

# Optimal Public Expenditure with Inefficient Unemployment

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This paper proposes a theory of optimal public expenditure when unemployment is inefficient. The theory is based on a matching model. Optimal public expenditure deviates from the Samuelson rule to reduce the unemployment gap (the difference between current and efficient unemployment rates). Such optimal “stimulus spending” is described by a formula expressed with three sufficient statistics: the unemployment gap, the unemployment multiplier (the decrease in unemployment achieved by increasing public expenditure), and the elasticity of substitution between public and private consumption. When unemployment is inefficiently high and the multiplier is positive, the formula yields the following results. (a) Optimal stimulus spending is positive and increasing in the unemployment gap. (b) Optimal stimulus spending is zero for a zero multiplier, increasing in the multiplier for small multipliers, largest for a moderate multiplier, and decreasing in the multiplier beyond that. (c) Optimal stimulus spending is zero if extra public goods have no value, it becomes larger as the elasticity of substitution increases, and it completely fills the unemployment gap if extra public goods are as valuable as extra private goods.

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## 1. Introduction

The theory of optimal public expenditure developed by Samuelson (1954) is a cornerstone of public economics. This theory shows that public goods should be provided to the point where the marginal rate of substitution between public and private consumption equals their marginal rate of transformation. While the theory has been expanded in numerous directions since its inception, one question has not been answered: how is the theory modified in the presence of unemployment, especially when unemployment is inefficient? This question is relevant because public expenditure is one of the main tools used by governments to tackle high unemployment.<sup>1</sup>

In this paper, we expand Samuelson's theory to situations with inefficient unemployment. We begin in section 2 by embedding Samuelson's theory into a matching model of the economy. This allows us to introduce inefficient unemployment into the analysis. Indeed, in a matching model, there is always some unemployment: not all labor services on offer are sold. Furthermore, productive efficiency usually fails: unemployment may be inefficiently high, when the price of labor services is too high, or inefficiently low, when the price is too low. When unemployment is inefficiently high, too many workers are idle; when unemployment is inefficiently low, too much labor is devoted to recruiting instead of producing.

In section 3, we find that when unemployment is efficient, the Samuelson rule remains valid; but when unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to bring unemployment closer to its efficient level. We denote the deviation of public expenditure from the Samuelson rule as "stimulus spending." We describe optimal stimulus spending with a formula expressed in terms of three sufficient statistics: (a) the unemployment gap, which is the difference between current and efficient unemployment rates; (b) the unemployment multiplier, which measures the reduction in unemployment achieved by increasing public expenditure; and (c) the elasticity of substitution between public and private consumption, which describes the utility derived from additional public consumption.<sup>2</sup>

Being expressed with sufficient statistics, our formula applies to a broad range of matching models, irrespective of the specification of the utility function, aggregate demand, and price

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<sup>1</sup>See Kreiner and Verdellin (2012) for a survey of the public-economic literature on optimal public expenditure. In macroeconomics, many papers estimate or simulate the effect of public expenditure on output, but only a handful (discussed in section 3) study optimal public expenditure. These papers, however, do not feature unemployment.

<sup>2</sup>See Chetty (2009) for an overview of the sufficient-statistic approach to optimal policy analysis.

mechanism. Furthermore, our formula addresses a common problem of sufficient-statistic formulas. The sufficient statistics usually are implicit functions of policy, so the formulas cannot explicitly characterize the optimal policy. We resolve this issue in two steps. First, we express our statistics as explicit functions of stimulus spending. Then, we back out optimal stimulus spending as a function of statistics independent of policy. The resulting explicit formula yields several results. (Here we discuss the case with positive unemployment multiplier and positive unemployment gap, but the paper considers all the cases.)

The first result is that when unemployment multiplier and unemployment gap are positive, optimal stimulus spending is positive. This result is simple to understand. By construction, at the Samuelson rule, an increase in public expenditure has no first-order effect on welfare when we ignore its effect on unemployment. Now, when the unemployment multiplier is positive, an increase in public expenditure lowers unemployment; and when the unemployment gap is positive, unemployment is inefficiently high, so lowering unemployment raises welfare. Hence, overall, an increase in public expenditure generates a positive first-order effect on welfare. It is therefore optimal to increase public expenditure above the Samuelson rule. Further, since the unemployment gap measures the welfare gain from reducing unemployment, optimal stimulus spending is increasing in the unemployment gap.

The second result is that optimal stimulus spending is zero for a zero multiplier, increasing in the multiplier for small multipliers, maximized for a moderate multiplier, and decreasing in the multiplier for larger multipliers. The intuition is the following. When the multiplier is small, optimal stimulus spending is determined by how much public expenditure reduces the unemployment gap. A larger multiplier means a larger reduction, so it warrants more stimulus spending. When the multiplier is large, this logic breaks down: it becomes optimal to fill the unemployment gap nearly entirely. As less spending is required to fill the gap when the multiplier is larger, optimal stimulus spending is decreasing in the multiplier.

The third result is that optimal stimulus spending is increasing in the elasticity of substitution between public and private consumption. This result is natural: a higher elasticity of substitution means that extra public goods are more valuable, making stimulus spending more desirable. There are two interesting limit cases: zero elasticity and infinite elasticity. With zero elasticity, extra public goods are useless. As public consumption always crowds out private consumption, it is never optimal to provide public goods beyond the Samuelson rule. With infinite elasticity, public and private goods

are interchangeable. It is therefore optimal to maximize the sum of public and private consumption. This is achieved by filling the unemployment gap entirely.

In addition, we establish that our formula remains the same whether the taxes used to finance public expenditure are distortionary or not. Nevertheless, distortionary taxation alters the design of stimulus spending. When taxes are nondistortionary, the unemployment multiplier and the output multiplier (the increase in output achieved by increasing public expenditure) are equal, so they can be used interchangeably in our formula. But with distortionary taxation, the unemployment and output multipliers are no longer the same, so the output multiplier cannot be used in our formula. Indeed, with distortionary taxation, raising taxes reduces labor supply, which reduces output but not unemployment. Hence, the output multiplier is smaller than the unemployment multiplier. With a strong labor-supply response, it is even possible for the output multiplier to be negative when the unemployment multiplier is positive. Accordingly, neither the size nor the sign of the output multiplier are useful to design stimulus spending. This point is important because the output multiplier plays a prominent role in the stimulus debate.

Since our sufficient statistics are estimable, we can use the formula to generate policy recommendations. As an illustration, in section 4, we apply the formula to the Great Recession in the United States. Estimates of the unemployment multiplier fall between 0.2 and 1, and according to research on state-dependent multipliers, they could be larger in bad times. (An unemployment multiplier of  $x$  means that raising public expenditure by one percent of GDP reduces unemployment by  $x$  percentage points). Estimates of the elasticity of substitution between public and private consumption fall between 0.5 and 2. Given this uncertainty, we compute optimal stimulus spending for a range of multipliers and elasticities of substitution. For example, with an elasticity of substitution of 1 (Cobb-Douglas utility), we obtain the following results. Optimal stimulus spending is large even with a small multiplier of 0.2: about 2.8 percentage points of GDP. It is largest for a modest multiplier of 0.4: about 3.7 points of GDP. It then decreases for larger multipliers. It falls to about 1.9 points of GDP when the multiplier reaches 1.5. Of course, optimal stimulus spending has a different impact on unemployment with small and large multipliers: it has almost no effect on unemployment with a multiplier of 0.2 but almost fills the unemployment gap with a multiplier of 1.5.

Finally, in section 5, we calibrate and simulate a specific matching model. This exercise suggests that the matching model describes business cycles well: in response to aggregate-demand shocks, the model generates countercyclical fluctuations in unemployment rate and unemployment multiplier.

In addition, we find that although our formula is obtained using several first-order approximations, it remains accurate for sizable business-cycle fluctuations.

## 2. A Matching Model of Inefficient Unemployment

We present the model used for the analysis. The model combines the public-expenditure framework of Samuelson (1954) with the matching framework of Michaillat and Saez (2015). Because of the matching structure, the model features unemployment, and the rate of unemployment is generally inefficient.

### 2.1. Informal Description

The model is not standard, so to help readers understand its properties, we begin by describing it informally. In the analysis the demand side of the model is completely generic; here for concreteness we use a specific demand side.

In the model, there are people and a government. People perform services for pay: they garden, cook, clean, educate children, cut hair, do administrative work, and so on. People are very much like P.G. Wodehouse's butler, Jeeves: they can do everything. Since nobody can be their own butler, however, people work as butlers for others and use the income to hire their butlers. This assumption captures the fact that a modern economy is based on market exchange rather than home production.

Beside purchasing services, people buy land, which provides utility and is a vehicle for saving. As land is in fixed supply, the tradeoff between services and land determines the aggregate demand for services. The relevant price is the price of services in terms of land.

People are hired by other people and the government. The people hired by other people produce private services (cleaning or cooking) while those hired by the government produce public services (tending public spaces or policing the streets). People value both public and private services. The government finances its expenditure by levying a tax.

People are hired on a matching market. This means that while people are available to work for forty hours a week, they are not working the whole time. For simplicity, we assume that everybody is idle for the same number of hours each week. Since unemployment is equally spread over the population, everybody has the same consumption, and insurance is not an issue.

This also means that people and the government need to post help-wanted ads to hire services. Posting ads requires labor: workers have to create the ads, read applications, and interview applicants. The time devoted to recruiting by these human-resource workers depends on the number of positions to be filled and the time spent filling each position. The services supplied by human-resource workers are not consumed—in the sense that they do not provide utility—but they are necessary to hire other workers whose services are consumed (provide utility).

Once hired, everyone is paid the same price for their services. People work for an employer for a while, until the relationship stops. As services are sold by the hour, people usually work for several employers at the same time.

The state of the services market is described by a tightness variable—the ratio of help-wanted ads to unemployment. When tightness is higher, it is easier to find work but harder to recruit workers. Consequently, the unemployment rate is lower, and employers devote a larger share of their workforce to recruiting.

There is an efficient tightness, which maximizes the amount of services that are consumed (provide utility). When tightness is inefficiently low, workers are unemployed for too many hours, so the amount of services consumed is too low. When tightness is inefficiently high, too many hours are devoted to human-resource tasks, so the amount of services consumed is too low as well.

In this economy, two variables—tightness and price—equalize demand and supply. If the price is high, demand for services is low (as land is relatively more attractive). If tightness were high, people would find work easily and the supply of services would be high. But then demand could not equal supply. Hence, tightness must be low in equilibrium. If instead the price is low, demand is high, and tightness must be high. Effectively, for any price, tightness adjusts to equalize demand and supply. The price can be determined in many ways—bargained between employer and worker, fixed by a social norm, or set by government regulation—but once the price mechanism is specified, the equilibrium is unique. There is no guarantee, however, that the price ensures efficiency.

What happens then when the government hires more workers? In the simple situation where public hiring affects neither private demand nor price, public hiring mechanically stimulates aggregate demand, which raises tightness. In good times, tightness is too high, so raising tightness further reduces total consumption. Consequently, public consumption crowds out private consumption more than one-for-one. If tightness is efficient, raising tightness has no effect on total consumption, so crowding out is exactly one-for-one. Finally, in bad times, tightness is too low, so raising tightness

increases total consumption, and crowding out is less than one-for-one. In this simple case, therefore, public expenditure is more desirable in bad times than in good times.

## 2.2. Supply Side

We now formally describe the model. We start with the supply side.

The model is dynamic and set in continuous time. The economy consists of a government and a measure one of identical households. Households are self-employed: they produce services and sell them on a matching market.<sup>3</sup> There are two types of services: private services, purchased by households, and public services, purchased by the government and provided to all households. Public and private services are bought on the same matching market at the same price  $p$ .

Each household has a productive capacity  $k > 0$ ; the capacity indicates the maximum amount of services that could be sold at any point in time. (Here  $k$  is exogenous, but in section 3.3 we show that the results are unchanged when  $k$  is chosen by households to maximize utility.) Since there is a measure one of households, the aggregate capacity in the economy is  $k$ .

Because of the matching process, not all available services are sold at any point in time, so there is always some unemployment. At time  $t$ , households sell  $C(t)$  services to other households and  $G(t)$  services to the government. Output  $Y(t)$  is the sum of all sales:

$$Y(t) = C(t) + G(t).$$

As households cannot sell their entire capacity,  $Y(t) < k$ . The unemployment rate is the share of aggregate capacity that is idle:  $u(t) = (k - Y(t))/k$ .

Services are sold through long-term relationships. Once a seller and a buyer have matched, the seller serves the buyer at each instant until the relationship ends. Relationships separate at rate  $s > 0$ , for exogenous reasons. Since  $Y(t)$  services are committed to existing relationships at time  $t$ , the amount of services available for purchase at time  $t$  is  $k - Y(t)$ .

To buy new services, households and the government advertise a total of  $v(t)$  vacancies. (In section 2.3, we explain how households and the government form their demand for services.) A Cobb-Douglas matching function taking as arguments available services and vacancies determines

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<sup>3</sup>Michaillat and Saez (2015, sec. 3) show how the model can be extended to include firms hiring workers on a matching market and selling their production on another matching market.

the rate  $h(t)$  at which new long-term relationships are formed:

$$h(t) = \omega \cdot v(t)^{1-\eta} \cdot (k - Y(t))^\eta,$$

where  $\eta \in (0, 1)$  is the matching elasticity, and  $\omega > 0$  is the matching efficacy.

With constant returns to scale in matching, the rates at which sellers and buyers form new relationships is determined by the market tightness,  $x(t)$ . The market tightness is the ratio of the matching function's two arguments:  $x(t) \equiv v(t)/(k - Y(t))$ . Each of the  $k - Y(t)$  available services is sold at rate  $f(x(t)) = h(t)/(k - Y(t)) = \omega x(t)^{1-\eta}$ , and each of the  $v(t)$  vacancies is filled at rate  $q(x(t)) = h(t)/v(t) = \omega x(t)^{-\eta}$ . The selling rate  $f(x)$  is increasing in  $x$ , and the buying rate  $q(x)$  is decreasing in  $x$ . Hence, when tightness is higher, it is easier to sell services but harder to buy them.

In such a model, output follows the law of motion  $\dot{Y}(t) = f(x(t))(k - Y(t)) - sY(t)$ . The term  $f(x(t))(k - Y(t))$  is the number of new relationships forming at time  $t$ ; the term  $sY(t)$  is the number of existing relationships separating at time  $t$ . If  $f(x)$  and  $s$  are constant over time, output converges to the steady-state level

$$(1) \quad Y(x, k) = \frac{f(x)}{f(x) + s} k.$$

The unemployment rate is  $u = 1 - Y/k$ , so the steady-state unemployment rate is

$$(2) \quad u(x) = \frac{s}{s + f(x)}.$$

The function  $Y(x, k)$  is positive and increasing in  $x$  and  $k$ , and its elasticity with respect to  $x$  is  $(1 - \eta)u(x)$ . The function  $u(x)$  is positive and decreasing in  $x$ , and its elasticity with respect to  $x$  is  $-(1 - \eta)(1 - u(x))$ . Hence, when tightness is higher, output is higher and unemployment is lower.

In the United States, labor market flows are large, so unemployment reaches its steady-state level quickly. In fact, Hall (2005, fig. 1) shows that the unemployment rate obtained from (2) is indistinguishable from the actual employment rate. Thus, as Hall does, we ignore the transitional dynamics of output and unemployment and assume that the two variables satisfy (1) and (2) at all times. To simplify the analysis further, we abstract from transitional dynamics and randomness at the seller's level: we assume that at all times a seller exactly sells a share  $1 - u(x)$  of her capacity  $k$ ; the remaining share  $u(x)$  is idle.



Posting a vacancy costs  $\rho > 0$  services per unit of time. These services are devoted to matching with appropriate suppliers of services. Matching services do not directly provide utility to households, so we distinguish between services purchased and services providing utility. Households purchase  $C(t)$  services and the government purchases  $G(t)$  services. We refer to  $C(t)$  as private expenditure and to  $G(t)$  as public expenditure. But households only derive utility from  $c(t) < C(t)$  private services and  $g(t) < G(t)$  public services;  $c(t)$  and  $g(t)$  are computed by subtracting the matching services used by the households and the government from  $C(t)$  and  $G(t)$ . We refer to  $c(t)$  as private consumption, to  $g(t)$  as public consumption, and to  $y(t) = c(t) + g(t)$  as total consumption.

The wedge between expenditure and consumption is determined by tightness. As we did with sellers, we abstract from transitional dynamics and randomness at the buyer's level. This means that by posting  $v_0$  vacancies, a buyer establishes exactly  $v_0 q(x)$  new matches at any point in time. It also means that a buyer is always in a situation where the same number of relationships form and separate. So if a buyer wants to continuously purchase  $Y_0$  services,  $sY_0$  new matches must be continuously created to replace the matches that have separated. This requires  $v_0 = sY_0/q(x)$  vacancies and  $\rho v_0 = \rho sY_0/q(x)$  matching services. Hence, only  $y_0 = Y_0 - \rho sY_0/q(x)$  of the services purchased are actually consumed. This relationship can be rewritten as  $Y_0 = [1 + \tau(x)] y_0$ , where the wedge between consumption and expenditure is

$$(3) \quad \tau(x) \equiv \frac{\rho s}{q(x) - \rho s}.$$

The matching wedge  $\tau(x)$  is positive and increasing for  $x \in [0, x^m)$ , where  $x^m > 0$  is defined by  $q(x^m) = \rho s$  and  $\lim_{x \rightarrow x^m} \tau(x) = +\infty$ . The elasticity of  $\tau(x)$  with respect to  $x$  is  $(1 + \tau(x))\eta$ . Hence, when tightness is higher, the matching wedge is larger.

The reasoning holds for any consumption  $y_0$ . Thus, if a household or the government desire to consume one service, they need to purchase  $1 + \tau(x)$  services—one service for consumption plus  $\tau(x)$  services for matching. Hence, private consumption is related to private expenditure by  $c = C/[1 + \tau(x)]$  and public consumption to public expenditure by  $g = G/[1 + \tau(x)]$ . Accordingly, total consumption is a function of tightness and capacity:

$$(4) \quad y(x, k) = \frac{1 - u(x)}{1 + \tau(x)} k.$$

The function  $y(x, k)$  is positive for  $x \in [0, x^m)$  and  $k > 0$ . We refer to  $y(x, k)$  as aggregate supply; it plays a central role in the analysis because it gives the amount of services consumed for a given tightness. Equation (4) shows that aggregate supply is less than aggregate capacity because some services are not sold ( $u(x) > 0$ ) and some are used for matching instead of consumption ( $\tau(x) > 0$ ).

In such a matching model the rate of unemployment is generally inefficient—because prices generally fail to maintain productive efficiency (Michaillat and Saez 2015, pp. 525–529). The formal definition of efficiency is the following:

**DEFINITION 1:** *Tightness and unemployment are efficient if they maximize total consumption for a given aggregate productive capacity. The efficient tightness is denoted by  $x^*$  and the efficient unemployment rate by  $u^*$ .*

Equation (4) implies that the elasticity of  $y(x, k)$  with respect to  $x$  is  $(1 - \eta)u(x) - \eta\tau(x)$ . This elasticity is  $1 - \eta > 0$  for  $x = 0$ , strictly decreasing in  $x$ , and  $-\infty$  at  $x = x^m$ . Thus, there is a unique  $x^*$  where the elasticity is zero. Since the partial derivative of  $y(x, k)$  with respect to  $x$  is positive for  $x < x^*$ , zero at  $x^*$ , and negative for  $x > x^*$ , the tightness  $x^*$  maximizes  $y(x, k)$  for a given  $k$ . Efficient tightness and unemployment are therefore characterized as follows:

**LEMMA 1:** *The efficient tightness  $x^*$  is implicitly defined by*

$$(5) \quad (1 - \eta)u(x^*) - \eta\tau(x^*) = 0.$$

*The efficient unemployment rate is given by  $u^* = u(x^*)$ .*

An increase in tightness has two opposite effects on consumption: it increases consumption by reducing the amount of unsold services; and it decreases consumption by raising the amount of services devoted to matching. When (5) is satisfied, the increase in tightness reduces unsold services as much as it increases matching services, which indicates that consumption is maximized.

To measure how far from productive efficiency the economy operates, we introduce a first sufficient statistic:

**DEFINITION 2:** *The unemployment gap is  $u - u^*$ .*

The unemployment gap is positive when unemployment is inefficiently high and negative when unemployment is inefficiently low. Equation (5) is useful to determine the sign of the unemployment

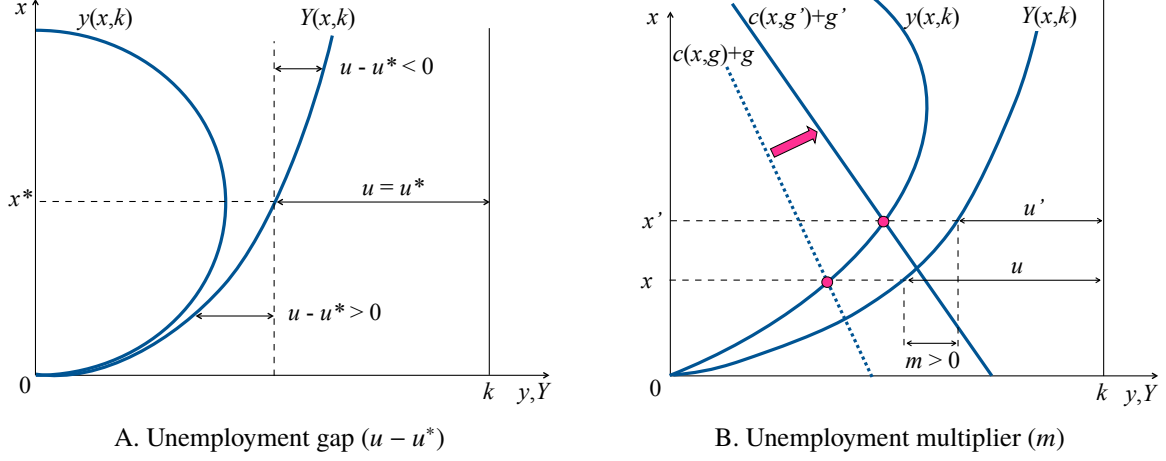


Figure 1. Aggregate Demand and Supply, and Sufficient Statistics

gap: when the unemployment rate  $u$  is high relative to the matching wedge  $\tau$ , such that  $u/\tau > \eta/(1-\eta)$ , tightness is inefficiently low, so the unemployment gap is positive.

Panel A in figure 1 summarizes the supply side of the model. It depicts how total consumption and output depend on tightness. It also depicts the efficient tightness and positive, zero, and negative unemployment gaps.

### 2.3. Demand Side and Equilibrium: General Case

We turn to the demand side and equilibrium of the model. While it is necessary to specify the supply side to compute social welfare and study optimal policy, the sufficient-statistic approach makes it unnecessary to specify demand side and equilibrium. We therefore keep them generic and look for sufficient statistics to summarize their relevant features.

The representative household derives instantaneous utility  $\mathcal{U}(c, g)$  from public and private consumption, where the function  $\mathcal{U}$  is strictly increasing in  $c$  and  $g$  and concave. The marginal rate of substitution between public and private consumption is

$$MRS_{gc} \equiv \frac{\partial \mathcal{U} / \partial g}{\partial \mathcal{U} / \partial c} > 0.$$

We assume that  $\mathcal{U}$  is such that  $MRS_{gc}$  is a decreasing function of  $g/c$ ; for example,  $\mathcal{U}$  could be a constant-elasticity-of-substitution utility function. We also assume that  $MRS_{gc}(0) > 1$ .

To measure how the marginal rate of substitution varies with  $g/c$ , we introduce a second sufficient

statistic:

**DEFINITION 3:** *The elasticity of substitution between public and private consumption, denoted  $\epsilon$ , is given by*

$$\frac{1}{\epsilon} = -\frac{d \ln(MRS_{gc})}{d \ln(g/c)}.$$

The elasticity of substitution is positive because  $MRS_{gc}$  is decreasing in  $g/c$ . When  $\epsilon < 1$  public and private services are gross complements; when  $\epsilon = 1$  public and private services are independent; and when  $\epsilon > 1$  public and private services are gross substitutes.<sup>4</sup>

The elasticity of substitution has two interesting limits:  $\epsilon \rightarrow 0$  and  $\epsilon \rightarrow +\infty$ . When  $\epsilon \rightarrow 0$ , public and private consumption are perfect complements. A certain number of public services are needed for a given level of private consumption, but beyond that, additional public services have zero value and the marginal rate of substitution falls to zero. At this point, public workers dig and fill holes. When  $\epsilon \rightarrow +\infty$ , the public and private consumption are perfect substitutes. The marginal rate of substitution is constant at 1, such that households are equally happy to consume private or public services.<sup>5</sup>

We assume that households save what they do not spend. We also assume that the asset used for saving is in fixed supply; consequently, there are no predetermined variables in the model, and the equilibrium immediately converges to its steady-state position. Since the equilibrium is always in steady state, the social welfare associated with the equilibrium is simply  $\mathcal{U}(c, g)$ .

Having introduced a second good in the economy—the asset—we can be more precise about the price  $p$ : it is the price of services relative to the asset.

The household chooses how much to spend and save to maximize utility. As a result, the household demands a quantity  $c(x, p, g)$  of consumption. The demand depends negatively on the price  $p$  because a higher price makes consumption of services more costly relative to saving. The demand depends negatively on tightness  $x$  because a higher tightness makes purchasing services more difficult. Finally, the demand depends on public consumption  $g$  because public consumption may affect the marginal utility of private consumption. To consume  $c(x, p, g)$  services, the household purchases a total of  $C(x, p, g) = [1 + \tau(x)] c(x, p, g)$  services; the extra  $\tau(x)c(x, p, g)$  services are used for matching.<sup>6</sup>

<sup>4</sup>The Cobb-Douglas function  $\mathcal{U}(c, g) = c^{1-\gamma} g^\gamma$  has  $\epsilon = 1$ .

<sup>5</sup>The Leontief function  $\mathcal{U}(c, g) = \min\{c, g\}$  has  $\epsilon = 0$ . The linear function  $\mathcal{U}(c, g) = c + g$  has  $\epsilon \rightarrow +\infty$ .

<sup>6</sup>To purchase  $C(x, p, g)$  services, households post  $sC(x, p, g)/q(x)$  vacancies.

Next, the government demands an amount  $g$  of consumption. This requires the purchase of  $G = [1 + \tau(x)]g$  services.<sup>7</sup> The government balances its budget at all time with a lump-sum tax  $T = G$ . The total demand for consumption then is  $g + c(x, p, g)$ . We refer to  $c(x, p, g)$  as private demand and to  $g + c(x, p, g)$  as aggregate demand.<sup>8</sup>

Finally, we specify a price mechanism:  $p = p(x, g)$ . The price of services appears as a function of tightness  $x$  and public consumption  $g$ ; but since  $x$  and  $g$  determine all other variables in a feasible allocation, the price could be any function of any variable—it is as generic as possible. The price mechanism generally fails to maintain efficiency. Hence, policies correcting prices could be useful to bring unemployment closer to its efficient level.<sup>9</sup> To capture this possibility, we assume that the function  $p(x, g)$  embeds all such policies. If price policies ensure that unemployment is always efficient, our analysis trivially applies. Our analysis is more interesting when price policies cannot keep unemployment at its efficient level; it explores how public expenditure can improve welfare, taking all price policies as given.

Given the price mechanism and public expenditure, tightness adjusts to equalize aggregate supply and demand:

$$(6) \quad y(x, k) = c(x, p(x, g), g) + g.$$

This equation implicitly defines equilibrium tightness as a function  $x(g)$  of public consumption. Panel B in figure 1 shows how  $x(g)$  is given by the intersection of the aggregate-demand and aggregate-supply curves. The information about  $x(g)$  relevant to the policy analysis is conveyed by a third sufficient statistic:

**DEFINITION 4:** *The unemployment multiplier is given by*

$$m = -y \cdot \frac{du}{dg}.$$

The unemployment multiplier measures the percentage-point decrease in the unemployment rate observed when public consumption increases by one percent of total consumption.

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<sup>7</sup>To purchase  $G$  services, the government posts  $sG/q(x)$  vacancies.

<sup>8</sup>We express demand in terms of consumption because consumption matters for welfare and aggregate supply (4) is expressed with consumption. We could equivalently describe demand in terms of expenditure.

<sup>9</sup>In some contexts, monetary policy could be such a policy. See appendix B.

As unemployment is determined by tightness (through (2)), the unemployment multiplier is determined by the response of tightness to public consumption. As showed in figure 1, panel B, public consumption affects tightness by shifting the aggregate-demand curve. This shift occurs through a mechanical channel, as public consumption directly contributes to aggregate demand; a private-demand channel, as public consumption may affect private demand in various ways (for instance, by altering the marginal utility from private consumption); and a price channel, as public consumption may affect the price of services and thus private demand. Depending on the relative strength of these channels, the multiplier can be negative, positive, below one, or above one.

## 2.4. Demand Side and Equilibrium: An Example with Land

To provide an example of demand side, we describe a model in which households save using land, as in Iacoviello (2005) and Liu, Wang, and Zha (2013). This example illustrates how demand-side parameters influence the sufficient statistics. Appendix A contains the derivations, and appendix B provides other examples.

The representative household purchases a quantity  $l(t)$  of land. Land is traded on a perfectly competitive market and is in fixed supply,  $l_0$ . In equilibrium the land market clears so  $l(t) = l_0$ .

The household derives utility from holding land, for instance from the housing services it provides. The household's instantaneous utility function is  $\mathcal{U}(c(t), g(t)) + \mathcal{V}(l(t))$ , where  $\mathcal{V}$  is strictly increasing and concave. We use a constant-elasticity-of-substitution specification for  $\mathcal{U}$ :

$$(7) \quad \mathcal{U}(c, g) = \left[ (1 - \gamma)^{\frac{1}{\epsilon}} c^{\frac{\epsilon-1}{\epsilon}} + \gamma^{\frac{1}{\epsilon}} g^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}.$$

The parameter  $\gamma \in (0, 1)$  indicates the value of public services relative to private services, and the parameter  $\epsilon > 0$  gives the elasticity of substitution between public and private consumption. The household's utility at time 0 is

$$(8) \quad \int_0^{+\infty} e^{-\delta t} [\mathcal{U}(c(t), g(t)) + \mathcal{V}(l(t))] dt,$$

where  $\delta > 0$  is the time discount rate. The law of motion of the household's land holding is

$$(9) \quad \dot{l}(t) = p(t) [1 - u(x(t))] k - p(t) [1 + \tau(x(t))] c(t) - T(t).$$

In the law of motion,  $p(t)[1 - u(x(t))]$  is the household's labor income,  $p(t)[1 + \tau(x(t))]$  is its spending on services, and  $T(t)$  is the lump-sum tax financing public expenditure.

The household takes  $l(0)$  and the paths of  $x(t)$ ,  $g(t)$ ,  $p(t)$ , and  $T(t)$  as given. It chooses the paths of  $c(t)$  and  $l(t)$  to maximize (8) subject to (9). Setting up the usual Hamiltonian, we obtain the first-order conditions of the maximization:

$$(10) \quad \frac{\partial \mathcal{U}}{\partial c}(c(t), g(t)) = \lambda(t)p(t)[1 + \tau(x(t))],$$

$$(11) \quad \mathcal{V}'(l(t)) = \delta\lambda(t) - \dot{\lambda}(t),$$

where  $\lambda(t)$  is the costate variable associated with land.

Given public consumption  $g$ , an equilibrium consists of paths for  $x(t)$ ,  $c(t)$ ,  $l(t)$ ,  $p(t)$ , and  $\lambda(t)$  that satisfy five equations: (10), (11),  $p(t) = p(x(t), g)$ ,  $l(t) = l_0$ , and  $y(x(t)) = c(t) + g$ . The fifth equation imposes that supply equals demand on the services market. All the variables can be recovered from the costate variable  $\lambda(t)$ , so the equilibrium reduces to a dynamical system of dimension one, with variable  $\lambda(t)$ . As  $\lambda(t)$  is nonpredetermined and the dynamical system is a source, the equilibrium jumps to its steady-state position at  $t = 0$ . Hence, the equilibrium is always in steady state.

In section 2.3, we introduce a generic private demand,  $c(x, p, g)$ , and a generic price mechanism,  $p(x, g)$ . We now compute the private demand in this model with land and propose a possible price mechanism. To compute the equilibrium, we would then plug this private demand and price mechanism into (6) and compute equilibrium tightness. Next, we would use tightness and various supply-side relationships to compute the other variables.

To compute the private demand, we combine (10) and (11):

$$(12) \quad \frac{\partial \mathcal{U}}{\partial c}(c, g) = (1 + \tau(x))p \frac{\mathcal{V}'(l_0)}{\delta}.$$

The equation says that the household is indifferent between purchasing one private service, which costs  $(1 + \tau(x))p$  units of land and yields utility  $\partial \mathcal{U} / \partial c$ , and purchasing  $(1 + \tau(x))p$  units of land, which costs the same amount and yields utility  $\mathcal{V}'(l_0) / \delta$  over a lifetime. We then combine (12) with (7) and find that private demand  $c$  is implicitly defined by

$$(13) \quad \left[ (1 - \gamma) + \gamma^{\frac{1}{\epsilon}} (1 - \gamma)^{\frac{\epsilon-1}{\epsilon}} \left( \frac{g}{c} \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{1}{\epsilon-1}} = (1 + \tau(x))p \frac{\mathcal{V}'(l_0)}{\delta}.$$

If the marginal utility of land goes up or the time discount rate goes down, households desire to save more and consume less, which depresses private demand. With price rigidity, such a negative demand shock leads to lower tightness and higher unemployment.

The price mechanism that we propose is rigid—in the sense that it does not respond to demand shocks—and yields a simple expression for the multiplier:

$$(14) \quad p(g) = p_0 \left[ (1 - \gamma) + \gamma^{\frac{1}{\epsilon}} \left( (1 - \gamma) \frac{g}{y^* - g} \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{1-r}{\epsilon-1}},$$

where  $p_0 > 0$  governs the price level,  $y^*$  is the efficient level of total consumption, and  $r$  determines the effect of public consumption on prices. In fact, if  $r < 1$ , the price is increasing in  $g$ ; if  $r = 1$ , the price is fixed; and if  $r > 1$ , the price is decreasing in  $g$  (this seems less realistic).

The parameter  $r$  is the main determinant of the unemployment multiplier:

$$(15) \quad m = \frac{(1 - u^*)r}{(1 - \gamma)\epsilon}.$$

The multiplier is positive, except if  $r < 0$ .<sup>10</sup> Another important determinant of the multiplier is the elasticity of substitutes between public and private consumption: the multiplier is smaller when public and private consumption are stronger substitutes (higher  $\epsilon$ ), and zero when they are perfect substitutes ( $\epsilon \rightarrow \infty$ ). When public and private consumption are stronger substitutes, public consumption replaces private consumption more easily, so an increase in public consumption reduces private demand more, explaining the lower multiplier.

## 2.5. Comparison with the Diamond-Mortensen-Pissarides Model

Our model shares many features with the standard matching model—the Diamond-Mortensen-Pissarides (DMP) model. Such features include the matching function, random search, long-term relationships, hiring through vacancies, fixed productive capacity, and the central role of market tightness. But it also differs from the DMP model on various aspects. Here we describe the differences and explain how they make our model more suited to the analysis of optimal public expenditure.

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<sup>10</sup>Expression (15) is valid when unemployment is efficient and public expenditure optimal. Otherwise the multiplier admits another expression, slightly more complicated but with the same properties. When  $r < 0$ , an increase in public consumption raises the price of services so much that it reduces private demand more than one-for-one.



Our reference is the textbook version of the DMP model, developed by Pissarides (2000).

First, our model is more general than the DMP model, making it more suited to the sufficient-statistic approach. The price mechanism is more general: it is not restricted to Nash bargaining. This generalization allows for a broader range of multipliers and unemployment gaps. Functional forms are also more general, allowing for a downward-sloping demand curve in the  $(y, x)$  plan. With such a demand curve, public spending usually affects tightness, and public consumption does not usually crowd out private consumption one-for-one. In contrast, in the DMP model, the demand curve is horizontal in the  $(y, x)$  plan. Hence, public spending does not usually affect tightness, and public consumption usually crowds out private consumption one-for-one (Michaillat 2014).

Second, our formulation of the efficiency condition is more general. In the DMP model the Hosios (1990) condition says that unemployment is efficient when workers' bargaining power equals the matching elasticity. Our efficiency condition, given by (5), is more general than the Hosios condition because it is not tied to Nash bargaining: it applies to any price mechanism. Instead of giving the bargaining power leading to efficiency, our condition gives the relationship satisfied by observable variables (unemployment and matching wedge) when unemployment is efficient.

Several additional, cosmetic differences make our matching model closer to the Walrasian model—the workhorse model in public economics. These differences make it easier to use public-economic tools and to compare our findings with canonical public-economic results.

First, we model a service economy instead of a labor market: services are traded instead of labor; the trading price is the price of services instead of the real wage; buyers are households (and the government) instead of firms; and sellers are self-employed workers instead of jobseekers.

Second, the Beveridge curve is recast as an aggregate-supply curve and the job-creation condition as an aggregate-demand curve.<sup>11</sup> The aggregate-supply curve is mathematically equivalent to the Beveridge curve, and the aggregate-demand curve to the job-creation condition, but our curves are closer to the Walrasian concepts of supply and demand.

Third, the condition determining equilibrium tightness is recast as a supply-equals-demand condition. In fact, it is useful to think of tightness as another price: in equilibrium both actual price and tightness ensure that supply equals demand (Michaillat and Saez 2015, pp. 526–529). The matching framework can thus be seen as a generalization of the Walrasian framework—where

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<sup>11</sup>In Pissarides (2000), the Beveridge curve is equation (1.5) and the job-creation condition is equation (1.9). In this paper, the aggregate-supply curve is (4) and in the example with land the aggregate-demand curve is (13).

only the price equalizes supply and demand. But unlike in the Walrasian model, where productive efficiency is respected whenever supply equals demand, equilibria in the matching model are generally inefficient.

Fourth, since we use the supply-demand formalism, the graphical representation of the equilibrium is different. In the DMP model the equilibrium is the intersection of the Beveridge and job-creation curves in an (unemployment, vacancy) plan. In our model the equilibrium is the intersection of the aggregate-supply and aggregate-demand curves in a (output, tightness) plan.<sup>12</sup>

Fifth, the recruiting cost takes a different form. In the DMP model, the vacancy-posting cost is measured in terms of final good, so there are effectively two goods in the economy—labor and final good. This complicates the welfare analysis. Here the cost is measured in terms of services, so there is a single good in the economy. This simplifies the welfare analysis: once consumption is defined as output net of recruiting services, welfare solely depends on consumption.

Sixth, while the DMP model focuses on atomistic workers and vacancies, our model studies households selling and buying many services. This brings the model closer to the Walrasian framework, in which agents buy and sell many goods. Furthermore, since households buy and sell many services, we can avoid heterogeneity across households and hence purge the welfare analysis from insurance problems.

### 3. A Sufficient-Statistic Formula for Optimal Public Expenditure

We use our matching model to derive a sufficient-statistic formula for optimal public expenditure. The main implication of the formula is that whenever unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to reduce the unemployment gap.

#### 3.1. Derivation

We determine the public consumption  $g$  that maximizes welfare  $\mathcal{U}(c, g)$ . In equilibrium,  $c = y(x, k) - g$  and  $x = x(g)$ . Thus, the optimal  $g$  maximizes  $\mathcal{U}(y(x(g), k) - g, g)$ . The first-order condition of the maximization is

$$(16) \quad 0 = \frac{\partial \mathcal{U}}{\partial g} - \frac{\partial \mathcal{U}}{\partial c} + \frac{\partial \mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

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<sup>12</sup>In Pissarides (2000), the equilibrium is depicted in figure 1.2. Here, the equilibrium is depicted in figure 1.

We assume that the maximization problem is well behaved:  $g \mapsto \mathcal{U}(y(x(g), k) - g, g)$  admits a unique extremum, and the extremum is an interior maximum. Under this assumption, (16) is a necessary and sufficient condition for optimality. Equation (16) shows that an increase in public consumption affects welfare through three channels: it mechanically raises welfare (first right-hand-side term); for a given level of total consumption, it reduces private consumption one-for-one, which lowers welfare (second right-hand-side term); and it affects tightness and thus total consumption, which further changes private consumption (third right-hand-side term).

Dividing (16) by  $\partial \mathcal{U} / \partial c$ , we obtain the following lemma:

**LEMMA 2:** *Optimal public expenditure satisfies*

$$(17) \quad \underbrace{1 = MRS_{gc}}_{\text{Samuelson rule}} + \underbrace{\frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}.$$

Equation (17) shows that in a matching model the Samuelson rule needs to be corrected. The correction term is the effect of public consumption on tightness,  $dx/dg$ , times the effect of tightness on total consumption,  $\partial y / \partial x$ , so it measures the effect of public consumption on total consumption,  $dy/dg$ . The correction term is positive whenever an increase in public consumption leads to an increase in total consumption.<sup>13</sup>

A first insight from (17) is that at the optimum, public consumption must be crowding out private consumption ( $dc/dg < 0$ ). Indeed, since  $MRS_{gc} > 0$ , (17) imposes that  $dy/dg < 1$  and  $dc/dg = dy/dg - 1 < 0$ . Our theory allows for either crowding in or crowding out of private consumption by public consumption; but if there is crowding in ( $dc/dg > 0$ ), public consumption cannot be optimal. From a situation of crowding in, the government can improve welfare by increasing public consumption until it starts crowding out private consumption. Crowding out necessarily happens at some point because once unemployment is efficient, total consumption is maximized and crowding out is one-for-one.

A second insight from (17) is that the Samuelson rule, which was originally derived in a neoclassical model, remains valid in a model with unemployment as long as unemployment is efficient. Indeed, when unemployment is efficient, consumption is maximized ( $\partial y / \partial x = 0$ ), so the

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<sup>13</sup>Formula (17) is closely related to the optimal unemployment-insurance formula in Landais, Michailat, and Saez (2018b, eq. (23)). The two formulas show that in matching models standard optimal policy formulas need to be corrected with a term that is positive whenever the policy improves welfare through tightness.

correction term is zero.

When unemployment is inefficient, consumption is below its maximum ( $\partial y/\partial x \neq 0$ ), and optimal public spending may deviate from the Samuelson rule. To describe such deviation, we decompose public spending in two components:

**DEFINITION 5:** *Samuelson spending  $(g/c)^*$  is given by the Samuelson rule:  $MRS_{gc}((g/c)^*) = 1$ . Stimulus spending is given by  $g/c - (g/c)^*$ .*

Since  $MRS_{gc}(0) > 1$  and  $MRS_{gc}$  is decreasing in  $g/c$ , Samuelson spending is always well defined.

Next, we express the elements of (17) with our three sufficient statistics: the elasticity of substitution between public and private consumption  $\epsilon$ , the unemployment gap  $u - u^*$ , and the unemployment multiplier  $m$ .

**LEMMA 3:** *The term  $1 - MRS_{gc}$  can be approximated as follows:*

$$(18) \quad 1 - MRS_{gc} \approx \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*},$$

where  $\epsilon$  is evaluated at  $g/c$ . The approximation is valid up to a remainder that is  $O\left([g/c - (g/c)^*]^2\right)$ . The term  $\partial y/\partial x$  can be approximated as follows:

$$(19) \quad \frac{x}{y} \cdot \frac{\partial y}{\partial x} \approx \frac{u - u^*}{1 - u^*}.$$

The approximation is valid up to a remainder that is  $O\left([u - u^*]^2\right)$ . Last, the term  $dx/dg$  satisfies

$$(20) \quad \frac{y}{x} \cdot \frac{dx}{dg} = \frac{m}{(1 - \eta)(1 - u)u}.$$

The proof of the lemma is relegated to appendix C. Equations (18) and (20) immediately follow from the definitions of the elasticity of substitution and the unemployment multiplier, but the derivation of equation (19) is more complex.

Using lemma 3, we prove in appendix C that (17) can be rewritten as follows:

**LEMMA 4:** *Optimal stimulus spending satisfies*

$$(21) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx z_0 \cdot \epsilon \cdot m \cdot \frac{u - u^*}{u^*},$$

where  $\epsilon$  and  $m$  are evaluated at  $[g/c, u]$  and  $z_0 = 1 / [(1 - \eta)(1 - u^*)^2]$ . The approximation is valid up to a remainder that is  $O([u - u^*]^2 + [g/c - (g/c)^*]^2)$ .

If the current values of stimulus spending and our three sufficient statistics satisfy (21), then stimulus spending is optimal. Thus (21) is useful to assess whether current stimulus spending is optimal or not. But (21) cannot be used to compute optimal stimulus spending. The root of the problem is that the sufficient statistics (especially the unemployment gap) are implicit functions of stimulus spending. To understand this problem, imagine that we plug the current values of the statistics in (21); the formula indicates some stimulus spending. The government could then adjust current public spending to achieve the indicated stimulus spending. As public spending changes, however, the sufficient statistics also change. Once the indicated stimulus spending is reached, it is very likely that (21) does not hold any longer. Hence, the stimulus spending initially indicated by (21) is not optimal. This is a typical limitation of the sufficient-statistic approach (Chetty 2009), which we now address by developing a new sort of sufficient-statistic formula.

We assume that public expenditure is at the Samuelson level  $(g/c)^*$  and unemployment is at an inefficient rate  $u_0 \neq u^*$ . We have in mind the following scenario. Initially everything is going well: unemployment is efficient, and public expenditure satisfies the Samuelson rule. Then a shock occurs, pushing unemployment to  $u_0$ . The shock could be anything: aggregate-demand shock, aggregate-supply shock, shock to the price of services, shock to the matching function, or shock to the separation rate.

Given the initial unemployment gap  $u_0 - u^*$ , we aim to compute optimal stimulus spending  $g/c - (g/c)^*$ . As  $g/c$  deviates from  $(g/c)^*$ , unemployment responds, so as we have just discussed, we cannot plug  $u_0 - u^*$  into (21) to compute the optimal policy. Instead, we take the response of unemployment into account, and we transform (21) into an explicit formula—a formula expressed with sufficient statistics independent of policy.

**PROPOSITION 1:** *Suppose that the economy is initially at an equilibrium  $[(g/c)^*, u_0]$ . Then*

optimal stimulus spending satisfies

$$(22) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx \frac{z_0 \epsilon m}{1 + z_1 z_0 \epsilon m^2} \cdot \frac{u_0 - u^*}{u^*},$$

where  $\epsilon$  and  $m$  are evaluated at  $[(g/c)^*, u_0]$ ,  $z_0 = 1 / [(1 - \eta)(1 - u^*)^2]$ , and  $z_1 = (g/y)^*(c/y)^*/u^*$ .

Under the optimal policy, the unemployment rate is

$$(23) \quad u \approx u^* + \frac{u_0 - u^*}{1 + z_1 z_0 \epsilon m^2}.$$

The approximations (22) and (23) are valid up to a remainder that is  $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$ .

The formal proof, presented in appendix C, builds on a simple argument: since the unemployment multiplier  $m$  is proportional to  $du/dg$ , a first-order Taylor expansion of  $u$  at  $u_0$  yields

$$u \approx u_0 - \text{constant} \cdot m \cdot \frac{g/c - (g/c)^*}{(g/c)^*}.$$

Substituting  $u$  by this expression in (21) yields (22).

Formula (22) expresses optimal stimulus spending  $g/c - (g/c)^*$  as a function of three sufficient statistics: initial elasticity of substitution between public and private consumption ( $\epsilon$ ), initial unemployment multiplier ( $m$ ), and initial unemployment gap ( $u_0 - u^*$ ). Formula (23) expresses the unemployment rate under optimal public expenditure as a function of the same statistics. The advantage of (22) over (21) is that its sufficient statistics are independent of policy. Thus, we can compute optimal stimulus spending by plugging the current values of the statistics into (22).

The policy debate on stimulus spending often revolves around unemployment gaps and multipliers (for example, Romer and Bernstein 2009). Formula (22) confirms that optimal stimulus spending is indeed a function of the unemployment gap and a multiplier—the unemployment multiplier. Yet, these statistics are not sufficient to measure the effect of public expenditure on welfare because an increase in public expenditure also modifies the composition of households' consumption. Consequently, optimal stimulus spending also depends on the elasticity of substitution between public and private consumption; this statistic should probably play a more prominent role in the policy debate.

As policy discussions usually focus on the output multiplier, we translate our results in terms of

that multiplier. This is not difficult because unemployment and output multipliers are closely related. We start by introducing a new unemployment multiplier.

**DEFINITION 6:** *The empirical unemployment multiplier is given by*

$$M = -\frac{Y}{1-u} \cdot \frac{du}{dG}.$$

The empirical unemployment multiplier measures the percent increase in the employment rate observed when public expenditure increases by one percent of GDP. In practice  $1-u \approx 1$ , so the multiplier approximately measures the percentage-point decrease in the unemployment rate observed when public expenditure increases by one percent of GDP.

The empirical multiplier  $M$  plays an important role in the applications of sections 4 and 5 because it is much easier to estimate than the multiplier  $m$  in our formula. The multiplier  $m$  is difficult to estimate because it measures the response of unemployment to a change in public consumption, which is not directly observable. The issue is that public expenditure on matching resources—which are unobserved—must be subtracted from total public expenditure to obtain public consumption. The multiplier  $M$ , on the other hand, measures the response of unemployment to a change in public expenditure, which is reported in national accounts. As  $m$  and  $M$  are closely related, we will use empirical estimates of  $M$  to calibrate  $m$  in the formulas.<sup>14</sup>

Additionally, the empirical unemployment multiplier acts as a bridge between the unemployment multiplier in our formula and the output multiplier:

**LEMMA 5:** *The unemployment multipliers  $m$  and  $M$  are related by*

$$(24) \quad m = \frac{(1-u) \cdot M}{1 - \frac{G}{Y} \cdot \frac{\eta}{1-\eta} \cdot \frac{\tau}{u} \cdot M}.$$

*Furthermore, empirical unemployment multiplier and output multiplier are equal:  $M = dY/dG$ .*

The proof of the lemma is in appendix C. However, it is easy to see why empirical unemployment multiplier and output multiplier are the same: because public expenditure must necessarily reduce the unemployment rate to raise output.

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<sup>14</sup>Although public consumption  $g$  and private consumption  $c$  are not observable, the consumption ratio  $g/c$  can be measured from national accounts, because  $g/c = G/C$  and both public expenditure  $G$  and private expenditure  $C$  are observable in national accounts. Thus, our formula is usable once  $m$  is replaced by  $M$ .

The lemma implies that  $m$  and  $M$  always have the same sign, and  $m$  is increasing in  $M$ . Moreover, the lemma shows that  $M$  and the output multiplier are equal. Thus, our formula can be expressed with the output multiplier by replacing  $m$  by the function of  $M$  given by (24) and then replacing  $M$  by  $dY/dG$ . After this manipulation, it appears that the output multiplier affects optimal stimulus spending in the same way as the unemployment multiplier  $m$ .

A caveat is that the output multiplier is only useful when taxation is nondistortionary. Section 3.3 shows that when taxation is distortionary, the link between unemployment and output multipliers breaks down, and it is necessary to use the unemployment multiplier to design stimulus spending.

## 3.2. Implications

Using our sufficient-statistic formula, given by (22), we explore how the sign and amplitude of optimal stimulus spending depend on the unemployment gap, unemployment multiplier, and elasticity of substitution between public and private consumption. We also use (23) to describe the properties of the unemployment gap under optimal stimulus spending.

**Sign of Optimal Stimulus Spending.** Formula (22) gives the sign of optimal stimulus spending in various situations:

**PROPOSITION 2:** *If the unemployment multiplier is zero ( $m = 0$ ), or the unemployment gap is zero ( $u_0 = u^*$ ), optimal stimulus spending is zero ( $g/c = (g/c)^*$ ). In all other situations, optimal public expenditure deviates from the Samuelson rule to partially fill the initial unemployment gap. Consider first a positive unemployment multiplier ( $m > 0$ ). If the unemployment gap is positive ( $u_0 > u^*$ ), optimal stimulus spending is positive ( $g/c > (g/c)^*$ ) but does not completely fill the unemployment gap ( $u > u^*$ ). If the unemployment gap is negative ( $u_0 < u^*$ ), optimal stimulus spending is negative ( $g/c < (g/c)^*$ ) but does not completely eliminate the unemployment gap ( $u < u^*$ ). If the unemployment multiplier is negative ( $m < 0$ ), the sign of optimal stimulus spending is the opposite.*

It is only if the unemployment multiplier is always zero or the unemployment gap is always zero that public expenditure should always be at the Samuelson level. In all other situations, optimal public expenditure deviates from the Samuelson rule.



The general pattern is that public expenditure should deviate from the Samuelson level to partially fill the initial unemployment gap. To understand these results, imagine that public expenditure is at the Samuelson level, the unemployment multiplier is positive, and unemployment is inefficiently high. Keeping total consumption constant, increasing public consumption by one service reduces private consumption by one service. Since the marginal utilities of public and private consumption are equalized at the Samuelson rule, the increase in public expenditure has no first-order effect on welfare so far. Yet, since the unemployment multiplier is positive, increasing public consumption lowers unemployment; and since unemployment is inefficiently high, reducing unemployment raises total consumption. Hence, through its effect on unemployment, the increase in public expenditure raises welfare. We conclude that increasing public expenditure above the Samuelson level, and thus reducing the unemployment gap, is optimal.

Why is it not optimal to completely fill the unemployment gap? If the government did that, we would reach a situation where one unit of public consumption costs one unit of private consumption (since crowding out is one-for-one when the unemployment gap is zero) but is less valuable than one unit of private consumption (since public spending is above Samuelson spending). As welfare can be increased by reducing public consumption, the situation is suboptimal.

These results have implications for the optimal cyclicality of public expenditure. Under the presumptions that the unemployment gap is positive in slumps but negative in booms, and that the unemployment multiplier is nonzero with a constant sign, then optimal stimulus spending changes sign over the business cycle. Thus optimal public expenditure fluctuates over the business cycle around the Samuelson level.

**Role of the Unemployment Multiplier.** We determine how optimal stimulus spending depends on the unemployment multiplier. From (22) and (23), we obtain the following proposition:

**PROPOSITION 3:** *Assume that the initial unemployment gap is positive ( $u_0 > u^*$ ). Let  $m^\dagger = 1/\sqrt{z_1 z_0 \epsilon}$ . Then optimal stimulus spending is a hump-shaped function of the unemployment multiplier: it is 0 when  $m = 0$ , increasing in  $m$  for  $m \in [0, m^\dagger]$ , maximized at  $m = m^\dagger$ , decreasing in  $m$  for  $m \in [m^\dagger, +\infty)$ , and 0 for  $m \rightarrow +\infty$ . The maximum optimal stimulus spending (reached at  $m = m^\dagger$ ) is*

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1}{2} \cdot \sqrt{\frac{z_0 \epsilon}{z_1}} \cdot \frac{u_0 - u^*}{u^*}.$$

*Furthermore, the unemployment gap under optimal stimulus spending is a decreasing function of the unemployment multiplier: it falls from  $u_0 - u^*$  when  $m = 0$  to 0 when  $m \rightarrow +\infty$ .*

The proposition shows that for a given unemployment gap, optimal stimulus spending is a hump-shaped function of the unemployment multiplier: it is increasing in the multiplier until the multiplier reaches  $1/\sqrt{z_1 z_0 \epsilon}$ , and decreasing after that. The proposition also gives the maximum optimal stimulus spending. For concreteness this proposition and the next only consider positive unemployment multipliers and unemployment gaps, but we could of course derive corresponding results with negative multipliers and gaps.

What is the intuition behind the hump shape? When public expenditure is optimal, the marginal social cost from consuming too many public services and too few private services equals the marginal social value from reducing unemployment. This marginal social value is determined by two factors: the current unemployment multiplier, which measures how much unemployment can be reduced by additional expenditure, and the current unemployment gap, which measures the social value from lower unemployment. For a given amount of stimulus spending and a given initial unemployment gap, a larger initial multiplier has conflicting effects on the two factors: it means a larger current multiplier (a higher marginal social value) but a smaller current unemployment gap (a lower marginal social value). The first effect advocates for more spending but the second for less spending. It turns out that the first effect dominates for small multipliers, so optimal stimulus spending is increasing in the multiplier; but the second effect dominates for large multipliers, so optimal stimulus spending is decreasing in the multiplier. In fact, for large multipliers, it becomes optimal to nearly entirely fill the unemployment gap; since less spending is required to fill the gap when the multiplier is larger, optimal stimulus spending is decreasing in the multiplier.

Our results qualify the view that a larger multiplier entails a larger stimulus spending—the bang-for-the-buck logic often used in policy discussions (see Mankiw and Weinzierl 2011, p. 212). Stimulus skeptics usually believe in small multipliers and infer that stimulus spending should be small or zero in slumps; similarly, stimulus advocates usually believe in large multipliers and infer that stimulus spending should be large in slumps. Our theory shows that the bang-for-the-buck logic does hold for small multipliers, but not for large ones. Therefore, a large multiplier is not a justification for a large stimulus. Instead, the relationship between multiplier and optimal stimulus spending is hump-shaped; as a result, optimal stimulus spending is quite similar for some small and

large multipliers.

**Role of the Elasticity of Substitution Between Public and Private Consumption.** We determine how optimal stimulus spending depends on the elasticity of substitution between public and private consumption. From (22), we immediately obtain the following results:

**PROPOSITION 4:** *Assume that the unemployment multiplier and initial unemployment gap are positive ( $m > 0$  and  $u_0 - u^* > 0$ ). Then optimal stimulus spending is an increasing function of the elasticity of substitution between public and private consumption: it rises from 0 when  $\epsilon = 0$  to*

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1}{z_1 m} \cdot \frac{u_0 - u^*}{u^*}$$

*when  $\epsilon \rightarrow +\infty$ . The unemployment gap under the optimal policy is a decreasing function of the elasticity of substitution: it falls from  $u_0 - u^*$  when  $\epsilon \rightarrow 0$  to 0 when  $\epsilon \rightarrow +\infty$ .*

The proposition shows that both optimal stimulus spending and the share of the unemployment gap filled under the optimal policy are increasing in the elasticity of substitution between public and private consumption with two interesting polar cases.

The first is  $\epsilon \rightarrow 0$ . In this case, additional public services have zero value: additional public workers “dig and fill holes in the ground”, and optimal stimulus spending is zero, irrespective of the unemployment rate and multiplier. Intuitively, public consumption beyond the Samuelson level is useless. Since public consumption crowds out private consumption in the vicinity of  $u^*$ , it is never optimal to provide more public consumption than in the Samuelson rule.<sup>15</sup>

The second special case is  $\epsilon \rightarrow +\infty$ . Then, the public services provided by the government perfectly substitute for the private services purchased by households. In this case, optimal stimulus spending completely fills the unemployment gap such that  $u = u^*$ . This result holds even if the

<sup>15</sup>The results in proposition 4 are based on (22), which is a first-order approximation around  $[u^*, (g/c)^*]$ . When  $u_0$  is relatively far from  $u^*$  and  $g/c$  from  $(g/c)^*$ , the equation may not be accurate and results may change. In particular, when  $\epsilon \rightarrow 0$  the first-order approximation of  $MRS_{gc}$  works well only very close to  $(g/c)^*$ , so (22) works well only if the optimal  $g/c$  is close to  $(g/c)^*$ . However, when public consumption crowds in private consumption ( $dc/dg > 0$ ), the optimal  $g/c$  is not close to  $(g/c)^*$  and (22) does not work well. (Note that having  $dc/dg > 0$  requires a large deviation from  $u^*$  because at  $u^*$  total consumption is maximized so  $dc/dg = -1$ .) Equation (22) suggests that stimulus spending should be zero in this situation. But going back to (17)—which can be written  $0 = MRS_{gc} + dc/dg$ —we see that optimal stimulus spending is positive when  $dc/dg > 0$ , even if  $\epsilon \rightarrow 0$ . Indeed then  $MRS_{gc}(g/c) = 0$  as soon as  $g/c > g/c^*$ ; nevertheless, since  $dc/dg > 0$  at  $(g/c)^*$ , it is optimal to increase spending above  $(g/c)^*$  and continue spending until  $dc/dg$  falls to 0, which necessarily occurs when unemployment is close enough to  $u^*$ .

multiplier is very small and public expenditure severely crowds out private consumption. Intuitively, public and private consumption are interchangeable, so it is optimal to maximize total consumption, irrespective of its composition. Accordingly, it is optimal to completely fill the unemployment gap.

Overall, proposition 4 clarifies the link between usefulness of public expenditure and optimal stimulus spending. A concern of stimulus skeptics is that additional public expenditure could be wasteful. Our theory develops this argument. It is true that when the elasticity of substitution between public and private consumption is zero, public expenditure should remain at the Samuelson level. But in the more realistic case where the elasticity of substitution is positive, some stimulus spending is indeed desirable in slumps.

### 3.3. Distortionary Taxation

So far taxation has been nondistortionary because labor supply was fixed. We now introduce endogenous labor supply: households supply a productive capacity  $k$  at utility cost  $\mathcal{W}(k)$ , where the function  $\mathcal{W}$  is strictly increasing and convex. We examine how distortionary taxation affects optimal public expenditure. Here we present a summary of the results; appendix D contains the complete analysis.

The government uses a linear income tax  $\tau^L$  to finance public expenditure. With such a tax, the household's labor income is  $(1 - \tau^L)Y(x, k) = (1 - \tau^L)(1 - u(x))k$ . To finance public expenditure  $G$ , the tax rate must be  $\tau^L = G/Y = g/y$ .

The household chooses  $k$  to maximize utility. The labor supply decision is distorted by the income tax: a higher tax reduces the returns to supplying labor and thus reduces the capacity  $k$  supplied by the household through substitution effects. Because of this distortion, the formula for optimal public expenditure is modified as follows:

$$1 - \underbrace{\frac{d \ln(k)}{d \ln(g)}}_{\text{modified Samuelson rule}} = \underbrace{MRS_{gc} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}.$$

This optimality condition differs from (17), but the two have the same structure once the Samuelson rule is modified to account for distortionary taxation.<sup>16</sup> The statistic  $1 - d \ln(k)/d \ln(g) > 1$  is the

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<sup>16</sup>The modified Samuelson rule was developed by Stiglitz and Dasgupta (1971), Diamond and Mirrlees (1971), and Atkinson and Stern (1974) to describe optimal public expenditure with a linear income tax. A large literature has built on these papers (see Ballard and Fullerton 1992; Kreiner and Verdellin 2012).

marginal cost of funds. It is more than one because the linear income tax distorts the labor supply. Because the marginal cost of funds is greater than one, the modified Samuelson rule recommends lower public expenditure than the regular Samuelson rule.

With a linear income tax, Samuelson spending is lower, but the correction to the Samuelson rule remains the same. Accordingly, our sufficient-statistic formula remains valid. As long as the statistic  $z_1$  is generalized to allow for supply-side responses, optimal stimulus spending satisfies (22) and the optimal unemployment rate satisfies (23).

Section 3.1 shows that when taxes are nondistortionary, unemployment and output multipliers are equal, so all our results can be reformulated with the output multiplier. Things are different when taxes are distortionary. Higher taxes reduce labor supply, which reduces output but not unemployment. The output multiplier becomes smaller than the unemployment multiplier. Therefore the output multiplier cannot be used to design optimal public expenditure—only the unemployment multiplier can be used.

Stimulus skeptics are concerned that output is already too low in slumps and that increasing taxes to fund stimulus spending would further reduce output through supply-side responses.<sup>17</sup> Nonetheless, our theory shows that if the unemployment multiplier is positive, stimulus spending should be positive in slumps—even if the output multiplier is negative. How can such a policy be optimal? Starting from the modified Samuelson rule, a small increase in public expenditure reduces unemployment, reduces labor supply, and increases public consumption, which are all good for welfare; but it reduces output and thus private consumption, which is bad for welfare. At the modified Samuelson rule, the cost of lower private consumption offsets the benefits of higher public consumption and lower labor supply; the only remaining effect on welfare is the positive effect from lower unemployment. Therefore increasing public expenditure above the modified Samuelson rule is indeed desirable. The key is that the negative welfare effect of the reduction in output stemming from lower labor supply has already been taken into account by the modified Samuelson rule; what has not been accounted for is the positive welfare effect of the increase in output stemming from lower unemployment.<sup>18</sup>

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<sup>17</sup>Barro and Redlick (2011) find in US data that the deficit-financed output multiplier is positive (around 0.5), but because taxation significantly depresses output, the balanced-budget output multiplier is negative (around -0.6).

<sup>18</sup>Here we have considered the traditional approach to taxation: public spending is funded with a linear income tax. In appendix D, we also consider the modern approach to taxation, which follows the benefit principle. This principle, introduced by Hylland and Zeckhauser (1979) and fully developed by Kaplow (1996, 1998), is an important result in modern public-economic theory: it states that the optimal provision of public expenditure is disconnected

### 3.4. Comparison with a Keynesian, Fixprice Model

We have shown that the Samuelson rule breaks down when productive efficiency fails, which happens when the price of services is not at the efficient level. We treat the failure of productive efficiency using a matching model, but in macroeconomics failures of productive efficiency are usually studied using models in which prices are fixed at some inefficient level, in the tradition of Barro and Grossman (1971) and Bénassy (1993).<sup>19</sup> Here we apply our methodology to such a fixprice model and compare the results of the two approaches.

The economy has the same structure as in the model of section 2, except that services are traded on a perfectly competitive market instead of a matching market. The price of services is fixed at a level  $p$ , which may not be the market-clearing level. The private demand for services is given by a function  $c(p, g)$ , with  $\partial c / \partial p < 0$ . The aggregate demand for services is  $y(p, g) = c(p, g) + g$ . The aggregate supply of services is fixed at  $k$ . Since there is no wedge between output and consumption,  $y$ ,  $c$ , and  $g$  are both output and consumption of services. This fixprice model can be seen as the limit case of our matching model when matching costs become vanishingly small ( $\rho \rightarrow 0$ ). Hence, all the equations from the matching model apply once we set  $\tau(x) = 0$ .<sup>20</sup>

Since  $c = y - g$ , the optimal  $g$  maximizes  $\mathcal{U}(y - g, g)$ . The first-order condition of the maximization is

$$(25) \quad 1 = MRS_{gc} + \frac{dy}{dg}.$$

This is the same condition as (17) in the matching model. The implications of the optimality condition are different than in the matching model because of the values taken by the output multiplier  $dy/dg$  in the fixprice model.

In equilibrium, the price of services clears the market for services, so aggregate demand equals aggregate supply. In that case,  $y = k$  so  $dy/dg = 0$  and  $MRS_{gc} = 1$ : the Samuelson rule holds.

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from distortionary taxation. Extra public expenditure is financed by a tax change leaving all individual utilities unchanged—thus not altering labor supply. This is done by changing a nonlinear income tax schedule to absorb the benefits derived from extra public spending, individual by individual. Under the benefit principle, even though capacity is endogenous, all the formulas obtained with fixed capacity hold.

<sup>19</sup>New Keynesian models build upon this tradition, but replace fixed prices with slowly adjusting prices. Slow price adjustments make the equilibrium dynamics more interesting but the theoretical analysis more difficult. We focus on fixed prices for tractability and consistency with our earlier analysis.

<sup>20</sup>Michaillat and Saez (2015, pp. 539–540) discuss the link between the two models in more detail.

What happens in disequilibrium, when the price of services is fixed at a level that does not clear the market for services? When there is excess demand,  $y(p, g) > k$ . The demand-side of the market has to be rationed, such that output is determined by the supply side:  $y = k$ . Thus  $dy/dg = 0$  and  $MRS_{gc} = 1$ : the Samuelson rule also holds. However, when there is excess supply, the supply-side of the market has to be rationed, and output is determined by the demand side:  $y = y(p, g) < k$ . The output multiplier can be one, above one, or below one, depending on the effect of public consumption on the marginal utility from private consumption; that is, depending on whether  $c$  and  $g$  are complements or substitutes in the Edgeworth-Pareto sense.<sup>21</sup>

If the output multiplier is greater or equal to one, then for any public expenditure,  $MRS_{gc} + dy/dg > 1$ . Thus, it is optimal for the government to spend until the output gap is filled, irrespective of the usefulness of additional public expenditure. Intuitively, with such large multipliers, there is either no effect of public consumption on private consumption, or crowding in of private consumption by public consumption, so increasing public consumption raises all inputs into the welfare function, until the output gap is filled. Clearly, public consumption should fill the output gap. As showed in appendix E, this implies that up to a second-order remainder, optimal stimulus spending is

$$(26) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx z_2 \cdot \frac{1 - (g/y)^*(dy/dg)}{dy/dg} \cdot \frac{k - y_0}{y_0},$$

where  $y_0$  is the initial level of output,  $dy/dg$  is evaluated at  $[(g/c)^*, y_0]$ , and  $z_2 = 1 / [(c/y)^*(g/y)^*]$ .

The results when the output multiplier is greater or equal to one are consistent with results obtained in other fixprice models. For instance, using a fixprice model with a multiplier of one, Mankiw and Weinzierl (2011, pp. 232–234) find that it is optimal to completely fill the output gap. This result has three implications. First, optimal stimulus spending grows in proportion to the output gap. Second, optimal stimulus spending is decreasing in the output multiplier: with a larger multiplier, less spending is required to fill the output gap. Finally, the value of additional public spending (measured by the elasticity of substitution between public and private consumption) is irrelevant: whether additional public workers provide services that substitute perfectly for private

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<sup>21</sup>To see this, consider the demand side with land described in section 2.4. The output multiplier is  $dy/dg = 1 + dc/dg$ , so we need to determine the sign of  $dc/dg$ . Since there is excess supply, private consumption is determined by private demand, so we study the response of private demand to  $g$ . Private demand  $c(p, g)$  is defined by  $\partial \mathcal{U} / \partial c = p' \mathcal{V}'(l_0) / \delta$ , so  $dc/dg = -(\partial^2 \mathcal{U} / \partial c \partial g) / (\partial^2 \mathcal{U} / \partial c^2)$ , and  $dy/dg = 1 - (\partial^2 \mathcal{U} / \partial c \partial g) / (\partial^2 \mathcal{U} / \partial c^2)$ . We infer that the multiplier is one when  $c$  and  $g$  are unrelated in the Edgeworth-Pareto sense ( $\partial^2 \mathcal{U} / \partial c \partial g = 0$ ). The multiplier is above one when  $c$  and  $g$  are substitutes in the Edgeworth-Pareto sense ( $\partial^2 \mathcal{U} / \partial c \partial g < 0$ ). And the multiplier is below one when  $c$  and  $g$  are complements in the Edgeworth-Pareto sense ( $\partial^2 \mathcal{U} / \partial c \partial g > 0$ ).

services or they dig and fill holes, optimal stimulus spending is the same.

If the output multiplier is lower than one, it may not be optimal to fill the output gap. At the Samuelson level of spending,  $MRS_{gc} = 1$  so  $MRS_{gc} + dy/dg > 1$ : it is optimal to increase public expenditure to reduce the output gap. As public expenditure increases,  $MRS_{gc}$  decreases. If  $MRS_{gc} + dy/dg$  is above 1 once the output gap is filled, then it is optimal to completely fill the output gap. If  $MRS_{gc} + dy/dg$  reaches 1 before the output gap is filled, however, optimal public expenditure does not completely fill the output gap. In that case, optimal stimulus spending satisfies

$$(27) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx \epsilon \cdot (dy/dg).$$

This equation applies only if optimal stimulus spending is small enough that it does not completely fill the output gap—so only if multiplier and elasticity of substitution are small enough. The equation implies that as long as public consumption is valuable at the margin and the output multiplier is positive, stimulus spending should be positive. Additionally, optimal stimulus spending grows in proportion to the elasticity of substitution between public and private consumption and to the output multiplier, but is independent of the output gap.

Overall, when the economy is slack, we reach similar qualitative insights with the fixprice and matching models. This is good news: irrespective of how productive inefficiency is modeled, in slumps stimulus spending obeys similar principles. One difference between the two models lies in the shape of optimal stimulus spending as a function of the sufficient statistics. In the matching model that function is smooth. In the fixprice model, the function has two different parts. If the multiplier is large enough, stimulus spending should completely fill the output gap, so it is increasing in the output gap, strictly decreasing in the multiplier, and independent of the elasticity of substitution between public and private consumption. But when the multiplier is sufficiently small, optimal stimulus spending is given by (27), so it is independent of the output gap, strictly increasing in the multiplier, and strictly increasing in the elasticity of substitution between public and private consumption.

When the economy is tight, however, fixprice and matching recommendations differ: the matching model recommends to cut public spending in order to reduce the (negative) unemployment gap; the fixprice model recommends to keep public spending at the Samuelson level. Indeed, in the fixprice model, output is the same in the excess-demand and market-clearing regimes ( $y = k$ ). Hence the



welfare function and optimal policy are the same in the two regimes.<sup>22</sup>

## 4. Application to the Great Recession in the United States

We now complement our theoretical results with a numerical application. We calibrate our sufficient-statistic formula and compute optimal stimulus spending at the onset of the Great Recession in the United States. This exercise illustrates how much optimal public expenditure may deviate from the Samuelson rule, and how the deviation depends on the values of the sufficient statistics. Since the formula is valid whether taxes are distortionary or not, the numerical results apply in both cases.

Our starting point is 2008:Q3 in the United States: the unemployment rate is  $u = 6\%$  and public expenditure is  $G/C = 19.7\%$ . For simplicity, we assume that in 2008:Q3 the unemployment rate is efficient and public expenditure satisfies the Samuelson rule:  $u^* = 6\%$  and  $(G/C)^* = 19.7\%$ . These assumptions seem reasonable as unemployment and public expenditure in 2008:Q3 are close to their 25-year averages, and there is a presumption, going back at least to Okun (1963), that the economy is efficient on average.<sup>23</sup>

In 2008, an adverse shock hits the US economy and unemployment starts rising toward an inefficient level  $u_0 > u^*$ .<sup>24</sup> In our model unemployment immediately reaches the higher level  $u_0$ , but in reality unemployment slowly rises to  $u_0$ . The challenge for policymakers is to forecast  $u_0$  in advance. In the winter 2008–2009, when the US government designed the stimulus package, they forecast  $u_0 = 9\%$ , so we use  $u_0 = 9\%$  (Romer and Bernstein 2009, fig. 1). Then, to apply formula (22), we collect estimates of the two main statistics: the elasticity of substitution between public and private consumption ( $\epsilon$ ) and the unemployment multiplier ( $m$ ).

A literature attempts to estimate the elasticity of substitution. The empirical strategy is to isolate variations in the ratio of public-consumption price to private-consumption price and to assess their impact on the ratio of public consumption to private consumption. For example, if the consumption

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<sup>22</sup>In appendix E we analyze a fixprice model with endogenous capacity  $k$ . While it has other problems, such a model offers a symmetric treatment of the excess-demand and excess-supply regimes, much like the matching model.

<sup>23</sup>We set  $u$  to the seasonally adjusted unemployment rate constructed by the Bureau of Labor Statistics from the Current Population Survey. To construct  $G/C$ , we set  $G$  to the seasonally adjusted employment level in the government industry and  $C$  to the seasonally adjusted employment level in the private industry. Both series constructed are by the Bureau of Labor Statistics from the Current Employment Statistics survey. Over the 1990–2014 period, average unemployment rate is  $u = 6.1\%$  and average public expenditure is  $G/C = 19.7\%$ .

<sup>24</sup>Our formula accommodates any type of shock. But since we calibrate  $u^*$  and  $(G/C)^*$  using preshock observations, the shock should not affect  $u^*$  and  $(G/C)^*$ . So it could be a shock to aggregate demand, to aggregate supply, or to prices, but not to the matching process (matching function,  $s$ , or  $\rho$ ).

ratio stays constant in spite of secular variations in the price ratio, then the elasticity is about one.<sup>25</sup> To tackle the challenging identification problem, the modern literature uses the co-integration approach developed by Ogaki (1992). With US data, Amano and Wirjanto (1997, 1998) estimate elasticities of 0.9 and 1.56. Using data for nine East Asian countries, Kwan (2007, p. 52) obtains estimates ranging from 0.57 to 1.05.<sup>26</sup> These estimates are somewhat sensitive to the specification and time period chosen, but virtually all estimates fall in the range 0.5–2, with 1 as a plausible midrange estimate. To span the range to available estimates, we consider three values for the elasticity of substitution:  $\epsilon = 0.5$ ,  $\epsilon = 1$ , and  $\epsilon = 2$ .

Next, we determine plausible values for the unemployment multiplier  $m$ . Since  $m$  is not directly observable, we report estimates for the empirical multiplier  $M$  and then translate  $M$  into  $m$ . Using (24), the above values for  $G/C$  and  $u$ , and the calibration of  $\eta$  and  $\tau$  in Landais, Michailat, and Saez (2018a), we find  $m = 0.91 \times M / (1 - 0.046 \times M)$ . Hence  $m$  is almost identical to  $M$ .<sup>27</sup>

The unemployment multiplier  $M$  is estimated by measuring the percentage-point change in the unemployment rate when public expenditure increases by one percent of GDP. Monacelli, Perotti, and Trigari (2010, pp. 533–536) estimate a structural vector autoregression (SVAR) on US data and find multipliers between 0.2 and 0.6. Ramey (2013, pp. 40–42) estimates SVARs on US data with various identification schemes and sample periods. She finds multipliers between 0.2 and 0.5, except in one specification where the multiplier is 1.

Overall, the average unemployment multiplier is estimated to be in the 0.2–1 range. If multipliers are larger when unemployment is higher, as suggested by recent research on state-dependent multipliers, the multiplier entering our formula could even be larger. For instance, using regime-switching SVARs on US data, Auerbach and Gorodnichenko (2012, table 1, rows 1–3) find that while the output multiplier is 0.6 in expansions and 1 on average, it is as high as 2.5 in recessions. To account for the uncertainty about the exact value of the multiplier, we compute optimal stimulus spending for  $M$  between 0 and 2.

The results are displayed in figure 2. Panel A displays optimal stimulus spending as a share

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<sup>25</sup>This is plausible in light of the fairly stable ratio of government consumption to GDP in OECD countries since 1980 (see <https://data.worldbank.org/indicator/>).

<sup>26</sup>Earlier work finds estimates of 1.1 for Taiwan (Chiu 2001) and 1.39 for Japan (Okubo 2003, pp. 79–80).

<sup>27</sup>We calibrate (24) as follows. We set  $G/Y$  and  $u$  to their values after the shock but before the stimulus:  $G/Y = (G/C)/(1 + G/C) = 0.197/(1 + 0.197) = 16.5\%$  and  $u = 9\%$ . Landais, Michailat, and Saez (2018a, fig. 1) measure labor devoted to matching. When the unemployment rate is 9%, as in 2009:Q2, they find  $\tau = 1.7\%$ . We use this value. Last, following Landais, Michailat, and Saez (2018a, online appendix D), we set  $\eta = 0.6$ .

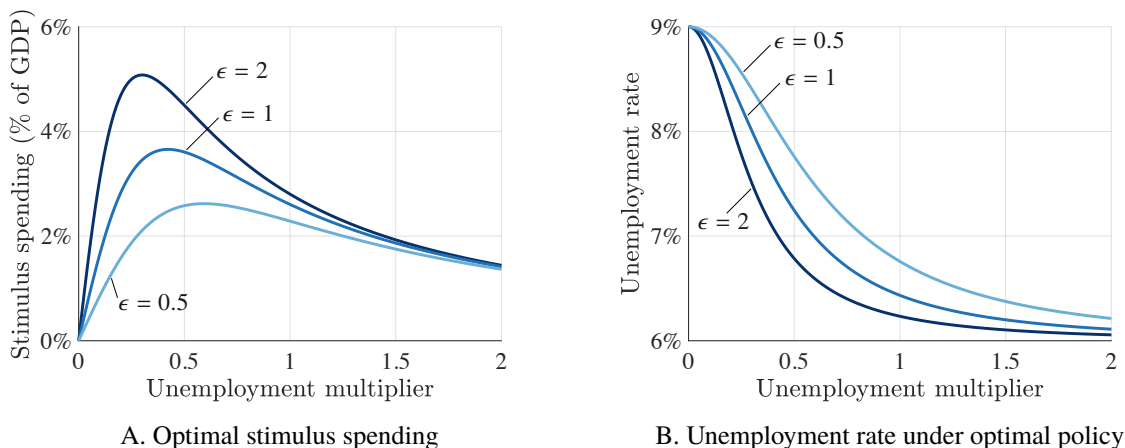


Figure 2. Optimal Stimulus Spending During the Great Recession in the United States

Notes: In the United States in 2008:Q3 unemployment was 6% and public expenditure was 16.5% of GDP. (We assume that in 2008:Q3 unemployment and public expenditure were efficient.) The Great Recession shock hit the US economy at that time, raising unemployment to a projected level of 9%. Panel A displays optimal stimulus spending as a share of GDP for various values of the empirical unemployment multiplier ( $M$ ) and elasticity of substitution between public and private consumption ( $\epsilon$ ). Panel B displays the corresponding unemployment rate. Panel A is computed using (22), and panel B is computed using (23).

of GDP ( $G/Y - (G/Y)^*$ ), constructed using (22). Panel B displays the unemployment rate under optimal stimulus spending, constructed using (23).<sup>28</sup> Several observations stand out.

First, even with a small multiplier of 0.2, optimal stimulus spending is significant. With  $\epsilon = 0.5$ , optimal stimulus spending is 1.6 percentage points of GDP. With  $\epsilon = 1$ , it is 2.8 points of GDP. And with  $\epsilon = 2$ , it is 4.7 points of GDP.

Second, the multiplier warranting the largest stimulus is fairly modest. With  $\epsilon = 0.5$  the largest stimulus (2.6 points of GDP) occurs with a multiplier of 0.6. With  $\epsilon = 1$  the largest stimulus (3.7 points of GDP) occurs with a multiplier of 0.4. And with  $\epsilon = 2$  the largest increase (5.1 points of GDP) occurs with a multiplier of 0.3.

Third, optimal stimulus spending is the same for small and large multipliers. For instance, fix  $\epsilon = 1$ : optimal stimulus spending is the same for multipliers of 0.12 and 1.5 (1.9 points of GDP). Of course the resulting unemployment rates are very different.

Fourth, for small multipliers, unemployment barely falls below its initial level of 9% although optimal stimulus spending is large. With a multiplier of 0.2, unemployment only falls to 8.7% with

<sup>28</sup>We calibrate  $z_0$  and  $z_1$  in (22) and (23) as we have calibrated (24). We set  $(g/y)^* = 19.7\%$ ,  $(g/y)^* = 16.5\%$ ,  $(c/y)^* = 1 - (g/y)^* = 83.5\%$ ,  $u^* = 6\%$ , and  $\eta = 0.6$ . This values imply  $z_0 = 2.83$  and  $z_1 = 2.30$ . We also translate the ratio  $g/c$  given by (22) into a ratio  $G/Y$  using the identity  $G/Y = (g/c)/(1 + g/c)$ .

$\epsilon = 0.5$ , 8.5% with  $\epsilon = 1$ , and 8.1% with  $\epsilon = 2$ . This is because public expenditure has little effect on unemployment when the multiplier is small.

Fifth, with a multiplier above 1, optimal stimulus spending almost brings back unemployment to its efficient level of 6%. With a multiplier of 1, the unemployment rate falls below 6.8%, so the remaining unemployment gap is less than 0.8 percentage points. With a multiplier of 2, the remaining unemployment gap is less than 0.2 percentage points.

Sixth, the elasticity of substitution between public and private consumption plays a significant role for small to medium multipliers, but not for large multipliers. Consider first a multiplier of 0.4. With  $\epsilon = 0.5$ , optimal stimulus spending is 2.4 percentage points of GDP; with  $\epsilon = 1$ , public expenditure should increase by an additional 1.2 points of GDP; and with  $\epsilon = 2$ , public expenditure should increase by another 1.2 points of GDP. Hence,  $\epsilon$  significantly influences optimal stimulus spending. In contrast, for a multiplier above 1, optimal stimulus spendings with  $\epsilon = 0.5$ ,  $\epsilon = 1$ , and  $\epsilon = 2$  are nearly indistinguishable. This is because for large multipliers, the optimal policy is to fill the unemployment gap, so it is not influenced by the elasticity of substitution.

Finally, we calculate optimal stimulus spending at the onset of the Great Recession using midrange values for the unemployment multiplier and elasticity of substitution:  $M = 0.5$  and  $\epsilon = 1$ . Under this calibration, optimal stimulus spending is 3.6 points of GDP; since US GDP in 2008 is \$14,700 billion, optimal stimulus spending is \$530 billion per year. How does this optimal stimulus package compare to the actual stimulus package? According to the Congressional Budget Office (CBO), the American Recovery and Reinvestment Act (ARRA), enacted into law in February 2009, is estimated to cost \$840 billion over ten years, with half of that amount spent in 2010.<sup>29</sup> So at the peak of the Great Recession in 2010, stimulus spending was \$420 billion. This is below but of the same order of magnitude as our optimal stimulus package of \$530 billion.

Yet, evaluating the adequacy of ARRA is more complicated than comparing these two numbers. Our model focuses on one stabilization policy: government expenditure on goods and services. ARRA was more complex; it was a blend of three policies: increase in government expenditure, increase in government transfers, and increase in government deficit. According to the CBO, only 30% of ARRA was devoted to government expenditure, so about  $0.3 \times \$420 \text{ billion} = \$130 \text{ billion}$ .<sup>30</sup> At the same time, government expenditure on goods and services was combined with

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<sup>29</sup>See <https://perma.cc/RJ6D-GZA8>.

<sup>30</sup>See <https://web.archive.org/web/20150905195457/http://www.recovery.gov/arra/Transparency/fundingoverview/>

other stabilization policies, so optimal stimulus spending on goods and services was less than \$530 billion. Determining whether the optimal stimulus was above or below \$130 billion would require a more sophisticated model describing jointly the effects of government transfers, government deficit, and government expenditure on goods and services.

## 5. Simulations

This section simulates a fully specified, structural matching model. The simulations show that the matching model provides a good description of the business cycle: in response to aggregate-demand shocks the model generates realistic, countercyclical fluctuations in the unemployment rate and unemployment multiplier. This realism suggests that the matching framework is adapted to study optimal policy over the business cycle. The simulations also show that our sufficient-statistic formula, obtained with first-order approximations, is accurate even for large business-cycle fluctuations. Indeed, in our matching model, the sufficient-statistic formula and the exact optimality condition deliver almost identical policies.<sup>31</sup>

### 5.1. Quantitative Properties of the Matching Model

We simulate the matching model with land developed in section 2.4. The model is calibrated to US data (see appendix A). We represent the business cycle as a succession of unexpected permanent aggregate-demand shocks. We use these shocks for two reasons: first, they generate inefficient fluctuations in unemployment; second, they generate the negative comovements between tightness and unemployment observed empirically (Michaillat and Saez 2015).

We parameterize aggregate demand with  $\alpha = \delta/V'(l_0)$ . Since the economy jumps to its new steady-state equilibrium in response to a shock, we only need to compute a collection of steady states parameterized by  $\alpha \in [0.97, 1.03]$ . We run two simulations: one in which  $G/Y$  is constant at 16.5%, its average value in the United States for 1990–2014, and one in which  $G/Y$  is at its optimal level, given by (17).

Figure 3 illustrates the simulations. The unemployment rate is countercyclical: when  $G/Y = 16.5\%$ , it rises from 4.4% when aggregate demand is highest ( $\alpha = 1.03$ ), to 6.1% (the average

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Pages/fundingbreakdown.aspx for a breakdown of ARRA's funding.

<sup>31</sup>We keep the simulation model simple to illustrate the theoretical results as transparently as possible. It would also be useful to simulate a richer model to obtain more precise quantitative results about optimal stimulus spending.

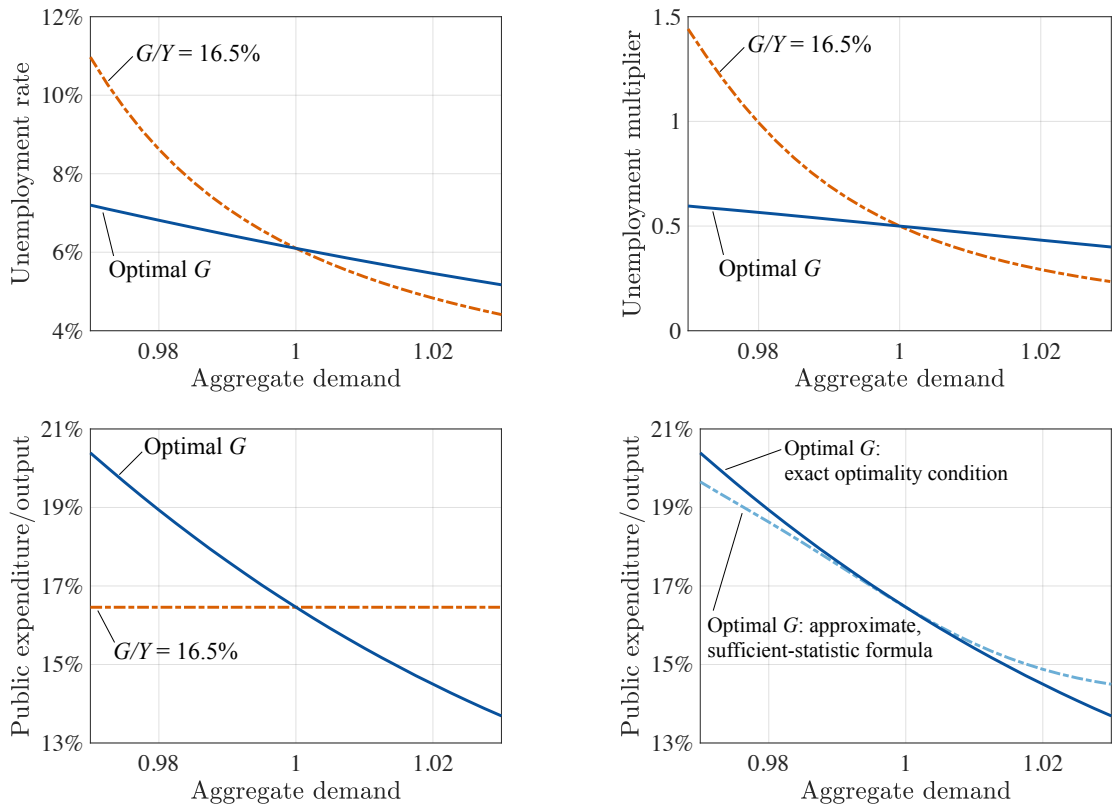


Figure 3. Business-Cycle Simulation in the Matching Model with Land

Notes: The figure displays business cycles both when public expenditure is  $G/Y = 16.5\%$  and when it is set using the optimality condition (17). The business cycles are generated by aggregate-demand shocks. The multiplier in the top-right panel is the empirical unemployment multiplier, also equal to the output multiplier. The bottom-right panel compares the results of the sufficient-statistic formula (22) and the exact optimality condition (17). The results are obtained by simulating the matching model with land from section 2.4 under the calibration in appendix A.

unemployment rate in the United States for 1990–2014) when aggregate demand is average ( $\alpha = 1$ ), and to 11.0% when aggregate demand is lowest ( $\alpha = 0.97$ ). Unemployment fluctuates in response to aggregate-demand shocks because of price rigidity: when  $\alpha$  goes up, the price of services does not adjust, which stimulates the aggregate-demand curve (13) and reduces unemployment.

The unemployment multiplier is also countercyclical: it increases from 0.2 when unemployment is 4.4%, to 0.5 (the midrange of US estimates) when unemployment is 6.1%, to 1.4 when unemployment is 11.0%. This countercyclicity is consistent with evidence suggesting that in the United States, multipliers are higher when unemployment is higher or output is lower (Auerbach and Gorodnichenko 2012; Candelon and Lieb 2013; Fazzari, Morley, and Panovska 2015). The mechanism behind this countercyclicity is described in Michailat (2014). When unemployment is high, there is a lot of idle capacity so the matching process is congested by sellers of services. Hence, increased

spending by the government has very little effect on other buyers of services. Crowding out of private expenditure by public expenditure is therefore weak, and the multiplier is large. When unemployment is low, the opposite occurs: matching is congested by buyers of services, crowding out of private expenditure by public expenditure is sharp, and the multiplier is small.

We also compute optimal public expenditure over the business cycle. We find that optimal public spending is markedly countercyclical, decreasing from  $G/Y = 20.4\%$  to  $G/Y = 13.7\%$  when  $\alpha$  increases from 0.97 to 1.03. This is unsurprising. The unemployment rate is efficient when  $\alpha = 1$ , inefficiently high when  $\alpha < 1$ , and inefficiently low when  $\alpha > 1$ ; furthermore, the unemployment multiplier is positive. Hence, public spending should be above Samuelson spending when  $\alpha < 1$  and below it when  $\alpha > 1$ .

Finally, unemployment responds when public expenditure is adjusted from  $G/Y = 16.5\%$  to its optimal level. When aggregate demand is low, optimal public expenditure is higher than  $G/Y = 16.5\%$  so unemployment falls below its original level. For instance, at  $\alpha = 0.97$  the unemployment rate falls from 11.0% to 7.2%. When aggregate demand is high, optimal public expenditure is below  $G/Y = 16.5\%$  so unemployment rises above its original level. For instance, at  $\alpha = 1.03$  the unemployment rate increases from 4.4% to 5.2%. The unemployment multiplier heavily depends on the unemployment rate, so it adjusts accordingly.

## 5.2. Accuracy of the Sufficient-Statistic Formula

Our sufficient-statistic formula, given by (22), is an approximation to the exact optimality condition, given by (17). Since unemployment fluctuations are large, the second-order remainder in our formula could be large, and the approximation could be inaccurate. In our simulations, however, this does not happen. Figure 3 shows that our formula is quite accurate: the public expenditure levels given by our formula and the exact condition never deviate by more than one percentage point of GDP. The largest deviations occur at  $\alpha = 0.97$ , where the exact condition gives  $G/Y = 20.4\%$  while our formula gives  $G/Y = 19.7\%$ , and at  $\alpha = 1.03$ , where the exact condition gives  $G/Y = 13.7\%$  while our formula gives  $G/Y = 14.5\%$ .

## 6. Conclusion

This paper has developed a theory of optimal public expenditure in the presence of unemployment. The theory shows that when unemployment is efficient, the Samuelson rule remains valid; but when unemployment is inefficient, optimal public expenditure deviates from the Samuelson rule to bring unemployment closer to its efficient level.

In the past few decades, monetary policy has been governments' preferred stabilization policy. Yet it has become clear that because of the zero lower bound—which was binding in Japan, the United States, and the eurozone after the Great Recession—governments cannot rely on monetary policy alone to stabilize the economy. Our theory suggests that public expenditure could contribute to stabilization whenever monetary policy is constrained.

In addition, public expenditure could be helpful to members of monetary unions, such as eurozone countries or US states. These governments have no control over monetary policy, so they cannot use it to tackle unemployment. But they can adjust public expenditure. Furthermore, since our theory focuses on budget-balanced spending, it applies both to US states, which cannot run budget deficits, and to eurozone countries, which face strict constraints on their public debt.

In this paper we have limited ourselves to static considerations. It would be useful to enrich the analysis with dynamic elements. Several such elements seem important: the use of government debt to finance public spending (Barro 1979); the distinction between temporary and permanent public spending (Barro 1981); public investment in infrastructure (Baxter and King 1993); the effects of public spending in a liquidity trap (Woodford 2011; Werning 2011); and the political process associated with the design of stimulus packages (Battaglini and Coate 2016).

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## Appendix A. The Model with Land

We derive several results that are useful in the analysis and simulation of the matching model with land presented in section 2.4. We also calibrate the model to US data.

### Utility Function

We compute the derivatives of the utility function (7):

$$\begin{aligned}\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} &= (1 - \gamma)^{\frac{1}{\epsilon}} \left(\frac{c}{\mathcal{U}}\right)^{\frac{\epsilon-1}{\epsilon}}, & \mathcal{U}_c &\equiv \frac{\partial \mathcal{U}}{\partial c} = \left(\frac{(1 - \gamma)\mathcal{U}}{c}\right)^{\frac{1}{\epsilon}} \\ \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} &= \gamma^{\frac{1}{\epsilon}} \left(\frac{g}{\mathcal{U}}\right)^{\frac{\epsilon-1}{\epsilon}}, & \mathcal{U}_g &\equiv \frac{\partial \mathcal{U}}{\partial g} = \left(\frac{\gamma\mathcal{U}}{g}\right)^{\frac{1}{\epsilon}} \\ \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} &= \frac{1}{\epsilon} \cdot \left(\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} - 1\right) \\ \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} &= \frac{1}{\epsilon} \cdot \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)}.\end{aligned}$$

When the Samuelson rule holds,  $MRS_{gc} = \mathcal{U}_g/\mathcal{U}_c = 1$ , so

$$(g/c)^* = \frac{\gamma}{1 - \gamma}, \quad (g/y)^* = \gamma, \quad (c/y)^* = 1 - \gamma,$$

and the derivatives simplify to

$$\begin{aligned}\frac{\partial \ln(\mathcal{U})}{\partial \ln(c)} &= 1 - \gamma, & \frac{\partial \ln(\mathcal{U})}{\partial \ln(g)} &= \gamma \\ \mathcal{U}_c &= 1, & \mathcal{U}_g &= 1 \\ \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} &= -\frac{\gamma}{\epsilon}, & \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} &= \frac{\gamma}{\epsilon}.\end{aligned}$$

### Household's Problem and Equilibrium

We solve the household's utility-maximization problem and analyze equilibrium dynamics.

The current-value Hamiltonian of the household's problem is

$$\mathcal{H}(t, c(t), l(t)) = \mathcal{U}(c(t), g(t)) + \mathcal{V}(l(t)) + \lambda(t) \{p(t) [1 - u(x(t))] k - p(t) [1 + \tau(x(t))] c(t) - T(t)\}.$$

It has control variable  $c(t)$ , state variable  $l(t)$ , and current-value costate variable  $\lambda(t)$ . The first-order conditions for an interior solution to the maximization problem are  $\partial\mathcal{H}/\partial c = 0$ ,  $\partial\mathcal{H}/\partial l = \delta\lambda(t) - \dot{\lambda}(t)$ , and the appropriate transversality condition. Since  $\mathcal{U}$  and  $\mathcal{V}$  are concave, these first-order conditions are not only necessary but also sufficient. These conditions yield (10) and (11).

Since all the equilibrium variables can be recovered from the costate variable  $\lambda(t)$ , the equilibrium can be represented as a dynamical system of dimension one, with variable  $\lambda(t)$ . The variable  $\lambda(t)$  satisfies the differential equation  $\dot{\lambda}(t) = \delta\lambda(t) - \mathcal{V}'(l_0)$ . The steady-state value of  $\lambda(t)$  is  $\lambda = \mathcal{V}'(l_0)/\delta > 0$ . Since  $\delta > 0$ , the dynamical system is a source. As  $\lambda(t)$  is a nonpredetermined variable, the system jumps to the steady state from any initial condition.

## Unemployment Multiplier

We compute the unemployment multipliers. We first express  $dx/dg$  as a function of the derivatives of the utility function. Then, we compute the unemployment multiplier  $m$  from  $dx/dg$  and the empirical unemployment multiplier  $M$  from  $m$ .

The price schedule is  $p(g) = p_0\mathcal{U}_c(y^* - g, g)^{1-r}$ . Its elasticity is

$$\frac{d \ln(p)}{d \ln(g)} = (1 - r) \left[ \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(g)} - \frac{g}{y^* - g} \cdot \frac{\partial \ln(\mathcal{U}_c)}{\partial \ln(c)} \right],$$

and the value of the elasticity at  $g^* \equiv \gamma y^*$  is

$$\frac{d \ln(p)}{d \ln(g)} = (1 - r) \cdot \frac{1}{\epsilon} \cdot \frac{\gamma}{1 - \gamma}.$$

The private demand  $c(x, g)$  is defined by  $\mathcal{U}_c(c, g) = p(g)(1 + \tau(x))\mathcal{V}'(l_0)/\delta$ . The elasticities of demand are

$$\begin{aligned} \frac{\partial \ln(c)}{\partial \ln(x)} &= \frac{\eta\tau(x)}{\partial \ln(\mathcal{U}_c)/\partial \ln(c)} \\ \frac{\partial \ln(c)}{\partial \ln(g)} &= -\frac{\partial \ln(\mathcal{U}_c)/\partial \ln(g) - \partial \ln(p)/\partial \ln(g)}{\partial \ln(\mathcal{U}_c)/\partial \ln(c)}. \end{aligned}$$

We calibrate the price level such that unemployment is efficient when public expenditure is at the Samuelson level, or equivalently  $x(g^*) = x^*$ . This means that  $c(x^*, g^*) = c^* \equiv (1 - \gamma)y^*$  and

$\eta\tau(x^*) = (1 - \eta)u^*$ . Thus, the elasticities of demand at  $x^*$  and  $g^*$  are

$$\frac{\partial \ln(c)}{\partial \ln(x)} = -(1 - \eta)u^* \frac{\epsilon}{\gamma} \quad \text{and} \quad \frac{\partial \ln(c)}{\partial \ln(g)} = \frac{r - \gamma}{1 - \gamma}.$$

The equilibrium condition determining tightness  $x(g)$  is  $y(x, k) = g + c(x, g)$ . In the simulations,  $k$  is fixed. Differentiating this equation with respect to  $g$ , we obtain the elasticity of  $x(g)$  with respect to  $g$ :

$$\frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} = \frac{g}{y} + \frac{c}{y} \left[ \frac{\partial \ln(c)}{\partial \ln(g)} + \frac{\partial \ln(c)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g)} \right]$$

so that

$$\frac{d \ln(x)}{d \ln(g)} = \frac{(g/y) + (c/y)(\partial \ln(c)/\partial \ln(g))}{\partial \ln(y)/\partial \ln(x) - (c/y)(\partial \ln(c)/\partial \ln(x))}.$$

As discussed above, we calibrate the model such that  $x(g^*) = x^*$  and  $c(x^*, g^*) = c^*$ . In addition,  $(c/y)^* = 1 - \gamma$  and  $(g/y)^* = \gamma$ . Hence, the value of the elasticity at  $g^*$  is

$$\frac{d \ln(x)}{d \ln(g)} = \frac{1}{(1 - \eta)u^*} \cdot \frac{r}{\epsilon} \cdot \frac{\gamma}{1 - \gamma}.$$

From the expression for  $d \ln(x)/d \ln(g)$ , we obtain  $m$  and  $M$  using (20) and (24):

$$m = (1 - \eta)(1 - u)u \cdot \frac{y}{g} \cdot \frac{d \ln(x)}{d \ln(g)} \quad \text{and} \quad M = \frac{m}{1 - u + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{u} \cdot m}.$$

Since  $(g/y)^* = \gamma$ , the values of  $m$  and  $M$  at  $g^*$  are

$$m = \frac{(1 - u^*)r}{(1 - \gamma)\epsilon} \quad \text{and} \quad M = \frac{r}{\gamma r + (1 - \gamma)\epsilon}.$$

## Calibration

We calibrate the matching model with land using evidence from the United States. The calibration is summarized in table A1. We use the calibration in the simulations presented in section 5.

We begin by calibrating the utility function. We arbitrarily set the elasticity of substitution between public and private consumption to  $\epsilon = 1$ . As showed above, the parameter  $\gamma$  determines Samuelson spending:  $(G/C)^* = \gamma/(1 - \gamma)$ . We assume that Samuelson spending is the average level of public expenditure in the United States for 1990–2014:  $(G/C)^* = 19.7\%$  (section 4). We therefore

Table A1. Parameter Values in Simulations

|                 | Description                                    | Source                              |
|-----------------|--|-------------------------------------|
| $\epsilon = 1$  | Elasticity of substitution between $g$ and $c$ | –                                   |
| $\gamma = 0.16$ | Parameter of utility function                  | Matches $(G/C)^* = 19.7\%$          |
| $s = 2.8\%$     | Monthly separation rate                        | Landais, Michailat, and Saez (2018) |
| $\eta = 0.6$    | Matching elasticity                            | Landais, Michailat, and Saez (2018) |
| $\omega = 0.60$ | Matching efficacy                              | Landais, Michailat, and Saez (2018) |
| $\rho = 1.4$    | Matching cost                                  | Matches $u^* = 6.1\%$               |
| $r = 0.46$      | Price rigidity                                 | Matches $M = 0.5$ at $\alpha = 1$   |
| $p_0 = 0.78$    | Price level                                    | Matches $u = u^*$ at $\alpha = 1$   |

set  $\gamma = 0.16$ .

Then we calibrate matching parameters. Landais, Michailat, and Saez (2018) use evidence from the US labor market for 1990–2014 and find a separation rate of  $s = 2.8\%$  (online appendix B), a matching elasticity of  $\eta = 0.6$  (online appendix D), and a matching efficacy of  $\omega = 0.60$  (online appendix G). We use these values. They also find that the average US unemployment rate and tightness for 1990–2014 are  $u = 6.1\%$  and  $x = 0.43$  (online appendix G). We assume that these averages are efficient:  $u^* = 6.1\%$  and  $x^* = 0.43$ . Then we set the matching cost from the relationship  $\tau = \rho s / [\omega x^{-\eta} - \rho s]$ , which implies  $\rho = \omega x^{-\eta} \tau / [(1 + \tau)s]$ . This relation holds for any  $\tau$  and  $x$ , in particular when tightness is efficient. But when tightness is efficient,  $\tau^* = (1 - \eta)u^* / \eta$  so  $\tau^* = 4.1\%$ . Plugging  $x^* = 0.43$  and  $\tau^* = 4.1\%$  in the expression for  $\rho$  yields  $\rho = 1.4$ .

Last we calibrate the price mechanism. The empirical evidence suggest that on average in the United States the unemployment multiplier is  $M = 0.5$  (section 4). Relying on (15), we set  $r = 0.46$  to match  $M = 0.5$ . We calibrate the price level such that  $u = u^* = 6.1\%$  when the demand parameter  $\alpha \equiv \delta / \mathcal{V}'(l_0) = 1$  and  $G/C = (G/C)^* = 19.7\%$ . Using (14), we find that  $p_0 = 0.78$ .

## Appendix B. Demand Side and Equilibrium: Other Examples

In section 2.4 we describe a demand side and equilibrium with land. Here we present two other demand sides: one with money and another one with government bonds. We find that they both yield equilibria with the same properties as the land equilibrium.

## Money in the Utility Function

We replace land by money and assume that households derive utility from real money balances. Introducing money in the utility function is a classical way to generate an aggregate demand: following Sidrauski (1967), a large number of business-cycle models with money in the utility function have been developed (for example, Barro and Grossman 1971; Blanchard and Kiyotaki 1987). Money is introduced in the utility function to capture the fact that money provides transaction services.

A household holds  $M(t)$  units of money and the supply of money is fixed at  $M_0$ . In equilibrium, the money market clears and  $M(t) = M_0$ . The price of services in terms of money is  $p(t)$ . We specify a general price mechanism that determines the price of services:  $p(t) = p(x(t), g(t))$ . The household's instantaneous utility function is  $\mathcal{U}(c(t), g(t)) + \mathcal{V}(M(t)/p(t))$ . The law of motion of the household's real money balances  $m(t) \equiv M(t)/p(t)$  is

$$\dot{m}(t) = [1 - u(x(t))]k - [1 + \tau(x(t))]c(t) - \pi(t)m(t) - T(t),$$

where  $\pi(t) \equiv \dot{p}(t)/p(t)$  is the inflation rate. In steady state,  $g$  and thus  $p$  are fixed so inflation is zero. The equilibrium immediately converges to steady state. In steady state the desired private consumption  $c(x, p, g)$  is given by

$$\frac{\partial \mathcal{U}}{\partial c} = (1 + \tau(x)) \frac{\mathcal{V}'(M_0/p)}{\delta}.$$

Equilibrium tightness  $x(g)$  is implicitly defined by

$$c(x, p(x, g), g) + g = y(x, k).$$

## Bonds in the Utility Function

Here we replace land by government bonds and assume that households derive utility from real bond holdings. Assuming that bonds enter the utility function is a simple way to generate an aggregate demand in a dynamic cashless economy. Several papers in macroeconomics and finance make this assumption: Poterba and Rotemberg (1987), Krishnamurthy and Vissing-Jorgensen (2012),

Michaillat and Saez (2014), Fisher (2015), Campbell et al. (2017), Del Negro et al. (2017), and Michaillat and Saez (2018). Compared to other assets, government bonds have special features: they are particularly safe and liquid (Krishnamurthy and Vissing-Jorgensen 2012); they are also useful to satisfy legal requirements or for “window dressing” (Fair and Malkiel 1971, sec. 2). Placing bonds in the utility function is a reduced-form way to capture these features.

Bonds are issued and purchased by households, and they have a price of one in terms of money. Money only plays the role of a unit of account. A household holds  $B(t)$  bonds and bonds are in zero net supply. In equilibrium, the bond market clears and  $B(t) = 0$ . The rate of return on bonds is the nominal interest rate  $i(t)$ . The nominal interest rate is determined by the central bank, which sets an interest rate  $i(t) = i(x(t), g(t))$ . Since the interest rate depends on tightness public consumption, the central bank potentially responds to both economic activity and fiscal policy.

The price of services in terms of money is  $p(t)$ . The inflation rate is  $\pi(t) \equiv \dot{p}(t)/p(t)$ . In the economy there are two goods—services and bonds—and hence one relative price (public and private services have the same price). The price of bonds relative to services is determined by the real interest rate,  $i(t) - \pi(t)$ . Since the nominal interest rate is determined by the central bank, it is the inflation rate that determines the real interest rate. The inflation rate is determined by a general price mechanism:  $\pi(t) = \pi(x(t), g(t))$ . Given the inflation rate, the price of services moves according to  $\dot{p}(t) = \pi(t)p(t)$ . The initial price  $p(0)$  is given. Given the inflation rate and nominal interest rate, tightness adjusts such that supply equals demand on the market for services.

The household’s instantaneous utility function is  $\mathcal{U}(c(t), g(t)) + \mathcal{V}(B(t)/p(t))$ . The law of motion of the household’s real wealth  $b(t) \equiv B(t)/p(t)$  is

$$\dot{b}(t) = [1 - u(x(t))]k - [1 + \tau(x(t))]c(t) + [i(t) - \pi(t)]b(t) - T(t).$$

As earlier, the equilibrium immediately converges to steady state. In steady state, the desired amount of private consumption  $c(x, i, \pi, g)$  is given by

$$\frac{\partial \mathcal{U}}{\partial c} = (1 + \tau(x)) \frac{\mathcal{V}'(0)}{\delta - i + \pi}.$$

This equation is the usual consumption Euler equation modified by the utility of wealth and evaluated in steady state. The demand for saving arises in part from intertemporal consumption-smoothing



considerations and in part from the utility provided by wealth. The equation implies that at the margin, the household is indifferent between consuming and holding real wealth. Equilibrium tightness  $x(g)$  is implicitly defined by

$$c(x, i(x, g), \pi(x, g), g) + g = y(x, k).$$

## Appendix C. Long Proofs

This appendix provides the longer proofs; the shorter proofs are incorporated in the main text.

### Proof of Lemma 3

Since  $MRS_{gc}$  is a function of  $g/c$ , the first-order Taylor expansion of  $MRS_{gc}(g/c)$  at  $(g/c)^*$  is

$$MRS_{gc}(g/c) = MRS_{gc}((g/c)^*) + \frac{dMRS_{gc}}{dg/c} \cdot (g/c - (g/c)^*) + O\left([g/c - (g/c)^*]^2\right).$$

In addition,  $MRS_{gc}((g/c)^*) = 1$  and  $dMRS_{gc}/d(g/c) = -1/[\epsilon \cdot (g/c)^*]$ . Hence,

$$(A1) \quad 1 - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right).$$

The  $1/\epsilon$  in the Taylor expansion is evaluated at  $(g/c)^*$ . But we can replace it by the  $1/\epsilon$  evaluated at  $g/c$  because the difference between the two is proportional to  $g/c - (g/c)^*$ . So once the difference is multiplied by  $g/c - (g/c)^*$  in (A1), it is absorbed in  $O\left([g/c - (g/c)^*]^2\right)$ . Thus, equation (A1) yields equation (18).

Next, we write  $\partial \ln(y)/\partial \ln(x)$  as a function of  $u$ :

$$\frac{\partial \ln(y)}{\partial \ln(x)} = (1 - \eta)u - \eta\tau(u).$$

The function  $\tau(u)$  is defined by  $\tau(u) = \tau(x(u))$ , where  $\tau(x)$  is given by (3) and  $x(u) = u^{-1}(u)$  is the inverse of the function  $u(x)$  given by (2). We have

$$\tau'(u) = \tau'(x) \cdot x'(u) = \frac{\tau'(x)}{u'(x(u))} = \frac{(1 + \tau)\eta\tau/x}{-(1 - \eta)(1 - u)u/x} = -\frac{(1 + \tau)\eta\tau}{(1 - \eta)(1 - u)u}.$$

Since  $(1 - \eta)u^* = \eta\tau(u^*)$ , we have  $\tau'(u^*) = -(1 + \tau(u^*))/(1 - u^*)$  and

$$-\eta\tau'(u^*) = \frac{\eta + \eta\tau(u^*)}{1 - u^*} = \frac{\eta + (1 - \eta)u^*}{1 - u^*} = \eta + \frac{u^*}{1 - u^*}.$$

Hence, the derivative of  $\partial \ln(y)/\partial \ln(x)$  with respect to  $u$  at  $u^*$  is  $(1 - \eta) - \eta\tau'(u^*) = 1/(1 - u^*)$ . Furthermore,  $\partial \ln(y)/\partial \ln(x) = 0$  at  $u^*$ . Thus, a first-order Taylor expansion of  $\partial \ln(y)/\partial \ln(x)$  at  $u^*$  yields (19).

Finally, since the elasticity of  $u(x)$  with respect to  $x$  is  $-(1 - \eta)(1 - u)$ , we find that

$$m = -y \cdot \frac{u}{g} \cdot \frac{d \ln(u)}{d \ln(g)} = \frac{y}{g} \cdot (1 - \eta) \cdot u \cdot (1 - u) \cdot \frac{d \ln(x)}{d \ln(g)} = \frac{y}{x} \cdot (1 - \eta) \cdot u \cdot (1 - u) \cdot \frac{dx}{dg}.$$

We obtain (20) by rearranging this equation.

### Proof of Lemma 4

We start from (17). First, we approximate  $1 - MRS_{gc}$  with (18). Next, we rewrite  $dx/dg$  with (20) and approximate  $\partial y/\partial x$  with (19). This yields

$$(A2) \quad \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1 - \eta} \cdot \frac{(u - u^*)}{u \cdot (1 - u) \cdot (1 - u^*)} + 0([g/c - (g/c)^*]^2 + [u - u^*]^2).$$

We can rewrite this as

$$\frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} = \frac{m}{1 - \eta} \cdot \frac{(u - u^*)}{u^* \cdot (1 - u^*)^2} + 0([g/c - (g/c)^*]^2 + [u - u^*]^2).$$

This is because the difference between  $1/[u \cdot (1 - u) \cdot (1 - u^*)]$  and  $1/[u^* \cdot (1 - u^*)^2]$  is  $O(u - u^*)$ . Once this difference is multiplied by  $u - u^*$  in (A2), it is absorbed by the term  $0([g/c - (g/c)^*]^2 + [u - u^*]^2)$ . We obtain (21) from this last equation.

### Proof of Proposition 1

The economy starts at an equilibrium  $[(g/c)^*, u_0]$ , where the unemployment rate  $u_0$  is inefficient. Since  $u_0 \neq u^*$ , the optimal  $g/c$  departs from  $(g/c)^*$ . In (21), the multiplier  $m$  and unemployment rate  $u$  are functions of  $g/c$ , so they respond as  $g/c$  moves away from  $(g/c)^*$ , and we cannot read the

optimal  $g/c$  off the formula. In this proof, we derive a formula giving the optimal  $g/c$  as a function of fixed quantities.

First, we express the equilibrium values of all variables as functions of  $[u, g/c]$ . The proof of lemma 3 showed that  $x$  and  $\tau$  can be written as functions of  $u$ . Since  $y = (1 - u) \cdot k/(1 + \tau)$ , we can also write  $y$  as a function of  $u$ . Since  $g = y \cdot (g/c)/[1 + g/c]$ ,  $g$  can be written as a function of  $u$  and  $g/c$ . As  $c = y - g$ ,  $c$  can also be written as a function of  $u$  and  $g/c$ . Last, since  $C = c \cdot (1 + \tau)$ ,  $C = c \cdot (1 + \tau)$ , and  $C = c \cdot (1 + \tau)$ , we can write  $C$ ,  $G$ , and  $Y$  as functions of  $u$  and  $g/c$ .

Among all pairs  $[u, g/c]$ , the only pairs describing an equilibrium are those consistent with the equilibrium condition  $u = u(x(g))$ , where  $g$  is the function of  $u$  and  $g/c$  described above,  $x(g)$  is the function defined by (6), and  $u(x)$  is the function defined by (2). This equilibrium condition defines the unemployment rate as an implicit function of  $g/c$ , denoted  $u(g/c)$ . Then, the pairs  $[u(g/c), g/c]$  for all  $g/c > 0$  are the equilibria for all possible levels of public expenditure.

We start by linking  $u$  to  $u_0$  and  $g/c$ . We write a first-order Taylor expansion of  $u(g/c)$  around  $u((g/c)^*) = u_0$ , subtract  $u^*$  on both sides, and divide both sides by  $u^*$ :

$$(A3) \quad \frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} + \frac{1}{u^*} \cdot \frac{du}{d \ln(g/c)} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right).$$

To compute  $du/d \ln(g/c)$  at  $[u_0, (g/c)^*]$ , we decompose the derivative:

$$\frac{du}{d \ln(g/c)} = \frac{du}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

First, the definition of the unemployment multiplier implies that

$$\frac{du}{d \ln(g)} = -m \cdot (g/y)^*,$$

where  $m$  is evaluated at  $[u_0, (g/c)^*]$ . Second, we compute  $d \ln(g)/d \ln(g/c)$ . We have

$$\ln(g/c) = \ln(g) - \ln(y(x(g/c), k) - g).$$

Differentiating with respect to  $\ln(g/c)$  yields

$$(A4) \quad 1 = \frac{d \ln(g)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)} + \frac{g}{c} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

Reshuffling the terms, we obtain

$$\frac{d \ln(g)}{d \ln(g/c)} = \frac{c}{y} + \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)}.$$

At  $u^*$ ,  $\partial \ln(y)/\partial \ln(x) = 0$ , so at  $u_0$ ,  $\partial \ln(y)/\partial \ln(x)$  is  $O(u_0 - u^*)$ . Once this term is multiplied by  $g/c - (g/c)^*$  in (A3), it creates a term that is  $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$ . Thus, we omit the term  $(\partial \ln(y)/\partial \ln(x)) \cdot (d \ln(x)/d \ln(g/c))$  and set

$$\frac{d \ln(g)}{d \ln(g/c)} = (c/y)^*.$$

So far, we have shown that

$$(A5) \quad \frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} - mz_1 \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right),$$

where

$$z_1 \equiv \frac{(g/y)^*(c/y)^*}{u^*}.$$

Equation (21) includes a remainder that is  $O\left([u - u^*]^2 + [g/c - (g/c)^*]^2\right)$ . Equation (A5) implies that  $(u - u^*)^2$  is  $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$ . Thus the remainder in formula (21) is  $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$ . Combining (21) and (A5), we therefore obtain

$$\frac{g/c - (g/c)^*}{(g/c)^*} = z_0 \epsilon m \left[ \frac{u_0 - u^*}{u^*} - mz_1 \frac{g/c - (g/c)^*}{(g/c)^*} \right] + O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right),$$

where  $\epsilon$  and  $m$  in the first bracket are evaluated at  $[u, g/c]$ . But instead we can use the values of  $\epsilon$  and  $m$  evaluated at  $[u_0, (g/c)^*]$  because the difference between the two values of each statistic is  $O([u - u_0] + [g/c - (g/c)^*])$ . So once the differences are multiplied by  $g/c - (g/c)^*$  and  $u_0 - u^*$  in the above equation, they are absorbed by the term  $O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right)$ . Thus, this last equation yields equation (22).

To finish the proof, we derive (23). With the arguments that we have just used, (21) can be written

$$\frac{g/c - (g/c)^*}{(g/c)^*} = z_0 \epsilon m \cdot \frac{u - u^*}{u^*} + O\left([u_0 - u^*]^2 + [g/c - (g/c)^*]^2\right),$$

where  $\epsilon$  and  $m$  are evaluated at  $[u_0, (g/c)^*]$ . Replacing the left-hand side of the last equation by the

right-hand side in (22), and dividing everything by  $z_0 \in m$ , we obtain (23).

### Proof of Lemma 5

As  $G = [1 + \tau(x(g))]g$  and the elasticity of  $1 + \tau(x)$  with respect to  $x$  is  $\eta\tau$ , we have

$$(A6) \quad \frac{d \ln(G)}{d \ln(g)} = 1 + \eta\tau \frac{d \ln(x)}{d \ln(g)} = 1 + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{u \cdot (1 - u)} \cdot m,$$

where the last equality is obtained using (20). Furthermore, the definitions of  $m$  and  $M$  imply

$$m = -y \frac{du}{dg} = -\frac{Y}{1 + \tau(x)} \cdot \frac{du}{dG} \cdot \frac{dG}{dg} = \frac{g}{G} (1 - u) M \frac{dG}{dg} = (1 - u) M \frac{d \ln(G)}{d \ln(g)}.$$

We now plug into this equation the expression for  $d \ln(G)/d \ln(g)$  obtained in (A6):

$$m = (1 - u) \cdot M + \frac{g}{y} \cdot \frac{\eta}{1 - \eta} \cdot \frac{\tau}{u} \cdot M \cdot m.$$

We obtain (24) by rearranging this equation.

Next, consider a change in public expenditure  $dG$ . This change leads to a change  $du$  in unemployment and, since  $Y = (1 - u)k$ , to a change  $dY = -du \cdot k$  in output. Hence,

$$\frac{dY}{dG} = -k \frac{du}{dG} = -\frac{Y}{1 - u} \cdot \frac{du}{dG} = M.$$

## Appendix D. Distortionary Taxation

We introduce endogenous labor supply and a distortionary income tax to study how distortionary taxation affects optimal public expenditure. We compare two approaches to taxation: the traditional approach in public economics and macroeconomics, which consists in using a linear income tax; and the modern approach in public economics, which consists in using a nonlinear income tax implemented following the benefit principle. With both approaches, the formula for optimal stimulus spending remains the same as when taxes are nondistortionary.

## Traditional Approach

In the traditional approach to taxation, the government uses a linear income tax  $\tau^L$  to finance public expenditure. With the linear income tax, the household's labor income becomes  $(1 - \tau^L)Y(x, k) = (1 - \tau^L)(1 - u(x))k$ . To finance public expenditure  $G$ , the tax rate must be  $\tau^L = G/Y = g/y$ .

The household chooses  $k$  to maximize utility. Let  $MRS_{kc} \equiv \mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$  be the marginal rate of substitution between labor and private consumption. As usual, the household supplies labor until the marginal rate of substitution between labor and consumption equals the post-tax real wage:

$$(A7) \quad MRS_{kc} = (1 - \tau^L) \frac{1 - u(x)}{1 + \tau(x)}.$$

Indeed, one unit of labor is only sold with probability  $1 - u(x)$ . When it is sold, it only yields  $1/(1 + \tau(x))$  units of consumption. Hence, the effective real wage is  $(1 - u(x))/(1 + \tau(x))$ , and the post-tax real wage is  $(1 - \tau^L)(1 - u(x))/(1 + \tau(x))$ .<sup>32</sup>

The supply decision is distorted by the income tax: a higher  $\tau^L$  implies a lower  $k$ . In fact, (A7) implicitly defines a function  $k(g)$  describing how productive capacity responds to a change in public expenditure and the associated tax change. As the income tax is distortionary, the function  $k(g)$  is decreasing in  $g$ .

The welfare of an equilibrium is  $\mathcal{U}(c, g) - \mathcal{W}(k)$ . Given a tightness function  $x(g)$  and a capacity function  $k(g)$ , the government chooses  $g$  to maximize  $\mathcal{U}(y(x(g), k(g)) - g, g) - \mathcal{W}(k(g))$ . The first-order condition of the government's problem is

$$0 = \frac{\partial\mathcal{U}}{\partial g} - \frac{\partial\mathcal{U}}{\partial c} - \mathcal{W}'(k) \frac{dk}{dg} + \frac{\partial\mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial k} \cdot \frac{dk}{dg} + \frac{\partial\mathcal{U}}{\partial c} \cdot \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Dividing the condition by  $\partial\mathcal{U}/\partial c$ , we obtain

$$1 = MRS_{gc} - \left( MRS_{kc} - \frac{\partial y}{\partial k} \right) \cdot \frac{dk}{dg} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

Households' optimal labor supply, given by (A7), implies that  $MRS_{kc} = (1 - \tau^L)(\partial y/\partial k)$ . The government's budget constraint implies that  $\tau^L = g/y$ . Last, from equation (4),  $\partial y/\partial k = y/k$ .

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<sup>32</sup>Formally, for all the models in section 2 and appendix B, the first-order condition with respect to  $k$  is  $\mathcal{W}'(k) = (1 - \tau^L)(1 - u(x))\lambda$ , where  $\lambda$  is the costate variable associated with real wealth in the household's Hamiltonian. We combine this equation and the first-order condition with respect to  $c$ , given by (10), and obtain (A7).

Hence,  $-(MRS_{kc} - \partial y/\partial k) = \tau_{LY}/k = g/k$  and we have proved the following:

**LEMMA A1:** *With a linear income tax, optimal public expenditure satisfies*

$$\underbrace{1 - \frac{d \ln(k)}{d \ln(g)}}_{\text{modified Samuelson rule}} = \underbrace{MRS_{gc} + \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}}_{\text{correction}}.$$

This optimality condition differs from (17), but the two have the same structure once the Samuelson rule is modified to account for distortionary taxation. Indeed, this condition can be written as the modified Samuelson rule plus a correction equal to  $(\partial y/\partial x) \cdot (dx/dg)$ . The statistic  $1 - d \ln(k)/d \ln(g) > 1$  in the modified Samuelson rule is the marginal cost of funds. It is more than one because the linear income tax distorts the labor supply. Because the marginal cost of funds is greater than one, the modified Samuelson rule recommends a lower level of public expenditure than the regular rule.

We generalize the definition of Samuelson spending with distortionary taxation:

**DEFINITION 7:** *With a linear income tax, Samuelson spending  $(g/c)^*$  is given by the modified Samuelson rule:*

$$MRS_{gc}((g/c)^*) = 1 - \frac{d \ln(k)}{d \ln(g)}.$$

*The elasticity  $d \ln(k)/d \ln(g) < 0$  is evaluated at optimal public expenditure.*

Although Samuelson spending is lower with a linear income tax, the correction to the Samuelson rule is the same, so our sufficient-statistic formula for optimal stimulus spending remains the same:

**PROPOSITION A1:** *Suppose that the economy is initially at an equilibrium  $[(g/c)^*, u_0]$ . Then, with a linear income tax, optimal stimulus spending satisfies (22) and the unemployment rate under the optimal policy satisfies (23), where the statistic  $z_1$  is generalized to allow for supply-side responses:*

$$z_1 = \frac{(g/y)^*(c/y)^*}{u^*} \cdot \frac{1}{1 - d \ln(k)/d \ln(g)}.$$

*The elasticity  $d \ln(k)/d \ln(g)$  is evaluated at  $[(g/c)^*, u^*]$ .*

**Proof:** With a linear income tax, Samuelson spending satisfies

$$MRS_{gc}(g/c^*) = 1 - \frac{d \ln(k)}{d \ln(g)},$$

so lemma A2 implies that optimal public expenditure satisfies

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{\partial y}{\partial x} \cdot \frac{dx}{dg}.$$

As in lemma 3, we have

$$MRS_{gc}((g/c)^*) - MRS_{gc}(g/c) = \frac{1}{\epsilon} \cdot \frac{g/c - (g/c)^*}{(g/c)^*}.$$

Moreover, (19) and (20) remain valid. Combining these results, we obtain (21).

Since formula (21) remains valid, the proof follows the same steps as the proof of proposition 1. The only difference occurs once we reach equation (A4). With a supply-side response to taxation, the equation becomes

$$1 = \frac{d \ln(g)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(x)} \cdot \frac{d \ln(x)}{d \ln(g/c)} - \frac{y}{c} \cdot \frac{\partial \ln(y)}{\partial \ln(k)} \cdot \frac{d \ln(k)}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)} + \frac{g}{c} \cdot \frac{d \ln(g)}{d \ln(g/c)}.$$

Using the same argument as in the proof of proposition 1, we can omit the term containing the factor  $\partial \ln(y)/\partial \ln(x)$ . Since  $\partial \ln(y)/\partial \ln(k) = 1$ , we therefore obtain  $d \ln(g)/d \ln(g/c) = (c/y)^*/(1 - d \ln(k)/d \ln(g))$ . Using the new expression for  $d \ln(g)/d \ln(g/c)$ , we conclude the proof like the proof of proposition 1.  $\square$

The unemployment multiplier in formulas (22) and (23) is a policy elasticity, in the sense of Hendren (2016). It measures the change in unemployment for a change in public expenditure accompanied by the change in taxes maintaining a balanced government budget. In section 3 taxes are not distortionary, so the unemployment multiplier should be measured using a policy reform in which taxes are nondistortionary. Here taxes are distortionary, so the unemployment multiplier should be measured using a policy reform in which the tax change distorts the labor supply.

When taxation is nondistortionary, equation (24) shows that the unemployment multiplier  $m$  in our sufficient-statistic formula is closely related to the empirical unemployment multiplier  $M$ . Furthermore, the output multiplier is equal to  $M$ , so all our results remain the same if we reformulate them with the output multiplier instead of  $m$ . But when taxation is distortionary, things are different, and the output multiplier cannot be used to design optimal public expenditure. With distortionary taxation, equation (24) remains valid, but the link between the output multiplier and  $M$  break down.



Indeed, output is  $Y = (1 - u)k$  so

$$\frac{dY}{dG} = -k \frac{du}{dG} + (1 - u) \frac{dk}{dG} = -\frac{Y}{1 - u} \cdot \frac{du}{dG} + \frac{Y}{k} \cdot \frac{dk}{dG} = M + \frac{Y}{k} \cdot \frac{dk}{dG}.$$

Since taxes are distortionary,  $dk/dG < 0$  and

$$M = \frac{dY}{dG} - \frac{Y}{k} \cdot \frac{dk}{dG} > \frac{dY}{dG}.$$

Thus, when a change in taxes distort the capacity supplied by households, the unemployment multiplier  $M$  is the output multiplier net of the supply-side response  $(Y/k)(dk/dG)$ . The supply-side response measures the percent change in labor supply when public expenditure increases by one percent of GDP. As taxation is distortionary, the supply-side response is negative and the unemployment multiplier is larger than the output multiplier. The unemployment multiplier is the correct sufficient statistic whether taxation is distortionary or not. With distortionary taxation, there is a wedge between unemployment and output multipliers equal to the supply-side responses, so the output multiplier is not useful to calibrate optimal stimulus spending.

Intuitively, an increase in public expenditure affects unemployment and the associated increase in taxes reduces labor supply. The negative effect on labor supply determines the marginal cost of fund and Samuelson spending but has nothing to do with the correction to the Samuelson rule and thus stimulus spending. The effect on unemployment, on the other hand, determines the correction to the Samuelson rule and thus stimulus spending. Since the unemployment multiplier measures the effect of public spending on unemployment, it governs optimal stimulus spending. Since the output multiplier conveys information about the effect of public spending on labor supply, it is not directly relevant to stimulus spending.

## Modern Approach

We turn to the modern approach to taxation in public economics, which consists in using a nonlinear income tax implemented according to the benefit principle. The benefit principle, which was introduced by Hylland and Zeckhauser (1979) and fully developed by Kaplow (1996, 1998), is an important result in modern public-economic theory: it states that the optimal provision of

public expenditure should be disconnected from distortionary taxation.<sup>33</sup> Hence, extra public expenditure should be financed by a change in the nonlinear tax schedule leaving all individual utilities unchanged—thus not altering further labor supply.

We assume that the government finances any increase in public expenditure by an increase in nonlinear income tax following the benefit principle: the tax schedule is changed to offset the extra benefit received by any individual from the extra public expenditure. Thus, the change in public expenditure leaves all individual utilities unchanged and does not further alter labor supply. In this case, although taxation is distortionary, we obtain the same results as with a fixed labor supply.

More precisely, we assume that households choose capacity  $k$  to maximize utility, and that public expenditure is funded by a distortionary, nonlinear income tax  $T(k)$ . We start from an equilibrium  $[c, g, x, k]$ . To ease notation, we introduce  $\phi(x) \equiv (1 - u(x))/(1 + \tau(x))$ . With the income tax, the household's disposable income becomes  $(1 - u(x))(k - T(k))$ . In equilibrium, households' disposable income equals their expenses:  $(1 - u(x))(k - T(k)) = (1 + \tau(x))c$  so  $c = \phi(x)(k - T(k))$ .

We implement a small change in public expenditure  $dg$  funded by a small tax change  $dT(k)$  that satisfies the benefit principle. This change triggers a small change  $dx$  in tightness. By the benefit principle, the tax change  $dT(k)$  is designed to keep the household's utility constant for any choice of  $k$ . For all  $k$ ,  $dT(k)$  satisfies

$$(A8) \quad \mathcal{U}(\phi(x) \cdot (k - T(k)), g) = \mathcal{U}(\phi(x + dx) \cdot (k - T(k) - dT(k)), g + dg).$$

The left-hand and right-hand sides of this equation define two identical functions of  $k$ . This implies that the household does not change his choice of  $k$  after the reform: the labor supply is unaffected by a change  $dg$  funded by the benefit principle.

Taking a first-order expansion of the right-hand side of (A8) and subtracting the left-hand side from the right-hand side, we obtain

$$\frac{\partial \mathcal{U}}{\partial c} \cdot [\phi'(x) \cdot (k - T(k)) \cdot dx - \phi(x) \cdot dT(k)] + \frac{\partial \mathcal{U}}{\partial g} \cdot dg = 0.$$

Dividing by  $\partial \mathcal{U} / \partial c$  and re-arranging yields

$$T(k) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k) = MRS_{gc} \cdot dg + \phi'(x) \cdot k \cdot dx.$$

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<sup>33</sup>See Kaplow (2004) and Kreiner and Verdellin (2012) for a survey of the benefit-principle approach.

Accordingly, the effect of the reform on the government budget balance  $R = \phi(x)T(k) - g$  is

$$dR = T(k) \cdot \phi'(x) \cdot dx + \phi(x) \cdot dT(k) - dg = (MRS_{gc} - 1) \cdot dg + \phi'(x) \cdot k \cdot dx = (MRS_{gc} - 1) \cdot dg + \frac{\partial y}{\partial x} \cdot dx.$$

(We used  $dk = 0$  and  $\phi'(x)k = \partial y / \partial x$ .) At the optimum,  $dR = 0$ , so we have proved the following:

**LEMMA A2:** *Under the benefit principle, optimal public expenditure satisfies (17).*

Under the benefit principle, (17) remains valid and capacity  $k$  is not affected by changes in public expenditure. Thus, our sufficient-statistic formula remains valid:

**PROPOSITION A2:** *Suppose that the economy is initially at an equilibrium  $[(g/c)^*, u_0]$ . Then, under the benefit principle, optimal stimulus spending satisfies (22) and the unemployment rate under the optimal policy satisfies (23).*

Finally, under the benefit principle, there are no labor-supply distortions for a marginal increase in public expenditure; therefore, output and unemployment multipliers are equal, and the output multiplier can be used to design optimal stimulus spending.

## Appendix E. Fixprice Model

Optimal public expenditure in the fixprice model presented in section 3.4 is straightforward. Only one result is cumbersome to obtain: the amount of stimulus spending required to completely fill the output gap, given by equation (26). We derive this result here. In addition, we present an extension of the fixprice model in which productive capacity is endogenous, not fixed, and we derive a sufficient-statistic formula for optimal public expenditure in that model. Compared to the fixprice model with fixed capacity, three differences arise: (a) the model offers a symmetric treatment of excessive production and insufficient production; (b) it is never optimal to completely fill the output gap; and (c) optimal stimulus spending is a smooth function of the sufficient statistics.

### Stimulus Spending Required to Fill the Output Gap

We derive (26). The economy starts at an equilibrium  $[(g/c)^*, y_0]$ , where output  $y_0 < k$  is inefficiently low. We compute the stimulus spending  $g/c - (g/c)^*$  required to fill the output gap  $k - y_0$ . To that

end, we link  $y$  to  $g/c$ . We write a first-order Taylor expansion of  $y(g/c)$  around  $y((g/c)^*) = y_0$ , evaluate it at  $y(g/c) = k$ , and divide it by  $y_0$ :

$$(A9) \quad \frac{k - y_0}{y_0} = \frac{d \ln(y)}{d \ln(\ln(g/c))} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right).$$

Next we compute  $d \ln(y)/d \ln(g/c)$  using the following decomposition:

$$(A10) \quad \frac{d \ln(y)}{d \ln(g/c)} = \frac{d \ln(y)}{d \ln(g)} \cdot \frac{d \ln(g)}{d \ln(g/c)} = (g/y)^* \cdot \frac{dy}{dg} \cdot \frac{d \ln(g)}{d \ln(g/c)},$$

where  $dy/dg$  is evaluated at  $[(g/c)^*, y_0]$ . The last step is to compute  $d \ln(g)/d \ln(g/c)$ . We have  $\ln(g/c) = \ln(g) - \ln(y - g)$ . Differentiating this equation with respect to  $\ln(g/c)$  yields

$$1 = \frac{d \ln(g)}{d \ln(g/c)} - (y/c)^* \frac{d \ln(y)}{d \ln(g/c)} + (g/c)^* \frac{d \ln(g)}{d \ln(g/c)}.$$

Using (A10) and reshuffling the terms, we obtain

$$\frac{d \ln(g)}{d \ln(g/c)} = \frac{1}{1 + (g/c)^* - (g/c)^*(dy/dg)}.$$

And using (A10) again, we find

$$(A11) \quad \frac{d \ln(y)}{d \ln(g/c)} = \frac{(g/y)^*(dy/dg)}{1 + (g/c)^* - (g/c)^*(dy/dg)} = \frac{(c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)}.$$

Combining (A11) with (A9), we finally obtain

$$\frac{g/c - (g/c)^*}{(g/c)^*} = \frac{1 - (g/y)^*(dy/dg)}{(c/y)^*(g/y)^*(dy/dg)} \cdot \frac{k - y_0}{y_0} + O\left([g/c - (g/c)^*]^2\right),$$

where the output multiplier  $dy/dg$  is evaluated at  $[(g/c)^*, y_0]$ . This equation yields (26).

## Endogenous Productive Capacity

We extend the fixprice model by introducing endogenous productive capacity, and we describe optimal public expenditure in that model. We could introduce endogenous capacity by assuming that households are price-takers: they supply capacity  $k$  to maximize utility given the price of services.

But proceeding this way has a downside: it introduces an internal inconsistency in the model when there is excess supply. Indeed, aggregate supply would describe how much households desire to work for a given price, assuming that they can sell all the services that they supply to the market. But in fact they would not be able to sell all the services because there is excess supply. To be consistent, the model should allow households to revise their supply decision given that in fact the probability to sell a given service is less than one.<sup>34</sup>

Here we address this issue as in the New Keynesian literature. We assume that households are price-setters: they set the price of services to maximize profits and supply the amount of services demanded at the profit-maximizing price. When the price is fixed, households simply supply as many services as required to satisfy demand (for example, Nakamura and Steinsson 2014, p. 773). Let  $y$  be aggregate output of services, which is demand-determined. Since households supply exactly the amount of services required by demand, aggregate supply of services is  $k = y$ .

The government now chooses  $g$  to maximize  $\mathcal{U}(y - g, g) - \mathcal{W}(y)$ . The first-order condition of the maximization is

$$(A12) \quad 1 = MRS_{gc} + \frac{dy}{dg} \cdot (1 - MRS_{kc}),$$

where  $MRS_{kc} \equiv \mathcal{W}'(k)/(\partial\mathcal{U}/\partial c)$  is the marginal rate of substitution between labor and private consumption. This equation is the same as equation (25), except that the output multiplier is multiplied by the labor wedge  $1 - MRS_{kc}$ .<sup>35</sup> This equation is also the same as equation (45) in Woodford (2011)—this is not surprising since our fixprice model has all the key ingredients of the New Keynesian model considered by Woodford.

The economy can be in three possible regimes, depending on the labor wedge : efficient production when  $1 - MRS_{kc} = 0$ , insufficient production when  $1 - MRS_{kc} > 0$  (a slump), and excessive production  $1 - MRS_{kc} < 0$  (a boom). When there is efficient production,  $MRS_{kc} = 1$  and the Samuelson rule remains valid. When there is excessive or insufficient production, things change:  $MRS_{kc} \neq 1$  so the correction to the Samuelson rule is nonzero.

We assume that the economy starts at  $[(g/c)^*, y_0]$ , with a marginal rate of substitution  $(MRS_{kc})_0 \neq 1$ . Following the procedure developed in the matching model, we obtain a formula expressed as a

<sup>34</sup>The matching model addresses this issue by introducing a matching function that gives the probabilities to sell services, and by letting households take these probabilities into account when they make their supply decisions.

<sup>35</sup>The labor wedge plays an important role in macroeconomics. See Shimer (2009) for a discussion.

function of fixed (not endogenous) sufficient statistics:

**PROPOSITION A3:** *Suppose that the economy is initially at an equilibrium  $[(g/c)^*, y_0]$ . Then optimal stimulus spending satisfies*

$$(A13) \quad \frac{g/c - (g/c)^*}{(g/c)^*} \approx \frac{\epsilon \cdot (dy/dg)}{1 + z_3 \cdot \epsilon \cdot \frac{(dy/dg)^2}{1 - (g/y)^*(dy/dg)}} [1 - (MRS_{kc})_0].$$

The statistics  $\epsilon$  and  $dy/dg$  are evaluated at  $[(g/c)^*, y_0]$ , and  $z_3 \equiv (MRS_{kc})_0 (c/y)^* (g/y)^* / \kappa$ , where  $\kappa \equiv 1/[d \ln(\mathcal{W}'(k))/d \ln(k)]$  is the Frisch elasticity of labor supply. Under the optimal policy, the labor wedge is

$$(A14) \quad 1 - MRS_{kc} \approx \frac{1}{1 + z_3 \cdot \epsilon \cdot \frac{(dy/dg)^2}{1 - (g/y)^*(dy/dg)}} [1 - (MRS_{kc})_0].$$

The approximations (22) and (23) are valid up to a remainder that is  $O([g/c - (g/c)^*]^2)$ .

**Proof:** Optimal stimulus spending satisfies (A12), which can be rewritten using (18):

$$(A15) \quad \frac{g/c - (g/c)^*}{(g/c)^*} = \epsilon \cdot (dy/dg) \cdot (1 - MRS_{kc}) + O([g/c - (g/c)^*]^2).$$

As in the matching model,  $MRS_{kc}$  responds to  $g/c$  when it deviates from  $(g/c)^*$ , so we cannot use (A15) to compute optimal stimulus spending. We follow the procedure developed in the matching model and re-express (A15) as a function of fixed sufficient statistics.

To that end, we analyze how  $MRS_{kc}$  respond to  $g/c$ . In this demand-determined economy, the aggregate-demand relationship always holds. Since the asset (land in our baseline model) is in fixed supply and prices are fixed, the marginal utility of private consumption ( $\partial \mathcal{U} / \partial c$ ) is fixed and does not change when public consumption changes.<sup>36</sup> Hence, we only consider how the marginal disutility of labor ( $\mathcal{W}'(k)$ ) reacts to public consumption. We find

$$\frac{d \ln(MRS_{kc})}{d \ln(g/c)} = \frac{d \ln(\mathcal{W}'(k))}{d \ln(g/c)} = \frac{1}{\kappa} \cdot \frac{d \ln(y)}{d \ln(g/c)},$$

<sup>36</sup>For example, in the demand side with land of section 2.4, the aggregate demand is given by  $\partial \mathcal{U} / \partial c = p \mathcal{V}'(l_0) / \delta$ . This relationship always holds since the economy is demand-determined. As  $l_0$  and  $p$  are fixed,  $\partial \mathcal{U} / \partial c$  does not respond to  $g$ .

where  $\kappa = 1/[d \ln(\mathcal{W}'(k))/d \ln(k)]$  is the Frisch elasticity of labor supply. Using (A11), we obtain

$$\frac{d \ln(MRS_{kc})}{d \ln(g/c)} = \frac{1}{\kappa} \cdot \frac{(c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)}.$$

Accordingly, the first-order Taylor expansion of  $MRS_{kc}(g/c)$  around  $(g/c)^*$  is

$$MRS_{kc} = (MRS_{kc})_0 + \frac{1}{\kappa} \cdot \frac{(MRS_{kc})_0 (c/y)^*(g/y)^*(dy/dg)}{1 - (g/y)^*(dy/dg)} \cdot \frac{g/c - (g/c)^*}{(g/c)^*} + O\left([g/c - (g/c)^*]^2\right).$$

In the equation the multiplier  $dy/dg$  and the elasticity  $\kappa$  are evaluated at  $[(g/c)^*, y_0]$ . To obtain (A13), we plug this expression for  $MRS_{kc}$  into (A15) and reshuffle the terms. In addition, combining equations (A15) and (A13), we obtain (A14).  $\square$

Formula (A13) is similar to formula (22) in the matching model; the principal difference is that the amount of inefficiency is not measured by the unemployment gap but by the labor wedge  $1 - (MRS_{kc})_0$ . Nonetheless the formula has very similar implications. First, with a positive output multiplier, then optimal stimulus spending is positive in slumps but negative in booms. Second, optimal stimulus spending is a hump-shaped function of the output multiplier. Third, optimal stimulus spending is larger when public consumption substitutes more easily for private consumption. Last, optimal stimulus spending only partially reduces the output gap:  $MRS_{kc}$  is brought closer to 1, but remains below 1.

Overall, the fixprice model with endogenous capacity leads to similar insights as the matching model. This is reassuring: irrespective of how productive inefficiency is modeled, stimulus spending obeys similar general principles.

Yet, for several reasons, the matching model seems more convenient than the fixprice model with endogenous capacity to think about optimal public expenditure. A first limitation of the fixprice model is that its description of booms is not fully satisfactory. When there is excessive production,  $MRS_{kc} > 1$  which implies  $\mathcal{W}'(k) > \partial \mathcal{U} / \partial c$ : people, constrained to supply the amount of services demanded, are working more than they want. If workers were not bound to supply whatever is demanded, all of them would stop providing services in booms, as the cost of providing each service is higher than the income received. In the matching model, in contrast, all relationships generate surplus for both buyer and seller.

Another limitation of the fixprice model is that the supply side is irrelevant, as the equilibrium

is demand-determined. Therefore, distortionary taxation has no effect at all, and the model is not useful to study distortionary taxation. In contrast, in the matching model, both supply and demand determine the equilibrium. The matching model is therefore well suited to study the effect of distortionary taxation on optimal public expenditure—something we do in appendix D.

A last limitation of the fixprice model is that the labor wedge  $1 - (MRS_{kc})_0$  is more challenging to measure than the unemployment gap  $u_0 - u^*$ . As a result, the fixprice formula (A13) is less convenient to apply than the matching formula (22). Since  $u_0$  is observable, measuring the unemployment gap only requires to measure the efficient unemployment  $u^*$ . This can be done from (5), following the method developed by Landais, Michailat, and Saez (2018). This can also be done by using historical unemployment data, since  $u^*$  does not respond to typical macroeconomic shocks and is therefore expected to be stable over time (see section 4). In contrast, it is difficult to measure the labor wedge because it is not possible to relate  $(MRS_{kc})_0$  to observable variables.<sup>37</sup> One strategy to measure  $(MRS_{kc})_0$  would be to assume that output is efficient before the shocks and that the utility functions  $\mathcal{W}$  and  $\mathcal{U}$  are stable. Then we could recover  $(MRS_{kc})_0$  from the observed change in output, the Frisch elasticity (to link the output change to the change in  $\mathcal{W}'(k)$ ), and a coefficient of risk aversion (to link the output change to the change in  $\partial\mathcal{U}/\partial c$ ). This strategy could work with aggregate-demand shocks but not with aggregate-supply shocks, as the disutility from labor  $\mathcal{W}$  varies under such shocks. Hence, it is generally impossible to measure the labor wedge.

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<sup>37</sup>For the same reason, it is difficult to measure the New Keynesian output gap in the data (Gali 2008, pp. 80–81).



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