Economics 101A – Spring 2005

A Revised Version of the Slutsky Equation Using the Expenditure Function
or, the expenditure function is our friend!

Brief review...

$e(p_1, p_2, u^0) \quad = \quad \min \ p_1 x_1 + p_2 x_2 \ \text{ s.t. } U(x_1, x_2) = u^0$

$\quad\quad = \quad p_1 x^C_1 (p_1, p_2, u^0) + p_2 x^C_2 (p_1, p_2, u^0)$

where x^C_1 and x^C_2 are the "compensated demands": the choices you would make to get utility level u^0 as cheaply as possible at prices (p_1, p_2).

Remember that the Lagrangean for the exp-min problem is:

$L(x_1, x_2, \mu) = p_1 x_1 + p_2 x_2 - \mu (U(x_1, x_2) - u^0)$

The foc are:

a) $p_1 - \mu U_1(x_1, x_2) = 0$
b) $p_2 - \mu U_2(x_1, x_2) = 0$
c) $U(x_1, x_2) = u^0$

What are the derivatives of the expenditure function w.r.t. (p_1, p_2)?

From the definition

$e(p_1, p_2, u^0) = p_1 x^C_1 (p_1, p_2, u^0) + p_2 x^C_2 (p_1, p_2, u^0)$, and so:

(†) $\partial e(p_1, p_2, u^0) / \partial p_1 = x^C_1 (p_1, p_2, u^0) + p_1 \partial x^C_1 (p_1, p_2, u^0) / \partial p_1 + p_2 \partial x^C_2 (p_1, p_2, u^0) / \partial p_1$.

In tutorial we discussed the "envelope theorem" which says that the 2nd and 3rd terms cancel out. A quick way to prove that:

Use the constraint: $U(x^C_1 (p_1, p_2, u^0) , x^C_2 (p_1, p_2, u^0) = u^0$, and differentiate w.r.t. p_1 to get

$$U_1 \ \partial x^C_1 (p_1, p_2, u^0) / \partial p_1 + U_2 \partial x^C_2 (p_1, p_2, u^0) / \partial p_1 = 0$$

But $U_1(x_1, x_2) = p_1/\mu$ and $U_2(x_1, x_2) = p_2/\mu$ from the f.o.c. Substituting we get

$$p_1/\mu \ \ \partial x^C_1 (p_1, p_2, u^0) / \partial p_1 + p_2/\mu \ \ \partial x^C_2 (p_1, p_2, u^0) / \partial p_1 = 0$$

which means that $p_1 \partial x^C_1 (p_1, p_2, u^0) / \partial p_1 + p_2 \partial x^C_2 (p_1, p_2, u^0) / \partial p_1 = 0$.

Thus we have:

$$\partial e(p_1, p_2, u^0) / \partial p_1 \quad = \quad x^C{}_1 (p_1, p_2, u^0) \ .$$

There is a story we tell to go along with this. If you are initially minimizing expenditure, and the price of good 1 goes up, what do you do? Your "first order" adjustment is to simply continue buying the old bundle – that will increase your spending by $x^C{}_1 \times \Delta p_1$. That is the first term in (†). But then you would like to re-adjust your choices of goods 1 and 2 to reflect the new prices. Those adjustments are the second and third terms in (†). But because your initial choice was optimal (satisfying the f.o.c) when you try and re-adjust x_1 and x_2 you don't save any more.

Now we are ready to analyze what happens to the **uncompensated or regular** demand when prices rise. Suppose we start at an intial situation with prices $(p_1{}^0, p_2{}^0)$ and income I^0. The initial choices are $x_1{}^0 = x_1 (p_1{}^0, p_2{}^0, I^0)$, $x_2{}^0 = x_2 (p_1{}^0, p_2{}^0, I^0)$ where the x_1 ( ) and x_2 ( ) functions *without superscripts* are the regular demand functions

We decompose the effect of a change in price $\Delta p_1 = p_1{}' - p_1{}^0$ as follows:

a) starting from $x_1{}^0$, $x_2{}^0$, think of the adjustment you would make if you could keep utility constant (remain on the old indifference curve). This gets you to a new position $x_1{}^*$, $x_2{}^*$. Since prices have risen this position costs more than you were initially spending. This move is called the "substitution effect" of the price increase.

b) then from $x_1{}^*$, $x_2{}^*$ think of the adjustment you make to get back to spending only the amount of income that you actually have. This is a movement inward along an income expansion path (IEP). You end up at $x_1{}'$, $x_2{}'$. This move is called the "income effect" of the price increase.

Note that the total change in x_1 is

$$\Delta x_1 \quad = \quad x_1{}' \quad - \quad x_1{}^0$$

$$= (x_1' - x_1^*) + (x_1^* - x_1^0)$$

$$= \Delta x_1^I + \Delta x_1^S .$$

How big are these two parts? To begin, notice that (x_1^0, x_2^0) and (x_1^*, x_2^*) are on the u^0 indifference curve.

$$x_1^0 = x_1(p_1^0, p_2^0, I^0)$$

But it is also true that

$$x_1^0 = x_1^C(p_1^0, p_2^0, u^0).$$

Also,

$$x_1^* = x_1^C(p_1', p_2^0, u^0)$$

So $\Delta x_1^S = x_1^* - x_1^0 = x_1^C(p_1', p_2^0, u^0) - x_1^C(p_1^0, p_2^0, u^0)$

$$\approx \partial x^C_1(p_1^0, p_2^0, u^0) / \partial p_1 \times \Delta p_1$$

The substitution effect depends on the rate at which compensated demands change: this is purely a function of the curvature of the indifference curve.

How about the income effect?

$$\Delta x_1^I = x_1' - x_1^*$$

First note that $x_1' = x_1(p_1', p_2^0, I^0)$: its just your regular demand choice at (p_1', p_2^0, I^0).

But what is x_1^* ? It is the choice when you have enough income to get to the old indifference curve at the new prices. How much money do you need to get there? That's just $e(p_1', p_2^0, u^0)$! So

$$x_1^* = x_1(p_1', p_2^0, e(p_1', p_2^0, u^0))$$ Make sure this makes sense to you!

Thus

$$\Delta x_1^I = x_1(p_1', p_2^0, I^0) - x_1(p_1', p_2^0, e(p_1', p_2^0, u^0))$$

$$\approx \partial x_1(p_1^0, p_2^0, I^0) / \partial I \times (I^0 - e(p_1', p_2^0, u^0))$$

So the income effect depends on the income derivative of demand *times* the size of the income change $\Delta I = I^0 - e(p_1', p_2^0, u^0)$. Note that $\Delta I < 0$, since you need more than I^0 to get to the u^0 indifference curve when prices are (p_1', p_2^0).

But how big is ΔI? We have to use one last trick.

We know that $I^0 = e(p_1^0, p_2^0, u^0)$.

So we can write

$$\begin{aligned}
\Delta I &= I^0 - e(p_1', p_2^0, u^0) . \\
&= e(p_1^0, p_2^0, u^0) - e(p_1', p_2^0, u^0) \\
&\approx \partial e(p_1^0, p_2^0, u^0) / \partial p_1 \times (p_1^0 - p_1') \\
&= \partial e(p_1^0, p_2^0, u^0) / \partial p_1 \times (-\Delta p_1) \\
&= - \partial e(p_1^0, p_2^0, u^0) / \partial p_1 \times \Delta p_1
\end{aligned}$$

Note that this is negative for a rise in the price of good 1. Finally (almost done) we have

$$\begin{aligned}
\partial e(p_1^0, p_2^0, u^0) / \partial p_1 &= x^C{}_1 (p_1^0, p_2^0, u^0) \quad \textit{from the first page of this lecture note} \\
&= x_1^0 \quad \text{because } x_1^0 \text{ is also the compensated demand choice}
\end{aligned}$$

All this together means that

$$\Delta I \approx - x_1^0 \Delta p_1 .$$

Note that the size of the income effect depends on how much x_1 you were buying.

Pulling it all together,

$$\begin{aligned}
\Delta x_1^I &= \partial x_1 (p_1, p_2, I^0) / \partial I \times \Delta I \\
&= - \partial x_1 (p_1, p_2, I^0) / \partial I \times x_1^0 \Delta p_1
\end{aligned}$$

Thus $\Delta x_1 = \Delta x_1^I + \Delta x_1^S$

$$= - \partial x_1 (p_1, p_2, I^0)/\partial I \times x_1^0 \Delta p_1 + \partial x^C{}_1 (p_1^0, p_2^0, u^0) / \partial p_1 \times \Delta p_1$$

or $\Delta x_1 / \Delta p_1 = - x_1^0 \; \partial x_1 (p_1^0, p_2^0, I^0)/\partial I + \partial x^C{}_1 (p_1^0, p_2^0, u^0) / \partial p_1$.

Now take the limit as Δp_1 gets small and the ratio $\Delta x_1 / \Delta p_1$ tells us the derivative of the regular demand function. We have established:

$$\partial x_1 (p_1^0, p_2^0, I^0) / \partial p_1 = - x_1^0 \; \partial x_1 (p_1^0, p_2^0, I^0)/\partial I + \partial x^C{}_1 (p_1^0, p_2^0, u^0) / \partial p_1$$

This is called Slutsky's equation, after the Russian economist who first proved it about 100 years ago. Slutsky's equation says that the derivative of the regular demand with respect to p_1 is a combination of the income and substitution effects. The income effect depends on the derivative of demand w.r.t income, *times* the amount of x_1 you initially consume. The substitution effect depends on the derivative of the compensated demand for good 1.

A neat thing about the Slutsky equation is that it gives us a way to recover information about indifference curves from the derivatives of demand w.r.t. prices and incomes. In principle, we can observe $\partial x_1 (p_1^0, p_2^0, I^0) / \partial p_1$ and $\partial x_1 (p_1^0, p_2^0, I^0)/\partial I$. So we can infer:

$$\partial x^C_1 (p_1^0, p_2^0, u^0) / \partial p_1 = \partial x_1 (p_1^0, p_2^0, I^0) / \partial p_1 + x_1^0 \ \partial x_1 (p_1^0, p_2^0, I^0)/\partial I$$

Suppose we get an estimate of $\partial x^C_1 (p_1^0, p_2^0, u^0) / \partial p_1$ that is close to 0. That means indifference curves must be almost like "right angles".