussion.

matching model and re-express (A15) as a function of fixed sufficient statistics.

To that end, we analyze how MRS_{kc} respond to g/c. In this demand-determined economy, the aggregate-demand relationship always holds. Since the asset (land in our baseline model) is in fixed supply and prices are fixed, the marginal utility of private consumption ($\partial \mathcal{U}/\partial c$) is fixed and does not change when public consumption changes.[41] Hence, we only consider how the marginal disutility of labor ($\mathcal{W}'(k)$) reacts to public consumption. We find

$$\frac{d\ln(MRS_{kc})}{d\ln(g/c)} = \frac{d\ln(\mathcal{W}'(k))}{d\ln(g/c)} = \frac{1}{\kappa} \cdot \frac{d\ln(y)}{d\ln(g/c)},$$

where $\kappa = 1/[d\ln(\mathcal{W}'(k))/d\ln(k)]$ is the Frisch elasticity of labor supply. Using (A11), we obtain

$$\frac{d\ln(MRS_{kc})}{d\ln(g/c)} = \frac{1}{\kappa} \cdot \frac{(c/y)^*(g/y)^*(dy/dg)}{1-(g/y)^*(dy/dg)}.$$

Accordingly, the first-order Taylor expansion of $MRS_{kc}(g/c)$ around $(g/c)^*$) is

$$MRS_{kc} = (MRS_{kc})_0 + \frac{1}{\kappa} \cdot \frac{(MRS_{kc})_0\,(c/y)^*(g/y)^*(dy/dg)}{1-(g/y)^*(dy/dg)} \cdot \frac{g/c-(g/c)^*}{(g/c)^*} + O\left([g/c-(g/c)^*]^2\right).$$

In the equation the multiplier dy/dg and the elasticity κ are evaluated at $[(g/c)^*, y_0]$. To obtain (A13), we plug this expression for MRS_{kc} into (A15) and reshuffle the terms. In addition, combining equations (A15) and (A13), we obtain (A14). □

Formula (A13) is similar to formula (20) in the matching model; the principal difference is that the amount of inefficiency is not measured by the unemployment gap but by the labor wedge $1-(MRS_{kc})_0$. Nonetheless the formula has very similar implications. First, with a positive output multiplier, then optimal stimulus spending is positive in slumps but negative in booms. Second, optimal stimulus spending is a hump-shaped function of the output multiplier. Third, optimal stimulus spending is larger when public consumption substitutes more easily for private consumption. Last, optimal stimulus spending only partially reduces the output gap: MRS_{kc} is brought closer to 1, but remains below 1.

Overall, the fixprice model with endogenous capacity leads to similar insights as the matching

[41]For example, in the demand side with land of Section 2.4, the aggregate demand is given by $\partial\mathcal{U}/\partial c = p\mathcal{V}'(l_0)/\delta$. This relationship always holds since the economy is demand-determined. As l_0 and p are fixed, $\partial\mathcal{U}/\partial c$ does not respond to g.

model. This is reassuring: irrespective of how productive inefficiency is modeled, stimulus spending obeys similar general principles.

Yet, for several reasons, the matching model seems more convenient than the fixprice model with endogenous capacity to think about optimal public expenditure. A first limitation of the fixprice model is that its description of booms is not fully satisfactory. When there is excessive production, $MRS_{kc} > 1$ which implies $\mathcal{W}'(k) > \partial \mathcal{U}/\partial c$: people, constrained to supply the amount of services demanded, are working are working more than they want. If workers were not bound to supply whatever is demanded, all of them would stop providing services in booms, as the cost of providing each service is higher than the income received. In the matching model, in contrast, all relationships generate surplus for both buyer and seller.

Another limitation of the fixprice model is that the supply side is irrelevant, as the equilibrium is demand-determined. Therefore, distortionary taxation has no effect at all, and the model is not useful to study distortionary taxation. In contrast, in the matching model, both supply and demand determine the equilibrium. The matching model is therefore well suited to study the effect of distortionary taxation on optimal public expenditure—something we do in Appendix D where we also extend the matching model to the case with endogenous supply.

A last limitation of the fixprice model is that the labor wedge $1 - (MRS_{kc})_0$ is more challenging to measure than the unemployment gap $u_0 - u^*$. As a result, the fixprice formula (A13) is less convenient to apply than the matching formula (20). Since u_0 is observable, measuring the unemployment gap only requires to measure the efficient unemployment u^*. This can be done from (5), following the method developed by Landais, Michaillat, and Saez (2017). This can also be done by using historical unemployment data, since u^* does not respond to typical macroeconomic shocks and is therefore expected to be stable over time (see Section 4). In contrast, it is difficult to measure the labor wedge because it is not possible to relate $(MRS_{kc})_0$ to observable variables.[42] One strategy to measure $(MRS_{kc})_0$ would be to assume that output is efficient before the shocks and that the utility functions $\mathcal{W}$ and $\mathcal{U}$ are stable. Then we could recover $(MRS_{kc})_0$ from the observed change in output, the Frisch elasticity (to link the output change to the change in $\mathcal{W}'(k)$), and a coefficient of risk aversion (to link the output change to the change in $\partial \mathcal{U}/\partial c$). This strategy could work with aggregate-demand shocks but not with aggregate-supply shocks, as the disutility from labor $\mathcal{W}$ varies under such shocks. Hence, it is generally impossible to measure the labor wedge.

[42]For the same reason, it is difficult to measure the New Keynesian output gap in the data (Galí 2008, pp. 80–81).

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