University of California, Berkeley
Department of Economics

Economics 230a
Fall 2005

Handout #2: Derivation of the User Cost of Capital

Consider a firm wishing to maximize its value, defined as:

$$(1)\qquad V_t = \int_t^{\infty} e^{-r(s-t)} X_s ds$$

where X_s is the firm's cash flow from real activities at date s,

$$(2)\qquad X_s = (1-\tau_s)p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{-\infty}^{s} D_u(s-u) q_u I_u du$$

In (2), K_s is the capital stock at date s, I_s is the investment flow, p_s is the price of output, and q_s is the price of capital. The corporate tax system has three components: τ_s, the corporate tax rate at date s, k_s, the initial subsidy to investment (e.g., an investment tax credit), and $D_u(s-u)$, the depreciation deduction at date s per dollar of investment at an earlier date u. This deduction depends not only on the age of the asset, $(s-u)$, but also on the tax rules that prevailed at date u.

Inserting (2) into (1) yields:

$$
\begin{aligned}
V_t &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s)p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{-\infty}^{s} D_u(s-u) q_u I_u du \right) ds \\
(3)\qquad &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s)p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{t}^{s} D_u(s-u) q_u I_u du + \tau_s \int_{-\infty}^{t} D_u(s-u) q_u I_u du \right) ds \\
&= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s)p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{t}^{s} D_u(s-u) q_u I_u du \right) ds + \overline{V}_t
\end{aligned}
$$

where the second line of (3) breaks the flows of depreciation allowances into two pieces: those attributable to investment after date t and before date t. The second piece, $\overline{V}_t$, affects the value of the firm at date t, but not its decisions from date t onward, and so may be ignored in the optimization process. The remaining expression for firm value can be simplified by changing the order of integration for depreciation allowances (first over date of allowances, then over date of investment, rather than starting with date of investment), leading to:

$$
\begin{aligned}
V_t &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s)p_s F(K_s) - q_s I_s (1-k_s) + q_s I_s \int_{s}^{\infty} e^{-r(u-s)} \tau_u D_s(u-s) du \right) ds + \overline{V}_t \\
(4)\qquad &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s)p_s F(K_s) - q_s I_s (1-\Gamma_s) \right) ds + \overline{V}_t
\end{aligned}
$$

where $\Gamma_s = k_s + \int_s^\infty e^{-r(u-s)} \tau_u D_s(u-s)du$ is the present value of tax benefits per dollar invested at date *s*.

The firm seeks to maximize its value at time *t*, as defined in expression (4). Determining the optimal investment policy requires further specification of the firm's technology. We assume that capital depreciates exponentially at rate δ, so that:

(5) $$\dot{K}_t = I_t - \delta K_t$$

and that the full marginal cost of investment, *q*, is not affected by the level of investment. Then, inserting (5) into (4) and solving for an optimum based on the Euler equation,

$$\frac{\partial V_t}{\partial K_s} - \frac{d\left(\partial V_t / \partial \dot{K}_s\right)}{ds} = 0,$$

yields:

(6) $$F'(K_s) = \frac{q_s^*}{p_s} \frac{\left(r + \delta - \frac{\dot{q}_s^*}{q_s^*}\right)}{(1-\tau_s)}$$

where $q_s^* = q_s(1-\Gamma_s)$, which one may think of as the effective price of capital goods, taking into account the present value of tax benefits directly associated with investment. The expression on the right-hand side of (6) is referred to as the *user cost of capital*.