Economics 101A
Section Notes
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Pareto Optimality and Public Goods with Two Agents

Consider the case where the case with $N = 2$ agents, indexed by $i = 1, 2$. Most of what we consider here is generalizable for larger N but working with 2 agents makes things much easier. Let agent 1's utility depends on his own action a_1 ("action" is defined very broadly here) as well as agent 2's action, so we can write $U_1(a_1, a_2)$, and similarly for agent 2 $U_2(a_1, a_2)$.

1 Pareto Optimality

1.1 Definition

The set of feasible actions (a_1^P, a_2^P) is *Pareto optimal (or efficient)* if there does not exist another of feasible actions $(\tilde{a}_1, \tilde{a}_2)$ such that

$$U_1(\tilde{a}_1, \tilde{a}_2) \geq U_1(a_1^P, a_2^P) \text{ and}$$
$$U_2(\tilde{a}_1, \tilde{a}_2) \geq U_2(a_1^P, a_2^P)$$

with at least one above inequality strict. In other words there does not exist an allocation that makes both as well off and making one strictly better off. A logically equivalent condition is that for any feasible set of actions $(\tilde{a}_1, \tilde{a}_2)$

$$U_1(\tilde{a}_1, \tilde{a}_2) > U_1(a_1^P, a_2^P) \Rightarrow U_2(\tilde{a}_1, \tilde{a}_2) < U_2(a_1^P, a_2^P)$$

A set of actions that makes agent 1 strictly better off must make agent 2 strictly worse off. *Important Note: Except for the trivial case of one person, Pareto optima and Nash equilibria do not necessarily coincide: plenty of Nash equilibria that are not Pareto optima and vice-versa (remember the Prisoner's Dilemma!)*

1.2 Utility Possibility Set

One can imagine the set of all pairs of utility (U_1, U_2) given by all of the different actions a_1 and a_2. The *utility possibility set* is that collection

$$\mathfrak{U} = \{(U_1, U_2) : U_1 = U_1(a_1, a_2), U_2 = U_2(a_1, a_2) \text{ for any feasible } a_1, a_2\}$$

which can usually be represented by a graph with U_1 on the x-axis and U_2 on the y-axis. By its very nature a Pareto optimum should be on the very edge of that set - that is its "frontier ". More formally the *utility possibility frontier* is the set

$$\mathfrak{U}_F = \{(U_1, U_2) \in \mathfrak{U} : \text{there is no } (\tilde{U}_1, \tilde{U}_2) \in \mathfrak{U} \text{ such that } \tilde{U}_1 \geq U_1 \text{ and } \tilde{U}_2 \geq U_2 \}$$

The difference between the utility possibility frontier and the set of Pareto optima, is that the set of Pareto optima refers to an outcome or allocation while the frontier refers only to utilities. Also, Pareto optima require that at least one inequality is strict while the frontier can include horizontal or vertical edges that are not Pareto. All Pareto optima will yield utilities on the frontier, however not quite all points on the frontier will relate to a Pareto optimum since it may contain points where one agent (not both) may do better without it costing the other agent.

1.3 Solving for Pareto Optima

1.3.1 Guaranteed Minimum Utility Formulation

One way of solving for Pareto optimum is to guarantee agent 1 a minimum amount of utility $\bar{u}_1$, which is taken parametrically, and then maximize agent 2's utility subject to this constraint. In other words

$$\max_{a_1, a_2} U_2(a_1, a_2) \quad \text{s.t. } U_1(a_1, a_2) \geq \bar{u}_1$$

If we let λ be the Lagrange multiplier on the constraint to get

$$\mathfrak{L}(a_1, a_2, \lambda) = U_2(a_1, a_2) + \lambda [U_1(a_1, a_2) - \bar{u}_1]$$

Assuming everything is smooth and the Pareto optimal actions are positive the following FOC must hold at (a_1^P, a_2^P)

$$\frac{\partial U_2}{\partial a_1} + \lambda \frac{\partial U_1}{\partial a_1} = 0 \qquad \text{and} \qquad \frac{\partial U_2}{\partial a_2} + \lambda \frac{\partial U_1}{\partial a_2} = 0 \tag{Pareto FOC}$$

along with the equation $U_1(a_1, a_2) = \bar{u}_1$. These FOC can be solved for the actions $a_1(\bar{u}_1)$, $a_2(\bar{u}_1)$, and $\lambda(\bar{u}_1)$ as well as the utility agent 2 actually receives: $u_2(\bar{u}_1) = U_2(a_1(\bar{u}_1), a_2(\bar{u}_1))$. Interestingly, in well-behaved cases where the utility possibility set is smooth and convex, the utility possibility frontier is given by the value function $u_2(u_1)$. By the envelope theorem the slope of the frontier is given by

$$\frac{du_2}{du_1} = \frac{\partial \mathfrak{L}}{\partial u_1} = -\lambda$$

Compare the Pareto FOC to the Nash FOC (see previous handout) you can see that the Pareto optimal actions take into account $\partial U_2/\partial a_1$ and $\partial U_1/\partial a_2$, i.e., that actions of agent 1 have an effect on agent 2 and vice-versa. These externalities are ignored in the Nash equilibrium and so the Nash equilibrium is only optimal if $\partial U_2/\partial a_1 = \partial U_1/\partial a_2 = 0$. Solving each FOC equation for $-\lambda$ and rearranging we see

$$-\lambda = \frac{\frac{\partial U_2}{\partial a_1}}{\frac{\partial U_1}{\partial a_1}} = \frac{\frac{\partial U_2}{\partial a_2}}{\frac{\partial U_1}{\partial a_2}} \Rightarrow \frac{\frac{\partial U_1}{\partial a_1}}{\frac{\partial U_1}{\partial a_2}} = \frac{\frac{\partial U_2}{\partial a_2}}{\frac{\partial U_2}{\partial a_2}} \tag{Pareto Tangency}$$

so the marginal rates of substitution between each action for each agent are equal, i.e. $MRS^1_{a_1 a_2} = MRS^2_{a_1 a_2}$. At the Nash equilibrium the marginal rates of substitution are typically perpendicular as $MRS^1_{a_1 a_2} = 0$ and $MRS^2_{a_1 a_2} = \infty$.

1.3.2 Maximizing Weighted Utilities Formulation

Another option is to consider a "social planner" who attaches a relative weight λ to agent 1 relative to agent 2 where $\lambda \gtreqless 1$ depending whether the planner values agent 1 more or less than agent 2. Unlike the above case we take λ as a fixed parameter and determine the utilities of both agents u_1 and u_2. A theorem from mathematics says that "pretty much" any Pareto optimal allocation can be found by maximizing the weighted utilities

$$\max_{a_1, a_2} \lambda U_1(a_1, a_2) + U_2(a_1, a_2)$$

for some λ. Different λ will give different Pareto optimal allocations. A popular favorite is to choose $\lambda = 1$, which corresponds to the Utilitarian social welfare function. With the exception of the constraint, the same FOC hold (implying the same Pareto Tangency), except now one solves for $a_1(\lambda)$, $a_2(\lambda)$, and the utilities $u_1(\lambda) = U_1(a_1(\lambda), a_2(\lambda))$ and $u_2(\lambda) = U_2(a_1(\lambda), a_2(\lambda))$.

The relationship between the two formulation considered can be seen by considering the problem

$$\max_{u_1} \lambda u_1 + u_2(u_1)$$

where $u_2(u_1)$ is the utility frontier derived in the guaranteed minimum utility formulation. Taking the FOC we get $\lambda + \frac{du_2(u_1)}{du_1} = 0$ or $\frac{du_2(u_1)}{du_1} = -\lambda$.[1] Thus we can imagine a social planner with straight, parallel indifference curves, each with slope $-\lambda$, in a graph. A Pareto optimum will be found where an indifference curve is tangent to the utility possibility frontier, with slope $\frac{du_2^P}{du_1^P}$, outlining $\mathfrak{U}$.

The FOC imply that we can solve for u_1 (and u_2) as a function of λ, giving $u_1(\lambda)$. Typically this function can be inverted to give the weight λ as a function of u_1, i.e. $\lambda(u_1)$. In this way a level of utility given to person 1 implies a certain relative weight person 1 gets relative to person 2, just as a given weight implies a level of utility to person 1 (and 2).

[1] The SOC imply that $\frac{d^2 u_2}{du_1^2} < 0$, i.e. the frontier must be concave so that the utility possibility set is convex.

2 Public Goods

Each agent has utility $U_i(G, x_i)$ where x_i is private consumption and public good $G = g_1 + g_2$ where g_i is agent i's provision of the public good. The public good, by definition is *nonrival*, consumption by one agent does not reduce it's benefit to another agent, and *nonexcludable*, i.e., it is prohibitively expensive to keep agents from consuming it. The price of private consumption is p_x and the price of the public good is p_G. Each agent has income I_i and thus has an individual budget constraint $p_x x_i + p_G g_i = I_i$.

2.1 Pareto Optimal Provision

Solving each person's budget constraint for x_i in terms of g_i we get $x_i = I_i/p_x - p_G g_i/p_x$. Substituting this expression and $G = g_1 + g_2$ into each individual's utility function we can find Pareto optima by solving

$$\max_{g_1, g_2} \lambda U_1\left(g_1 + g_2, \frac{I}{p_x} - \frac{p_G g_1}{p_x}\right) + U_2\left(g_1 + g_2, \frac{I}{p_x} - \frac{p_G g_1}{p_x}\right)$$

Note here that g_1 and g_2 are not each subject to a non-negativity constraint, only the total $G \geq 0$. This allows us to transfer money from one individual to another (e.g. person 1 can receive money if $g_1 < 0$ and $g_2 \geq -g_1$. Taking the FOC we get that the following two first order conditions must be satisfied at the optimum $\left(g_1^P, g_2^P\right)$

$$\begin{aligned} g_1 &: \lambda\left(\frac{\partial U_1}{\partial G} - \frac{\partial U_1}{\partial x_1}\frac{p_G}{p_x}\right) - \frac{\partial U_2}{\partial G} = 0 \\ g_2 &: \lambda\frac{\partial U_1}{\partial G} - \left(\frac{\partial U_2}{\partial G} + \frac{\partial U_2}{\partial x}\frac{p_G}{p_x}\right) = 0 \end{aligned}$$

Solving each equation for λ and solving for p_G/p_x we get

$$\lambda = \frac{\frac{\partial U_2}{\partial G}}{\frac{\partial U_1}{\partial G} + \frac{\partial U_1}{\partial x}\frac{p_G}{p_x}} = \frac{\frac{\partial U_2}{\partial G} + \frac{\partial U_2}{\partial x}\frac{p_G}{p_x}}{\frac{\partial U_1}{\partial G}} \Rightarrow \frac{\frac{\partial U_1}{\partial G}}{\frac{\partial U_1}{\partial x}} + \frac{\frac{\partial U_2}{\partial G}}{\frac{\partial U_2}{\partial x}} = MRS_{Gx}^1 + MRS_{Gx}^2 = \frac{p_G}{p_x} \qquad \text{(Samuelson's Rule)}$$

The condition that $p_G/p_x = MRS_{Gx}^1 + MRS_{Gx}^2$, is known as "Samuelson's Rule" (after the influential economist Paul Samuelson). This condition is different from that one derived with just private goods where we would have $MRS_{Gx}^1 = p_G/p_x$ which would be Pareto optimal if G were not a public good but a private good for person 1.

2.2 Nash equilibrium

In the Nash equilibrium we can expect individual 1 to maximize her own utility taking g_2 as given. Therefore we solve

$$\max_{g_1} U_1\left(g_1 + g_2, \frac{I}{p_x} - \frac{p_G g_1}{p_x}\right)$$

which leads to the FOC (assuming $g_1^N > 0$)

$$\frac{\partial U_1}{\partial G} - \frac{\partial U_1}{\partial x_1}\frac{p_G}{p_x} = 0 \Rightarrow MRS_{Gx}^1 = \frac{p_G}{p_x}$$

A similar condition applies for person 2, implying that $MRS_{Gx}^1 = p_G/p_x = MRS_{Gx}^2$ which does not satisfy samuelson's rule, resulting in a sub-optimal allocation of public goods as $G^N = g_1^N + g_2^N < g_1^P + g_2^P = G^P$.

2.3 Vertical Addition of Demand Curves

One way of visualizing Samuelson's rule is to use a graph (p_G, G) space much as one would in a typical demand diagram. Do this by solving the standard consumer problem (like the Nash equilibrium) except assume that each person faces an individualized price for the public good p_{Gi}. This way you will derive a demand curve

$G_i\left(p_{Gi}\right)$ by setting $MRS^i_{Gx} = p_{Gi}$ (set $p_x = 1$ to simplify matters - it is still the same for everyone). Now in aggregating the demand curves one should add up the inverse demand curves $p^D_{Gi}\left(G\right) = G_i^{-1}\left(G\right)$ where G_i^{-1} is the inverse function of G_i and should be downward sloping like most demand curves. Graphically this amounts to vertically adding up demand curves - something you should never do when dealing with private consumption (when you should add the actual demand curves, i.e. horizontally).

This idealized market will clear where $p^D_{G1}\left(G^*\right) + p^D_{G2}\left(G^*\right) = p^S_G\left(G^*\right)$ where $p^S_G\left(G\right)$ is just a standard inverse supply curve, resulting in an equilibrium price p^*_G. Note however that this implies that Samuelson's rule will hold as $p^D_{G1}\left(G^*\right) = MRS^1_{Gx}$ and $p^D_{G2}\left(G^*\right) = MRS^2_{Gx}$ and therefore $MRS^1_{Gx} + MRS^2_{Gx} = p^*_G$.

Example 1 *Assume that* $p_G = p_x = I_1 = I_2 = 1$ *and that utilities are Cobb-Douglas so* $U_1\left(G, x_1\right) = G^{1/2}x_1^{1/2}$ *and* $U_1\left(G, x_1\right) = G^{1/2}x_2^{1/2}$. *In this case the budget constraints imply that* $x_1 = 1 - g_1$ *and* $x_2 = 1 - g_2$ *and marginal rates of substitution are given by* $MRS^1_{Gx} = x_1/G$ *and* $MRS^2_{Gx} = x_2/G$. *Therefore Samuelson's Rule implies* $x^P_1/G^P + x^P_2/G^P = 1$ *so that* $x^P_1 + x^P_2 = G^P$. *Substituting in the budget constraints and using the fact that* $g^P_1 + g^P_2 = G^P$ *we get* $G^P = 1$. *Without specifying a* λ *we get the optimal level of the public good but we do not get a specific allocation of how much each should contribute. Substituting in* $G = 1$ *into the utility functions gives* $u_1 = x_1^{1/2}$ *and* $u_2 = x_2^{1/2}$. *Adding up the budget constraints gives* $x_1 + x_2 = 2 - G = 1$ *so* $x_2 = 1 - x_1$. *Therefore* $u_2 = \left(1 - x_1\right)^{1/2}$ *which combined with the fact that* $u_1 = x_1^{1/2} \Rightarrow x_1 = u_1^2$ *gives us* $u_2\left(u_1\right) = \left(1 - u_1^2\right)^{1/2}$ *which gives the utility possibility frontier (a quarter of a circle). Thus the solution to the minimum guaranteed utility formulation is*

$$x^P_1 = g^P_2 = u_1^2 \text{ and } x^P_2 = g^P_1 = 1 - u_1^2.$$

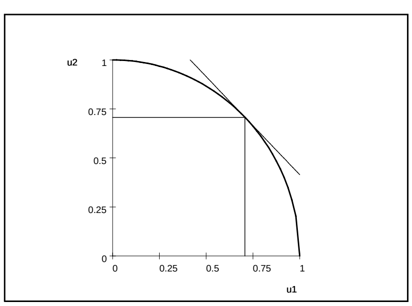
Putting a weight of λ *on person 1 and taking the FOC gives us that* $\lambda = u_1/\left(1 - u_1^2\right)^{1/2}$ *which solving for* u_1 *yields*

$$u^P_1 = \left(\frac{\lambda^2}{1+\lambda^2}\right)^{1/2},\ u^P_2 = \left(\frac{1}{1+\lambda^2}\right)^{1/2}, x^P_1 = g^P_2 = \frac{\lambda^2}{1+\lambda^2},\ \text{and } x^P_2 = g^P_1 = \frac{1}{1+\lambda^2}$$

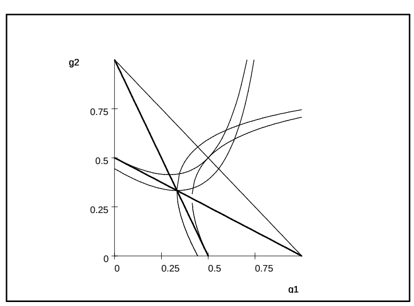
, which are the solutions to the maximizing weighted utilities formulation. If equal weights are given $\lambda = 1$ *and so* $x^P_1 = x^P_2 = g^P_1 = g^P_2 = 1/2$ *and* $u^P_1 = u^P_2 = 1/2^{1/2}$. *Solving for the Nash instead we get that person 1 will set* $x_1/\left(g_1 + g_2\right) = 1$, *and person 2 will set* $x_2/\left(g_1 + g_2\right) = 1$, *and therefore* $x_1 = x_2 = g_1 + g_2$. *With the budget constraints this implies* $1 - g_1 = 1 - g_2 = g_1 + g_2$ *which simplifying yields*

$$g^N_1 = g^N_2 = 1/3 \text{ and } G^N = x^N_1 = x^N_2 = 2/3$$

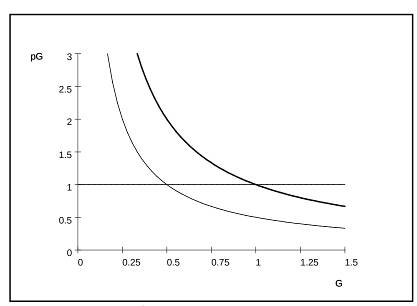
, resulting in a suboptimal allocation of public goods as $U_1\left(2/3, 2/3\right) = U_2\left(2/3, 2/3\right) = 2/3 < 1/2^{1/2} \cong 0.707$ *Note that the Nash conditions can be combined with each budget constraint to solve for the reaction curves as* $1 - g_1 = g_1 + g_2 \Rightarrow g_1\left(g_2\right) = \left(1 - g_2\right)/2$ *and similarly* $g_2\left(g_1\right) = \left(1 - g_1\right)/2$, *which can be combined in a standard graph. Indifference curves in* $\left(g_1, g_2\right)$ *are given by* $u_1 = \left(g_1 + g_2\right)^{1/2}\left(1 - g_1\right)^{1/2} \Rightarrow g_2^{IC_1} = u_1^2/\left(1 - g_1\right) - g_1$. *The idealized demand curves are given by* $x_i/G = p_{Gi}$ *and* $p_{Gi}G + x_i = 1 \Rightarrow x_i = p_{Gi}G = 1 - p_{Gi}G \Rightarrow G = 1/\left(2p_{Gi}\right)$ *or the inverse demand* $p_{Gi}\left(G\right) = 1/\left(2G\right)$. *Adding the two inverse demands* $p_{G1}\left(G\right) + p_{G2}\left(G\right) = 1/G$. *Inverting this equation we get an added up demand for the public good as* $G\left(p_G\right) = 1/p_G$. *Setting* $p_G = 1$ *we get* $G^* = 1$



Utility Possibility Set



Reaction Curves, etc.



Vertical Addition of Demands