**Economics 101A**
Section Notes
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# Consumer Theory and the Envelope Theorem

## 1 Utility Maximization Problem

The consumer problem looked at here involves

- Two goods: $x$ and $y$ with prices $p_x$ and $p_y$.
- Conusumers facing a *budget constraint* $p_x x + p_y y \leq I$, where $I$ is income.Consumers maximize *utility* $U(x,y)$ which is increasing in both arguments and quasi-concave in $(x,y)$. Also the non-negativity constraints $x \geq 0$ and $y \geq 0$ must hold as consumption cannot be negative.
- Competition in this world is *"perfect" or "pure"* so consumers are price takers, i.e. $p_x$ and $p_y$ are fixed for them. This may be justified by postulating a large number of consumers.

Since utility is increasing in both $x$ and $y$ we can safely assume that the budget constraint $p_x x + p_y y \leq I$ is satisfied with equality. We will also assume that the non-negativity constraints are slack so that $x > 0$ and $y > 0$. Setting the inequalities to equality, combining the two constraints, and rearranging a bit we get the standard **utility maximization problem** (UMP for short)

$$\max_{x,y} U(x,y) \quad \text{s.t. } p_x x + p_y y = I \tag{UMP}$$

The solution to this problem is typically found by writing the Lagrangean

$$\mathfrak{L}(x,y,l) = U(x,y) + \alpha(I - p_x x - p_y y)$$

and taking the first order conditions[1]

$$\frac{\partial \mathfrak{L}}{\partial x} = \frac{\partial U(x^d, y^d)}{\partial x} - \alpha^d p_x = 0$$
$$\frac{\partial \mathfrak{L}}{\partial y} = \frac{\partial U(x^d, y^d)}{\partial y} - \alpha^d p_y = 0$$
$$\frac{\partial \mathfrak{L}}{\partial \alpha} = I - p_x x^d - p_y y^d = 0$$

The superscript "$^d$" is used to refer to the fact that these solutions are the consumer's demands.

Solving yields the Lagrange multiplier $\alpha^d = \alpha^d(p_x, p_y, I)$ and the demand functions

$$x^d(p_x, p_y, I) \quad y^d(p_x, p_y, I)$$

To be more general we call these the **uncompensated (or Marshallian or Walrasian) demand functions**. These functions are "uncompensated" since price changes will cause utility changes: a situation that does not occur with compensated demand curves. Substituting these solutions back into the utility function,

[1] If the we did not assume the non-negativity constraints held the first two first order conditions would be

$$\frac{\partial \mathfrak{L}}{\partial x} = \frac{\partial U(x^D, y^D)}{\partial x} - \alpha^* p_x < 0, \text{ with " = " if } x^D > 0$$
$$\frac{\partial \mathfrak{L}}{\partial y} = \frac{\partial U(x^D, y^D)}{\partial y} - \alpha^* p_y < 0, \text{ with " = " if } y^D > 0$$

Note that this is a simplification of the Kuhn-Tucker conditions described previously.

the maximand, we get the actual utility achieved as a function of prices and income. This function is known as the **indirect utility function**

$$V(p_x, p_y, I) \equiv U\left[x^d(p_x, p_y, I), y^d(p_x, p_y, I)\right] \qquad \text{(Indirect Utility Function)}$$

This function says how much utility consumers are getting when they face prices $(p_x, p_y)$ and have income $I$.

An interesting question is how much utility changes when either prices or income change. As shown in the "Calculus and Optimization" hand-out we should expect that giving the person one dollar in income should increase her utility by the Lagrange multiplier $\alpha^d$

$$\frac{\partial V}{\partial I} = \alpha^d$$

Taking the total derivative of Indirect Utility Function with respect to $p_x$ yields

$$\frac{\partial V}{\partial p_x} = \frac{\partial U}{\partial x}\frac{\partial x^d}{\partial p_x} + \frac{\partial U}{\partial y}\frac{\partial y^d}{\partial p_x} \tag{1}$$

This expression does not tell us much, however it can be simplified with the two useful substitutions. First the the first two first order conditions tell us that

$$\frac{\partial U}{\partial x} = \alpha^d p_x \text{ and } \frac{\partial U}{\partial y} = \alpha^d p_y$$

so substituting these in to 1 we have

$$\frac{\partial V}{\partial p_x} = \alpha^d p_x \frac{\partial x^d}{\partial p_x} + \alpha^d p_y \frac{\partial y^d}{\partial p_x} = \alpha^d\left(p_x \frac{\partial x^d}{\partial p_x} + p_y \frac{\partial y^d}{\partial p_x}\right) \tag{2}$$

Second take the total derivative of the budget constraint $p_x x^d + p_y y^d = I$ with respect to $p_x$ to get

$$x^d + p_x \frac{\partial x^d}{\partial p_x} + p_y \frac{\partial y^d}{\partial p_x} = 0 \Rightarrow p_x \frac{\partial x^d}{\partial p_x} + p_y \frac{\partial y^d}{\partial p_x} = -x^d$$

Substituting this in to 2 we get a result known as **Roy's Identity**

$$\frac{\partial V}{\partial p_x} = -\alpha^d x^d \qquad \text{(Roy's Identity)}$$

A basic (albeit somewhat flawed) intuition for this identity is straightforward: if $p_x$ goes up by one dollar then the consumer will lose $x^d$ number of dollars, which each have utility value $\alpha^d$, so that utility drops by $\alpha^d x^d$. The more correct intuition is that $x^d$ and $y^d$ are changing in response to the price increase, but because the consumer is optimizing the entire time and facing a budget constraint, these indirect effects sum up to zero. A similar result holds for $y$ that

$$\frac{\partial V}{\partial p_y} = -\alpha^d y^d$$

which you can prove for yourself. Note that if you were just given the indirect utility function you could solve for $x^d, y^d$, and $\alpha^d$, just be taking combining the derivatives of $V$ appropriately, e.g. $x^d = -(\partial V/\partial p)/\alpha^d = -(\partial V/\partial p)/(\partial V/\partial I)$.

**Example 1** ***Quasilinear utility*** *takes a form where one of the goods consumed enters linearly into the utility function, which in this case we take to be* $U(x, y) = x + h(y)$ *where* $h'(y) > 0$ *and* $h''(y) < 0$*. (The reader may like to try the case where* $h(x) = \sqrt{x}$*) The first order conditions for this problem simplify to* $1 = \alpha^d p_x$*,* $h'(y^d) = \alpha^d p_y$*, and* $p_x x^d + p_y y^d = I$ *. Solving yields*

$$\begin{aligned}
\alpha^d &= 1/p_x \\
y^d &= (h')^{-1}(\alpha^d p_y) = (h')^{-1}\left(\frac{p_y}{p_x}\right) \\
x^d &= \frac{I - p_y y^d}{p_x} = \frac{I}{p_x} - \left(\frac{p_y}{p_x}\right) \cdot (h')^{-1}\left(\frac{p_y}{p_x}\right)
\end{aligned}$$

*Notice that $y^d$ does not depend on income $I$, only on relative prices $p_y/p_x$. Indirect utility is given by*

$$
\begin{aligned}
V(p_x, p_y, I) &= x^d + h\left(y^d\right) \\
&= \frac{I}{p_x} - \left(\frac{p_y}{p_x}\right) \cdot (h')^{-1}\left(\frac{p_y}{p_x}\right) + h\left[(h')^{-1}\left(\frac{p_y}{p_x}\right)\right]
\end{aligned}
$$

*It is fairly easy to see that $\partial V/\partial I = 1/p_x = \alpha^d$. Roy's identity is more difficult to check*

$$
\begin{aligned}
\frac{\partial V}{\partial p_y} &= -\frac{1}{p_x} \cdot (h')^{-1}\left(\frac{p_y}{p_x}\right) - \left(\frac{p_y}{p_x}\right)\frac{1}{h''(p_y/p_x)}\frac{1}{p_x} + h'\left[(h')^{-1}\left(\frac{p_y}{p_x}\right)\right]\frac{1}{h''(p_y/p_x)}\frac{1}{p_x} \\
&= -\frac{1}{p_x} \cdot (h')^{-1}\left(\frac{p_y}{p_x}\right) - \left(\frac{p_y}{p_x}\right)\frac{1}{h''(p_y/p_x)}\frac{1}{p_x} + \left(\frac{p_y}{p_x}\right)\frac{1}{h''(p_y/p_x)}\frac{1}{p_x} \\
&= -\frac{1}{p_x} \cdot (h')^{-1}\left(\frac{p_y}{p_x}\right) \\
&= -\alpha^d y^d
\end{aligned}
$$

*The reader is invited to check Roy's Identity for themselves in the case of $x$.*

## 2 Expenditure Minimization Problem

The consumer problem can be approached in a different way which produces some useful tools. Instead of maximizing utility given a certain income, imagine how much income it would take to achieve a certain level of utility. In other words consider the following **expenditure minimization problem** (EMP for short), which as always take prices as given

$$\min_{x,y} p_x x + p_y y \quad \text{s.t. } U(x, y) \geq u \tag{EMP}$$

This problem looks very much like the UMP above except that the objective function and constraint have been switched around. We wish to minimize the income $I = p_x x + p_y y$ needed achieve a fixed level of income $u$, for given prices $(p_x, p_y)$. Our third parameter in parameter in this problem (after $p_x$ and $p_y$) is no longer $I$, but $u$. This problem can typically be solved by writing the Lagrangean

$$\mathfrak{L}(x, y, \beta) = p_x x + p_y y + \beta\left[u - U(x, y)\right]$$

Assuming the constraint holds with equality, the first order conditions are

$$
\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial x} &= p_x - \beta^c \frac{\partial U(x^c, y^c)}{\partial x^c} = 0 \\
\frac{\partial \mathfrak{L}}{\partial y} &= p_y - \beta^c \frac{\partial U(x^c, y^c)}{\partial y^c} = 0 \\
\frac{\partial \mathfrak{L}}{\partial \beta} &= u - U(x^c, y^c) = 0
\end{aligned}
$$

The first two FOC are quite similar to above replacing $\beta^c$ with $1/\alpha^d$, but the third constraint corresponding to the constraint is very different. Solving these three equations in the three unknowns yields the Lagrange multiplier $\beta^c = \beta^c(p_x, p_y, u)$, the shadow price in dollars of having to provide an extra unit of utility to this consumer, as well as the **compensated demands**

$$x^c(p_x, p_y, u) \qquad y^c(p_x, p_y, u)$$

which are a function now of the required utility $u$, not income $I$. Note here that even though utility stays the same, quantities demanded will change as $p_x$ and $p_y$ change since the individual is trying to minimize her expenditures on consumption. Levels of required income $I$ are assumed to automatically adjust to let make sure that individual can still achieve utility $u$, although not necessarily the bundle of goods previously consumed. The individual is fully compensated for changes in price which could otherwise affect her utility if $I$ were held fixed. Substituting in the solutions back into the objective function, the minimand, we get the **expenditure function**

$$E(p_x, p_y, u) \equiv p_x x^c(p_x, p_y, u) + p_y y^c(p_x, p_y, u) \quad \text{(Expenditure Function)}$$

which is precisely the amount $I$ needed to maintain utility level $u$, for given prices $p_x$ and $p_y$.

As before we can differentiate $E$ with respect to $u$ to get $\partial E/\partial u = \hat{\gamma}$. Taking the total derivative of $E$ with respect to $p_x$ we get

$$\frac{\partial E}{\partial p_x} = x^c + p_x \frac{\partial x^c}{\partial p_x} + p_y \frac{\partial y^c}{\partial p_x} \quad (3)$$

We can simplify this expression with two substitutions. First we substitute in the first order conditions which imply, $p_x = \beta^c \partial U/\partial x$ and $p_y = \beta^c \partial U/\partial y$ into 3 to get

$$\frac{\partial E}{\partial p_x} = x^c + \beta^c \frac{\partial U}{\partial x}\frac{\partial x^c}{\partial p_x} + \beta^c \frac{\partial U}{\partial y}\frac{\partial y^c}{\partial p_x} = x^c + \beta^c \left[\frac{\partial U}{\partial x}\frac{\partial x^c}{\partial p_x} + \frac{\partial U}{\partial y}\frac{\partial y^c}{\partial p_x}\right] \quad (4)$$

Second we differentiate the constraint $U(x^c, y^c) = u$ totally with respect to $p_x$ to get

$$\frac{\partial U}{\partial x}\frac{\partial x^c}{\partial p_x} + \frac{\partial U}{\partial y}\frac{\partial y^c}{\partial p_x}$$

which implies that the second term in 4 is zero. This implies the result known as **Shepard's Lemma** (the analogue to Roy's Identity) that

$$\frac{\partial E}{\partial p_x} = x^c \quad \text{(Shepard's Lemma)}$$

Again the (somewhat misleading) intuition for this is clear. If $p_x$ changes by a small amount then $x^c$ will not change by very much and so the increased cost of consuming these units is precisely $x^c$. The better intuition is that there are changes in $x^c$ and $y^c$, but because of optimizing behavior, the consumer avoids spending any more than $x^c$, although since she was optimizing before she cannot avoid spending any less. The case for $y$ is identical

$$\frac{\partial E}{\partial p_y} = y^c$$

Thus, given the expenditure function you can derive the compensated demands just be taking the partial derivatives of $E$.

The UMP and EMP are mathematically known as "dual" problems. What is a constraint in one is the objective in the other and vice-versa. Because of this a useful **identities** relating compensated and uncompensated demands holds, namely

$$x^d(p_x, p_y, I) = x^c(p_x, p_y, V(p_x, p_y, I)) \quad \text{(ID1)}$$

$$x^c(p_x, p_y, u) = x^d(p_x, p_y, E(p_x, p_y, u)) \quad \text{(ID2)}$$

Similarly there are two identities relating the indirect utility function with the expenditure function

$$V(p_x, p_y, E(p_x, p_y, u)) = u \quad \text{(ID3)}$$

$$E(p_x, p_y, V(p_x, p_y, I)) = I \quad \text{(ID4)}$$

These identities can be proven formally (the proof is a little bit beyond the scope of this course), but a slow reading of them should show them to be quite intuitive. These identities can be used to simplify problem solving. For instance, solving the UMP one gets $x^d, y^d, \alpha^d$ and $V$. Setting $V(p_x, p_y, I) = u$, and solving for $I$ yields the expenditure function $E$ which can then be differentiated to get $x^c, y^c$ and $\beta^c$ using Shepard's Lemma. Similarly one can proceed from the EMP, and solve the equation $E(p_x, p_y, u) = I$ for $u$ to get $V$, which can then be used to get $x^d, y^d$, and $\alpha^d$, using Roy's Identity.

**Example 2** *Continuing with quasilinear utility, the FOC imply that* $p_x = \beta^c$, $p_y = \beta^c h'(y^c)$, *and* $x^c + h(y^c) = u$. *Solving the system we get*

$$\begin{aligned}
\beta^c &= p_x \\
y^c &= (h')^{-1}\left(\frac{p_y}{p_x}\right) \\
x^c &= u - h\left[(h')^{-1}\left(\frac{p_y}{p_x}\right)\right]
\end{aligned}$$

*Notice that* $y^c = y^d$, *as neither involves either* $u$ *or* $I$, *so that the identity holds trivially. The expenditure function is given by*

$$\begin{aligned}
E(p_x, p_y, u) &= p_x x^c + p_y y^c \\
&= p_x u - p_x h\left[(h')^{-1}\left(\frac{p_y}{p_x}\right)\right] + p_y (h')^{-1}\left(\frac{p_y}{p_x}\right)
\end{aligned}$$

*Solving for* $u$ *in this equation will yield the indirect utility function derived above. Shepard's Lemma can also be verified rather similarly to how Roy's Identity was verified.*

# 3 The Envelope Theorem

The derivations of Roy's Identity and Shepard's Lemma, as well as the interpretation of the Lagrange multipliers are all special cases of what is known as the envelope theorem. Stated generally, say we wish to solve the maximization problem

$$\max_{x,y} f(x, y, \xi) \qquad \text{s.t. } g(x, y, \xi) \leq c$$

where $x$ and $y$ are control variables and $\xi$ is a given parameter which effects $f$ and/or $g$, but over which we do not maximize over, i.e. it is given exogenously. The Lagrangean is then written as

$$\mathfrak{L}(x, y, \lambda, \xi) = f(x, y, \xi) + \lambda [c - g(x, y, \xi)]$$

for which the first two first order conditions are given by[2]

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial x} &= \frac{\partial f(x^*, y^*, \xi)}{\partial x} - \lambda \frac{\partial g(x^*, y^*, \xi)}{\partial x} = 0 \\
\frac{\partial \mathfrak{L}}{\partial y} &= \frac{\partial f(x^*, y^*, \xi)}{\partial y} - \lambda \frac{\partial g(x^*, y^*, \xi)}{\partial y} = 0
\end{aligned}$$

Do not take the derivative with respect to $\xi$, as $\xi$ is given exogenously. The solution to this problem, given by $x^*(\xi)$ and $y^*(\xi)$, depends on $\xi$ as different values of $\xi$ will imply different solutions.

Substituting the solutions into the function $f$ gives the **value function**

$$F(\xi) = f[x^*(\xi), y^*(\xi), \xi]$$

[2] Equivalent is the minimization problem

$$\min_{x,y} f(x, y, \alpha) \qquad \text{s.t. } g(x, y, \alpha) \geq c$$

as this is equivalent to the maximization problem

$$\max_{x,y} -f(x, y, \alpha) \qquad s.t. g(x, y, \alpha) \geq c$$

Writing out the Lagrangean one gets

$$\mathfrak{L}(x, y, \lambda) = -f(x, y, \alpha) + \lambda [g(x, y, \alpha) - c]$$

which gives equivalent first order conditions as the negative signs cancel.

which is the maximized value of $f$, which ultimately depends on $\xi$. Taking the total derivative of $F(\xi)$ we get

$$\frac{dF(\xi)}{d\xi} = \frac{\partial f}{\partial x}\frac{dx^*}{d\xi} + \frac{\partial f}{\partial y}\frac{dy^*}{d\xi} + \frac{\partial f}{\partial \xi} \tag{5}$$

The last term represents the direct effect of $\xi$ on $f$, while the first two terms represent the indirect effect of $\xi$ on $f$ by changing $x^*$ and $y^*$. This expression can be simplified in two steps. First substituting in the first order conditions $\partial f/\partial x = \lambda \partial g/\partial x$ and $\partial f/\partial y = \lambda \partial g/\partial y$ into 5 yields

$$\frac{dF(\xi)}{d\xi} = \lambda^* \frac{\partial g}{\partial x}\frac{dx^*}{d\xi} + \lambda^* \frac{\partial g}{\partial y}\frac{dy^*}{d\xi} + \frac{\partial f}{\partial \xi} = \lambda^* \left(\frac{\partial g}{\partial x}\frac{dx^*}{d\xi} + \frac{\partial g}{\partial y}\frac{dy^*}{d\xi}\right) + \frac{\partial f}{\partial \xi} \tag{6}$$

Second, differentiating the constraint $g(x^*, y^*, \xi) = c$ totally with respect to $\xi$ yields

$$\frac{\partial g}{\partial x}\frac{dx^*}{\partial \xi} + \frac{\partial g}{\partial y}\frac{dy^*}{\partial \xi} + \frac{\partial g}{\partial \xi} = 0 \Rightarrow \frac{\partial g}{\partial x}\frac{dx^*}{\partial \xi} + \frac{\partial g}{\partial y}\frac{dy^*}{\partial \xi} = -\frac{\partial g}{\partial \xi}$$

which substituting into 6 and rearranging yields the **envelope theorem**

$$\frac{dF(\xi)}{d\xi} = \frac{\partial f}{\partial \xi} - \lambda^* \frac{\partial g}{\partial \xi} = \frac{\partial \mathfrak{L}}{\partial \xi} \tag{Envelope Theorem}$$

Therefore the change in the value function is given by the partial derivative of the Lagrangean with respect to $\xi$ - a helpful simplification.[3] This covers Roy's Identity, Shepard's Lemma, and the interpretation of the Lagrange multiple (check!).

[3] In the unconstrained case or in the case where the constraint is slack ($\lambda^* = 0$) is just a special case with

$$\frac{dF(\xi)}{d\xi} = \frac{\partial f}{\partial \xi}$$

The total effect is given by just the partial, direct effect seen in $f$.