Factor Price Frontier Derivation for the Two-Sector Incidence Model

By definition of the production elasticities of substitution, we know that \( \hat{K}_i - \hat{L}_i = \sigma_i (\hat{w} - \hat{r}) \) for \( i = X, Y \). For convenience, express \( K \) and \( L \) as ratios of output, e.g., \( k_X \equiv K_X/X \). It follows that

\[
(1) \quad \hat{k}_i - \hat{l}_i = \sigma_i (\hat{w} - \hat{r}) \quad i = X, Y
\]

By the envelope theorem, we know that derivatives of the cost function satisfy \( d(rk_i + wl_i) = k_idr + l_idw \), so \( rdk_i + wdl_i = 0 \). This implies that

\[
(2) \quad \left( \frac{rk_i}{P_i} \right) \hat{k}_i + \left( \frac{wl_i}{P_i} \right) \hat{l}_i = \theta_{ki} \hat{k}_i + \theta_{li} \hat{l}_i = 0 \quad i = X, Y
\]

where \( \theta_{ji} \) is the cost share of factor \( j \) in sector \( i \).

Finally, note that \( L_X + L_Y = l_XX + l_YY = \bar{L} \); \( K_X + K_Y = k_XX + k_YY = \bar{K} \); totally differentiating:

\[
(3a) \quad (\hat{L}_X + \hat{X}) \lambda_{LX} + (\hat{L}_Y + \hat{Y}) \lambda_{LY} = 0; \quad \text{also} \quad (3b) \quad (\hat{k}_X + \hat{X}) \lambda_{kX} + (\hat{k}_Y + \hat{Y}) \lambda_{KY} = 0
\]

where \( \lambda_{LX} = L_X/\bar{L} \) is the share of the economy’s labor that is used in sector \( X \), and the other terms are defined in the same manner.

Now, substitute (2) into (1) for both sectors to get expressions for \( \hat{L}_X \) and \( \hat{L}_Y \) and (using the fact that the labor and capital cost shares \( \theta \) add to one for each sector, and that \( \lambda_{LX} + \lambda_{LY} = 1 \)) substitute these expressions into (3a) to obtain:

\[
(4a) \quad \lambda_{LX} \hat{X} + \lambda_{LY} \hat{Y} = (\lambda_{LX} \theta_{kX} \sigma_X + \lambda_{LY} \theta_{kY} \sigma_Y) (\hat{w} - \hat{r})
\]

Follow the same procedure to get expressions for \( \hat{k}_X \) and \( \hat{k}_Y \) to substitute into (3b) to obtain:

\[
(4b) \quad \lambda_{kX} \hat{X} + \lambda_{kY} \hat{Y} = -(\lambda_{kX} \theta_{lX} \sigma_X + \lambda_{kY} \theta_{lY} \sigma_Y) (\hat{w} - \hat{r}), \quad \text{and subtract (4b) from (4a) to obtain:}
\]

\[
(\hat{w} - \hat{r}) = \frac{\lambda^*}{a_X \sigma_X + a_Y \sigma_Y} (\hat{X} - \hat{Y}) = \frac{\lambda^*}{\sigma} (\hat{X} - \hat{Y})
\]

where \( a_i (= \lambda_{kX} \theta_{lX} + \lambda_{kY} \theta_{lY}) \) measures the relative size of production sector \( i \) based on its shares of the economy’s capital and labor, and \( \lambda^* (= \lambda_{LX} - \lambda_{kX} ) \) is positive (negative) if sector \( X \) is more (less) labor intensive than sector \( Y \). Thus, a shift in production toward \( X \) will increase the relative return to the factor that \( X \) uses relatively intensively.