Answers to Problem Set #1

1. The household maximizes utility, \( U(c_1, c_2, l) = (c_1 - a_1)^{\beta_1} (c_2 - a_2)^{\beta_2} l^{1 - \beta_1 - \beta_2} \), subject to the budget constraint \( p_1 c_1 + p_2 c_2 + w l = w \).

A. To obtain an expression for excess burden, we need to start with an expression for the expenditure function. Setting up a Lagrangian,

\[
\mathcal{L} = (c_1 - a_1)^{\beta_1} (c_2 - a_2)^{\beta_2} l^{1 - \beta_1 - \beta_2} + \lambda \left( w(1 - l) - p_1 c_1 - p_2 c_2 \right),
\]

we get the first-order conditions with respect to \( c_1 \), \( c_2 \), and \( l \):

\[
\frac{\beta_i U}{c_i - a_i} = \lambda p_i, \quad i = 1, 2; \quad \frac{(1 - \beta_1 - \beta_2) U}{l} = \lambda w.
\]

Substituting these into the expression for the utility function \( U = U(c_1, c_2, l) \) and solving for \( \lambda \) yields:

\[
\lambda = \beta_1^{\beta_1} \beta_2^{\beta_2} (1 - \beta_1 - \beta_2) \left( (1 - \beta_1 - \beta_2)^{\beta_1} p_1^{\beta_1} p_2^{\beta_2} w^{-(1 - \beta_1 - \beta_2)} \right) = \left( \frac{1}{k} \right) p_1^{\beta_1} p_2^{\beta_2} w^{-(1 - \beta_1 - \beta_2)},
\]

or

\[
\lambda(p_1, p_2, w) = \left( \frac{1}{k} \right) p_1^{\beta_1} p_2^{\beta_2} w^{-(1 - \beta_1 - \beta_2)}
\]

where \( k = \beta_1^{\beta_1} \beta_2^{\beta_2} (1 - \beta_1 - \beta_2)^{-(1 - \beta_1 - \beta_2)} \).

Substituting this expression for \( \lambda \) into the first-order conditions for \( c_1 \), \( c_2 \) and \( l \) yields the compensated demand functions,

\[
c_1 = a_1 + \beta_1 k p_1^{\beta_1 - 1} p_2^{\beta_2} w^{-(1 - \beta_1 - \beta_2)} U; \quad c_2 = a_2 + \beta_2 k p_1^{\beta_1} p_2^{\beta_2 - 1} w^{(1 - \beta_1 - \beta_2)} U;
\]

\[
l = (1 - \beta_1 - \beta_2) k p_1^{\beta_1} p_2^{\beta_2} w^{-(1 - \beta_1 - \beta_2)} U
\]

Substituting these demand functions into the budget constraint yields the expenditure function,

\[
E(p_1, p_2, w, U) = p_1 a_1 + p_2 a_2 - w + k p_1^{\beta_1} p_2^{\beta_2} w^{(1 - \beta_1 - \beta_2)} U.
\]
The expression for excess burden equals the expenditure function evaluated with taxes minus the expenditure function evaluated without taxes minus revenue collected, all at the same level of utility. Using the above compensated demand functions to derive an expression for revenue at this utility level, we get (after a couple of steps):

\[ EB = E(p_1^1, p_2^1, w^1, U) - E(p_1^0, p_2^0, w^0, U) - Rev \]

\[
= kU \left[ p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \left( \beta_1 \left( \frac{p_1^0}{p_1^1} \right) + \beta_2 \left( \frac{p_2^0}{p_2^1} \right) + (1 - \beta_1 - \beta_2) \left( \frac{w^0}{w^1} \right) \right) - p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \right] ,
\]

where the superscript indicates with taxes (= 1) or without taxes (= 0).

B. If all prices are inflated by the term \((1+\theta)\), then excess burden is:

\[
= kU \left[ (1+\theta) p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \left( \frac{1}{1+\theta} \right) \left( \beta_1 + \beta_2 + (1 - \beta_1 - \beta_2) \right) - p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \right] = 0.
\]

C. Using the general expression for excess burden derived in part A, the excess burden for a tax or subsidy on good 2 is:

\[
= kU \left[ p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \left( \beta_1 + \beta_2 \left( \frac{p_2^0}{p_2^1} \right) + (1 - \beta_1 - \beta_2) \right) - p_1^{\beta_1} p_2^{\beta_2} w^{(1-\beta_1-\beta_2)} \right] = kU p_1^{\beta_2} w^{(1-\beta_1-\beta_2)} \left( \beta_2 p_2^0 - p_2^0 \right) = X \left( 1 - \beta_2 \right) p_2^0 \left( p_2^1 - p_2^0 \right)
\]

where \(X > 0\). Taking the derivative of this term for excess burden with respect to \(p_2\), we get

\[
X \left( 1 - \beta_2 \right) \beta_2 p_2^{\beta_2 - 1} + \beta_2 (\beta_2 - 1) p_2^0 \beta_2 \left( p_2^1 - p_2^0 \right) = X \left( 1 - \beta_2 \right) \beta_2 p_2^{\beta_2 - 2} \left( p_2^0 - p_2^1 \right)
\]

which has the same sign as \(p_2^1 - p_2^0\). Therefore, marginal excess burden is zero when there is no initial distortion and is increasing with the absolute value of the tax or subsidy.

2. A. Expected terminal wealth from holding is \(P_t(1+\bar{r})(1-c) + P_t c\). Expected terminal wealth from realizing and reinvesting equals \((P_t(1-c) + P_t c)(1+r^+ (1-c) + r^- (1-t))\), where \(r^+ = \int_{r_{\text{min}}}^{r_{\text{max}}} rf(r) dr\) and \(r^- = \int_{r_{\text{min}}}^{0} rf(r) dr\) are the expected values of the positive and negative portions of the return distribution, with \(\bar{r} = r^+ + r^-\). (Note that \(r^+ > \bar{r} > 0 > r^-\)). After a few steps, one can show that the expected terminal wealth from realizing is
higher than that from holding if and only if \( R = \frac{P_t}{P_0} < \frac{c - r^+c - r^-t}{1 - c} \), which we define as \( R^* \). Note that if \( c = t \), then \( R^* = 1 \) – the investor will realize if there is a loss and hold if there is a gain. But the decision is more complicated if \( c \neq t \).

B. After simplifying, one obtains: 

\[
\frac{dR^*}{dc} = R^* \left( \frac{1}{c} + \frac{1}{1-c} - \frac{r^+}{\bar{r} - r^+c - r^-t} - \frac{r^+}{r^+c + r^-t} \right)
\]

Setting \( c = t \), we get 

\[
\frac{dR^*}{dc} = R^* \left( \frac{1}{c} + \frac{1}{1-c} \right) \left( 1 - \frac{r^+}{\bar{r}} \right) < 0.
\]

A higher capital gains rate lowers \( R^* \) and therefore increases the lock-in effect.

C. For the tax rate on losses, we get: 

\[
\frac{dR^*}{dt} = -R^* r^\bar{r} \left( \frac{1}{\bar{r} - r^+c - r^-t} + \frac{1}{r^+c + r^-t} \right).
\]

Setting \( c = t \), we get 

\[
\frac{dR^*}{dt} = -R^* r^\bar{r} \left( \frac{1}{\bar{r}(1-t)} + \frac{1}{\bar{r}t} \right) > 0.
\]

That is, the opportunity to deduct losses at a higher rate in the future if gains are initially realized increases current realizations.

3. A. \( pc_2 = (wL(1-t) + B - pc_1)(1 + r(1-t)) \Rightarrow c_1 + \frac{1}{1+r(1-t)}c_2 = \frac{wL}{p} + \frac{B}{p} \).

B. \( pc_2 = (wL(1-t) + pA + B - pc_1)(1 + r(1-t)) \Rightarrow c_1 + \frac{1}{1+r(1-t)}c_2 = \frac{wL}{p} + A + \frac{B}{p} \).

C. \( p(1+\bar{\tau})c_2 = (wL + pA + B - p(1+\bar{\tau})c_1)(1 + r) \Rightarrow c_1 + \frac{1}{1+r}c_2 = \frac{1}{1+\bar{\tau}} \left( \frac{wL}{p} + A + \frac{B}{p} \right) \).

D. Because the consumer price level remains at \( p \), the producer price of consumption falls to \( p/(1+\bar{\tau}) \). The nominal wage therefore also falls, to \( w/(1+\bar{\tau}) \), so the budget constraint is: 

\[
pc_2 = \left( \frac{wL}{1+\bar{\tau}} + \frac{p}{1+\bar{\tau}} A - pc_1 \right)(1 + r) \Rightarrow c_1 + \frac{1}{1+r}c_2 = \left( \frac{wL}{1+\bar{\tau}} + A \right) + \frac{B}{p}.
\]

There is no longer an effective tax on \( B \), since the consumer price level hasn’t changed.