Economics 101A
Section Notes
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Supply and Demand: Partial Equilibrium and Comparative Statics

1 Equilibrium

1.1 Assumptions

Let $D(p)$ be a downward-sloping ($dD/dp < 0$) aggregate market demand and let $S(p)$ be an upward-sloping ($dS/dp > 0$) aggregate market supply for a certain (homogenous) good Q with price p.[1] In **partial equilibrium analysis** we generally ignore dependence of these functions on other output or input prices, as well as income, or if we do we only look at one of these variables at at time. Furthermore we assume that markets are **perfectly competitive.** All consumers face the same demand price p_D and purchase $Q_D = D(p_D)$. Similarly all producers face the same supply price p_S, selling a total of $Q_S = S(p_S)$. Absent any intervention the supply and demand prices will be equal $p_D = p_S$.[2].

1.2 The Equilibrium Condition

In the absence of any intervention the equilibrium price p^* is the price that makes supply and demand equal[3], or mathematically speaking.

$$S(p^*) = D(p^*) \qquad \text{(Equilibrium)}$$

This equation is enough to solve for p^*, and the equilibrium quantity bought and sold is just $Q^* = S(p^*) = D(p^*)$. In fully general terms however there are really four unknowns (Q_D, Q_S, p_D, p_S) which are set by four equations

$$Q_D = D(p_D) \qquad (1)$$
$$Q_S = S(p_S) \qquad (2)$$
$$p_D = p_S \qquad (3)$$
$$Q_D = Q_S \qquad (4)$$

(1) and (2) are basically definitions, while (3) is an assumption which can be altered, and (4) is the equilibrium condition. Combining all 4 produces the (Equilibrium) condition where the equilibrium values are $(Q_D^*, Q_S^*, p_D^*, p_S^*)$.

2 Applications

By altering the four equations above in various ways a number of interesting applications arise.

[1] Note that we can assume that either $dD/dp = 0$ (the demand curve flat) or $dS/dp = 0$ (the supply curve is flat), but not both.

[2] This assumption is sometimes known as the **law of one price**. If consumers faced multiple prices for the same good they would only buy at the lowest price. Similarly if producers faced multiple prices for the same good they would only sell at the highest price.

Perfect competition implies that all consumers and firms are price-takers. This may be justified by assuming that there are "many" "small" consumers and firms. Consumers may buy as much as they want at p_D and producers can sell as much as they like at p_S without having a noticeable effect on the prices as whatever quantities they buy will be negligible relative to total supply or demand.

Another implicit assumption is that information on goods and prices is perfect.

[3] The actual mechanics of how this occurs is a tricky subject. A founder of neo-classical economics, Leon Walras, posited that an auctioneer could find the price that make supply equal demand by iterated trial and error. However the auctioneer idea is undeniably unrealistic in most real world markets.

2.1 Quantity Tax

2.1.1 Equilibrium Condition

Say the government imposes a quantity tax t on Q, so that equation (3) is changed to $p_D = p_S + t$ (a subsidy can be seen as just as a negative tax) . Combining all of the equations now yields

$$D\left(p^* + t\right) = S\left(p^*\right) \tag{Tax Equilibrium}$$

where $p_S^* = p^*$, $p_D^* = p^* + t$, and $Q_S^* = Q_D^* = S\left(p^*\right)$.

2.1.2 Comparative Statics

An interesting question to answer here is how this tax affects output relative to a no tax equilibrium. To do this treat the equilibrium price as a function of the tax, which behaves as a parameter, i.e. $p^* = p_S\left(t\right)$ and consider the effect of a small tax $t = dt > 0$. Now differentiate the (Tax Equilibrium) condition $D\left(p\left(t\right) + t\right) = S\left(p\left(t\right)\right)$ totally with respect to t,

$$\frac{dD}{dp}\left(\frac{dp_S}{dt} + 1\right) = \frac{dS}{dp}\frac{dp_S}{dt}$$

and solve for dp_S/dt to get

$$\frac{dp_S^*}{dt} = \frac{\frac{dD}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \tag{p_S tax}$$

The supply price decreases as this derivative is negative since the numerator is negative and the denominator is positive. Using the fact that $p_D = p_S + t$, then $dp_D/dt = dp_S/dt + 1$, or

$$\frac{dp_D^*}{dt} = \frac{\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \tag{p_D tax}$$

and so the demand price increases as the numerator is now positive. The change in the quantity bought and sold can be found by using the chain-rule as $dQ_S^*/dt = dS/dp \cdot dp_S^*/dt$ and so

$$\frac{dQ_D^*}{dt} = \frac{dQ_S^*}{dt} = \frac{\frac{dS}{dp}\frac{dD}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \tag{Q tax}$$

This quantity is negative since the numerator is negative. Deriving results in this way is mathematically very similar to shifting curves in a supply and demand graph. This overall methodology is known as **comparative statics**.

2.1.3 Using Elasticities

Often times we wish to express changes in price and quantities in percentage terms using elasticities. Let $e^D = \frac{dD}{dp}\frac{p}{Q} < 0$, $e^S = \frac{dS}{dp}\frac{p}{Q} > 0$, and $\tau = t/p$ (like an ad valorem tax)[4] then we can re-express the conditions expressed above,

$$\frac{dp_S^*}{p^*} = \frac{\frac{dD}{dp}\frac{p}{Q}}{\frac{dS}{dp}\frac{p}{Q} - \frac{dD}{dp}\frac{p}{Q}}\frac{dt}{p} = \frac{e^D}{e^S - e^D}d\tau$$

and similarly

$$\frac{dp_D^*}{p^*} = \frac{e^S}{e^S - e^D}d\tau$$

For Q we have

$$\frac{dQ_S^*}{Q^*} = \frac{\frac{dS}{dp}\frac{dD}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}}\frac{dt}{Q} = \frac{e^S\frac{dD}{dp}\frac{p}{Q}}{e^S - e^D}\frac{dt}{p} = \frac{e^S e^D}{e^S - e^D}d\tau$$

[4] Note that because the tax is small p_D and p_S are small and so we do not differentiate between the two in the elasticity formulas to simplify the expressions.

Of course the value of these expressions depends on how good these approximations are.[5]

2.1.4 Dead-Weight Loss

The dead-weight loss of the tax is given by the triangle defined by (i) the demand curve, (ii) the supply curve, and (iii) a vertical line at Q^*. The area of this triangle is just one-half of the height (dt) times the length ($|dQ|$)of the triangle, i.e.

$$DWL = \frac{1}{2}dt \cdot |dQ| = -\frac{1}{2}\frac{\frac{dS}{dp}\frac{dD}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}}dt^2$$

Expressed the DWL as a fraction of revenue $R = pQ$ in terms of the elasticities

$$\frac{DWL}{R} = \frac{1}{2}\frac{dt}{p} \cdot \frac{|dQ|}{Q} = -\frac{1}{2}\frac{e^S e^D}{e^S - e^D}d\tau^2$$

where we use the fact that $d\tau = dt/p$.

2.2 Government Purchase

2.2.1 Equilibrium Condition

Now say the government purchases G in goods (which we assume does not go to consumers), changing equation (4) to $Q_D + G = Q_S$ (similarly a government sale can be expressed as a negative G) . Combining all of the equations now yields

$$D(p^*) + G = S(p^*) \qquad \text{(Purchase Equilibrium)}$$

where $p_S^* + p_D^* = p^*$, $Q_S^* = S(p^*)$, and $Q_D^* = S(p^*) - G$. Unlike the tax problem there is really only one price, but two quantities.

2.2.2 Comparative Statics

Now let's examine the effect of this purchase relative to a no purchase equilibrium. Now treat the equilibrium price as a function of the purchase, which behaves as a parameter, i.e. $p^* = p(G)$ and consider the effect of a small tax $G = dG > 0$. Now differentiate the (Purchase Equilibrium) condition totally with respect to G,

$$\frac{dD}{dp}\frac{dp}{dG} + 1 = \frac{dS}{dp}\frac{dp_S}{dG}$$

and solve for dp/dt to get

$$\frac{dp^*}{dG} = \frac{1}{\frac{dS}{dp} - \frac{dD}{dp}} \qquad (p \text{ purchase})$$

The price increases as this expression is positive - the government purchase drives up the price of the good. For the quantity supplied, use the chain rule for $dQ_S^*/dG = dS/dp \cdot dp^*/dG$ and so

$$\frac{dQ_S^*}{dG} = \frac{\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \qquad (Q_S \text{ purchase})$$

which is positive - firms increase output in response to the higher equilibrium price. Using the fact that $Q_D^* = Q_S^* - G$, $dQ_D^*/dp = dQ_S^*/dp - 1$ and so

$$\frac{dQ_D^*}{dG} = \frac{\frac{dD}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \qquad (Q_D \text{ purchase})$$

This expression is negative - demand decreases as government purchases have driven up the price of the good.

[5] Note that we can write the last expression as $\frac{dQ_S^*}{Q^*} = \bar{e}d\tau$ where $\bar{e}$ is the harmonic mean of the absolute value of the elasticities, i.e. $\frac{1}{\bar{e}} = \frac{1}{e^S} + \frac{1}{|e^D|}$.

2.2.3 Using Elasticities

Let's re-express the above quantities using percent changes with $g = G/Q$ and elasticities[6]

$$\frac{dp^*}{p^*} = \frac{1}{\frac{dS}{dp} - \frac{dD}{dp}} \frac{dG}{p} = \frac{1}{\frac{dS}{dp}\frac{p}{Q} - \frac{dD}{dp}\frac{p}{Q}} \frac{dG}{Q} = \frac{1}{e^S - e^D} dg$$

The quantities supplied and demanded are just

$$\frac{dQ_S^*}{Q^*} = \frac{e^S}{e^S - e^D} dg \quad \text{and} \quad \frac{dQ_D^*}{Q^*} = \frac{e^D}{e^S - e^D} dg$$

2.2.4 Dead-Weight Loss

The dead-weight loss of the government purchase is given by the triangle defined by (i) the demand curve, (ii) the supply curve, and (iii) a vertical line at Q_S^*. The area of this triangle is just one-half of the height (?) times the length (dQ_S) of the triangle. To figure out the height of the triangle we need figure out the demand price at the quantity Q_S, i.e. where $Q_S = D(\tilde{p}_D)$ or $\tilde{p}_D = D^{-1}(Q_S)$. Differentiating we get $d\tilde{p}_D = (dD/dp)^{-1} dQ_S$. Therefore the height of the triangle is given by

$$dp^* - d\tilde{p}_D = \frac{1}{\frac{dS}{dp} - \frac{dD}{dp}} dG - \frac{1}{\frac{dD}{dp}} \frac{\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} dG = \frac{1}{\frac{dD}{dp}} \frac{\frac{dD}{dp} - \frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} dG = -\frac{1}{\frac{dD}{dp}} dG$$

Deadweight loss is then

$$DWL = \frac{1}{2}(dp^* - d\tilde{p}_D) \cdot dQ_S = -\frac{1}{2} \frac{1}{\frac{dD}{dp}} \frac{\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} dG^2$$

Expressed the DWL as a fraction of revenue $R = pQ$ in terms of the elasticities

$$\frac{DWL}{R} = -\frac{1}{2} \frac{1}{\frac{dD}{dp} p} \frac{\frac{dS}{dp}}{\frac{dS}{dp} - \frac{dD}{dp}} \frac{dG^2}{Q} = -\frac{1}{2} \frac{1}{\frac{dD}{dp}\frac{p}{Q}} \frac{e^S}{e^S - e^D} \frac{dG^2}{Q^2} = -\frac{1}{2} \frac{e^S}{e^D (e^S - e^D)} dg^2$$

[6] Here we quietly assume that since G is small we can ignore the differences between Q_D^* and Q_S^* in the elasticity formulas.