Economics 230a **Fall 2011**

Factor Price Frontier Derivation for the Two-Sector Incidence Model

By definition of the production elasticities of substitution, we know that $\hat{K}_i - \hat{L}_i = \sigma_i(\hat{w} - \hat{r})$ for $i = X, Y$. For convenience, express K and L as ratios of output, e.g., $k_X \equiv K_X/X$. It follows that

(1) $\hat{k}_i - \hat{l}_i = \sigma_i(\hat{w} - \hat{r}) \qquad i = \text{X}, Y$

By the envelope theorem, we know that derivatives of the cost function satisfy $d(rk_i + wl_i) = k_i dr + l_i dw$, so $rdk_i + wdl_i = 0$. This implies that

(2) $\left(\frac{rk_i}{P_i}\right)\hat{k}_i + \left(\frac{wl_i}{P_i}\right)\hat{l}_i = \theta_{Ki}\hat{k}_i + \theta_{Li}\hat{l}_i = 0 \qquad i = \text{X}, Y$

where θ_{ji} is the cost share of factor j in sector i.

Finally, note that $L_X + L_Y = l_X X + l_Y Y = \bar{L}$; $K_X + K_Y = k_X X + k_Y Y = \bar{K}$; totally differentiating:

(3a) $(\hat{l}_X + \hat{X})\lambda_{LX} + (\hat{l}_Y + \hat{Y})\lambda_{LY} = 0$; also (3b) $(\hat{k}_X + \hat{X})\lambda_{KX} + (\hat{k}_Y + \hat{Y})\lambda_{KY} = 0$

where $\lambda_{LX} = L_X / \bar{L}$ is the share of the economy's labor that is used in sector X, and the other terms are defined in the same manner.

Now, substitute (2) into (1) for both sectors to get expressions for $\hat{l}_X$ and $\hat{l}_Y$ and (using the fact that the labor and capital cost shares θ add to one for each sector, and that $\lambda_{LX} + \lambda_{LY} = 1$) substitute these expressions into (3a) to obtain:

(4a) $\lambda_{LX}\hat{X} + \lambda_{LY}\hat{Y} = (\lambda_{LX}\theta_{KX}\sigma_X + \lambda_{LY}\theta_{KY}\sigma_Y)(\hat{w} - \hat{r})$

Follow the same procedure to get expressions for $\hat{k}_X$ and $\hat{k}_Y$ to substitute into (3b) to obtain:

(4b) $\lambda_{KX}\hat{X} + \lambda_{KY}\hat{Y} = -(\lambda_{KX}\theta_{LX}\sigma_X + \lambda_{KY}\theta_{LY}\sigma_Y)(\hat{w} - \hat{r})$, and subtract (4b) from (4a) to obtain:

$$\boxed{(\hat{w} - \hat{r}) = \frac{\lambda^*}{a_X\sigma_X + a_Y\sigma_Y}\left(\hat{X} - \hat{Y}\right) = \frac{\lambda^*}{\sigma}\left(\hat{X} - \hat{Y}\right)}$$

where $a_i (= \lambda_{Li}\theta_{Ki} + \lambda_{Ki}\theta_{Li})$ measures the relative size of production sector i based on its shares of the economy's capital and labor, and $\lambda^* (= \lambda_{LX} - \lambda_{KX})$ is positive (negative) if sector X is more (less) labor intensive than sector Y. Thus, a shift in production toward X will increase the relative return to the factor that X uses relatively intensively.