Handout #1

Derivation of Factor Price Frontier Expression for the Two-Sector Incidence Model

By definition of the elasticities of substitution in production, we know that

$$\hat{K}_X - \hat{L}_X = \sigma_X(\hat{w} - \hat{r}) \quad \text{and} \quad \hat{K}_Y - \hat{L}_Y = \sigma_Y(\hat{w} - \hat{r})$$

For convenience, express K and L as ratios of output, e.g., $k_X \equiv K_X/X$. It follows that

$$\text{(1a)} \quad \hat{k}_X - \hat{l}_X = \sigma_X(\hat{w} - \hat{r}) \quad \text{and} \quad \text{(1b)} \quad \hat{k}_Y - \hat{l}_Y = \sigma_Y(\hat{w} - \hat{r})$$

By the envelope theorem, we know $d(rk_X + wl_X) = k_X dr + l_X dw$, so $rdk_X + wdl_X = 0 \;\; \Rightarrow$

$$\text{(2a)} \quad \left(\frac{rk_X}{P_X}\right)\hat{k}_X + \left(\frac{wl_X}{P_X}\right)\hat{l}_X = \theta_{KX}\hat{k}_X + \theta_{LX}\hat{l}_X = 0; \text{ also (2b)} \quad \theta_{KY}\hat{k}_Y + \theta_{LY}\hat{l}_Y = 0$$

Finally, since $L_X + L_Y = l_X X + l_Y Y = \overline{L}$, we may totally differentiate to obtain:

$$\text{(3a)} \quad (\hat{l}_X + \hat{X})\lambda_{LX} + (\hat{l}_Y + \hat{Y})\lambda_{LY} = 0; \quad \text{also} \quad \text{(3b)} \quad (\hat{k}_X + \hat{X})\lambda_{KX} + (\hat{k}_Y + \hat{Y})\lambda_{KY} = 0$$

where $\lambda_{LX} = L_X / \overline{L}$ is the share of the economy's labor that is used in sector X, and the other terms are defined in the same manner.

Now, substitute (2a) into (1a) and (2b) into (1b) to get expressions for $\hat{l}_X$ and $\hat{l}_Y$ and (using the fact that the labor and capital cost shares θ add to one for each sector, and that $\lambda_{LX} + \lambda_{LY} = 1$) substitute these expressions into (3a) to obtain:

$$\text{(4a)} \quad \lambda_{LX}\hat{X} + \lambda_{LY}\hat{Y} = (\lambda_{LX}\theta_{KX}\sigma_X + \lambda_{LY}\theta_{KY}\sigma_Y)(\hat{w} - \hat{r})$$

Follow the same procedure to get expressions for $\hat{k}_X$ and $\hat{k}_Y$ to substitute into (3b) to obtain:

$$\text{(4b)} \quad \lambda_{KX}\hat{X} + \lambda_{KY}\hat{Y} = -(\lambda_{KX}\theta_{LX}\sigma_X + \lambda_{KY}\theta_{LY}\sigma_Y)(\hat{w} - \hat{r})$$, and subtract (4b) from (4a) to obtain:

$$\boxed{(\hat{w} - \hat{r}) = \frac{\lambda^*}{a_X\sigma_X + a_Y\sigma_Y}\left(\hat{X} - \hat{Y}\right) = \frac{\lambda^*}{\overline{\sigma}}\left(\hat{X} - \hat{Y}\right)}$$

where $a_X = \lambda_{LX}\theta_{KX} + \lambda_{KX}\theta_{LX}; \quad a_Y = \lambda_{LY}\theta_{KY} + \lambda_{KY}\theta_{LY}; \quad \lambda^* = \lambda_{LX} - \lambda_{KX}$