Part I

Duality of Production, Cost, and Profit Functions

Chapter I.1

COST, REVENUE, AND PROFIT FUNCTIONS

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1. Introduction

In the classical theory of cost and production, the firm is assumed to face fixed technological possibilities and competitive input markets, and to choose an input bundle to minimize the cost of producing each possible output. For fixed input prices, this behavior determines minimum cost as a function of output, yielding the standard cost curves of elementary textbooks. An immediate generalization is to allow input prices to vary and consider minimum cost as a function of both input prices and output. With this minor modification, the cost function becomes a powerful analytic tool in the theory of production, particularly in econometric applications.

The principal practical advantage of the cost function lies in its computationally simple relation to the cost minimizing input demand functions: the partial derivatives of the cost function with respect to input prices yield the corresponding input demand functions, and the sum of the values of the input demands equals cost. The useful analytic properties of the cost function derive from a fundamental duality

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between this function and the underlying production possibilities. The definition of the cost function as the result of an optimization yields strong mathematical properties, and establishes the cost function as a "sufficient statistic" for all the economically relevant characteristics of the underlying technology.

In econometric applications, use of the cost function as the starting point for developing models avoids the difficulty of deriving demand systems constructively from production possibilities, while at the same time insuring consistency with the hypothesis of competitive cost minimization. Further, under a number of econometric specifications of firm behavior, the cost function and its derivatives define the reduced form of the model.

Properties of the cost function can also be used to generalize and simplify the qualitative implications of cost minimization. In particular, a number of comparative statics results can be derived without assuming divisibility of commodities, or convexity and smoothness of production possibilities.

Two concepts are closely related to the cost function and also are useful in theory and applications. One is the revenue function of a multiple-product firm facing competitive markets, defining maximum revenue as a function of output prices and inputs. The second is the profit function of a firm facing competitive markets for inputs and outputs, defining maximum profits as a function of input and output prices. Cost, revenue, and profit functions can all be considered as special cases of a restricted profit function, defining maximum profits over a subset of inputs and outputs with competitive prices when quantities of the remaining inputs and outputs are fixed.

This chapter can be divided into two parts. The first part, consisting of Sections 2 to 12, is a self-contained treatment of the theory of cost functions and its applications. Mathematical rigor and generality are deemphasized for pedagogic simplicity, and economic interpretations are stressed. These sections will be accessible to readers with modest technical backgrounds. Proofs of more difficult results are postponed. The second part, consisting of Sections 13 to 20, gives a formal analysis of the properties of restricted profit functions for the more technical reader. Examples of restricted profit functions are discussed in Section 18. Appendix A.3 gives a self-contained survey of properties of convex sets and functions used in this chapter.

PART I. COST FUNCTIONS

2. History

The cost curve is a classical concept in economics, antedating even the concept of a production function. However, the systematic analysis of the properties of price derivatives of the cost function seems to have originated in a paper of Hotelling (1932) on the mathematically equivalent problem of minimizing consumer expenditure subject to a utility level constraint. The cost function and its properties were discussed in Samuelson (1947), and later led Samuelson to develop the concept of a factor–price frontier (which is a level curve of a cost function).

The properties of consumer expenditure functions were developed further by Roy (1942) and McKenzie (1957). McKenzie seems to have first noted that the properties of expenditure functions can be obtained as a consequence of optimization using the mathematical theory of convex functions with much weaker assumptions than were employed by the earlier authors.

The theory establishing the dual relation between cost functions and production functions was introduced into economics by Shephard (1953), who drew heavily on properties of convex sets discovered by Fenchel (1953). Additional contributions to economic applications of duality theory have been made by Uzawa (1964), McFadden (1962), Diewert (1974a), Hanoch (1975a), and Lau (1976a).

Perhaps because the theoretical results on cost functions were scattered and relatively inaccessible, their potential worth in econometric analysis was not recognized until Nerlove (1963) employed the Cobb–Douglas case in a study of returns to scale in electric utilities. Since the mid-1960s, a series of empirical studies, including papers by Diewert (1969a), and Jorgenson and Lau (1974a), have made systematic use of duality concepts.

3. Production Technologies

Basic to a model of the firm are descriptions of the commodities with which it deals and the technological limits on its actions. Following Debreu (1959), the concept of a commodity is taken generally to include both physical goods, such as wheat and fuel, and services, such as transportation and labor. Further, commodities are distinguished by location and date; e.g., trucks delivered at different locations and/or in different months will be considered distinct commodities. In particular, dated commodities extend over the planning horizon of the firm, and static and intertemporal theories of the producer are formally equivalent.

Occasionally, the same good will appear on both the input and output ledgers of a firm. If inputs are delivered temporally prior to outputs, these quantities are properly recorded as distinct commodities. However, if the ledgers of the firm also record intermediate goods in the production process (and this is particularly likely to be true if "intertemporal decentralization of accounts" is imposed on a firm having a lengthy production process), the same good may appear simultaneously as an input and an output. In this case, it is sometimes adequate for an economic problem to record net output. For other problems, it is convenient to treat the input and output as separate commodities in the firm's accounts. In the following analysis, we shall treat inputs and outputs as distinct commodities, making the artificial accounting distinction above necessary.

We consider a firm which uses N inputs indexed $n = 1,2,\ldots,N$, to produce M outputs, indexed $m = 1,2,\ldots,M$. An input bundle is an N-tuple of non-negative real numbers, $\mathbf{v} = (v_1,\ldots,v_N)$, as is an input price vector $\mathbf{r} = (r_1,\ldots,r_N)$. An output bundle is an M-tuple of non-negative real numbers, $\mathbf{y} = (y_1,\ldots,y_M)$. The cost of an input bundle $\mathbf{v}$ at an input price vector $\mathbf{r}$ is given by the inner product of $\mathbf{v}$ and $\mathbf{r}$, $c = \mathbf{r}\cdot\mathbf{v} = r_1v_1 + r_2v_2 + \cdots + r_Nv_N$.

The technological limits on the actions of the firm can be described by the set **Y** of pairs of input and output bundles (**v**,**y**) which are possible, in the sense that the firm can deliver the prescribed output bundle **y** by using the input bundle **v**; **Y** is termed the *production possibility set* of the firm. For example, a Cobb–Douglas production function $y_1 = v_1^{1/2}v_2^{1/2}$ corresponds to a production possibility set with one output and two inputs, $\mathbf{Y} = \{(v_1,v_2,y_1)|v_1,v_2 \geqq 0 \& v_1^{1/2}v_2^{1/2} = y_1\}$.

The production possibility set of a firm is determined first by the state of technological knowledge and physical laws. For example, the outputs of chemical refining processes are limited by chemical laws and the current knowledge of chemical engineers. There may be further limitations on the availability of techniques due to imperfect information and legal restrictions (e.g., patent agreements, pollution control regulations, safety standards). Non-transferable commodities, such as "managerial capacity", climate, and environmental factors, may also enter the determination of production possibilities. Finally, in most economic problems, the firm will be required to meet restrictions on some input and output quantities due to prior contracts, quotas, rationing, or "hardening" of commodities following ex ante design decisions. Common examples are commitments to fixed plant and equipment inputs, and contracts to purchase inputs (e.g., labor services) or supply outputs. It should be noted that "fixed" inputs or outputs can be either included or excluded from the commodity list facing the firm, depending on the economic problem. The sources of restrictions on the firm's production possibilities will be important in determining the economic interpretation of the cost function and its generalizations, but can be left undefined in the derivation of the formal properties of these functions.

With virtually no loss of economic generality, we usually assume that the production possibility set of a firm is non-empty and closed, and that a non-zero output bundle requires a non-zero input bundle. The condition that the production possibility set be closed requires that there be no "thresholds" at which discontinuities in required inputs or attainable outputs occur.[1] A production possibility set with these properties will be called *regular*.

In examining the cost function, it is convenient to work with "isoquants" rather than the production possibility set itself. First define the *producible output set* $\mathbf{Y}^*$ containing all the output bundles $\mathbf{y}$ which appear in some pair of input and output bundles in the production possibility set; i.e., $\mathbf{Y}^* = \{\mathbf{y} | (\mathbf{v},\mathbf{y}) \in \mathbf{Y} \text{ for some } \mathbf{v}\}$. Next, for each $\mathbf{y}$ in $\mathbf{Y}^*$, define the *input requirement* set $\mathbf{V}(\mathbf{y})$ containing all the input bundles $\mathbf{v}$ which can produce $\mathbf{y}$; i.e., $\mathbf{V}(\mathbf{y}) = \{\mathbf{v} | (\mathbf{v},\mathbf{y}) \in \mathbf{Y}\}$. The input requirement set corresponds to the conventional notion of an isoquant, except that it may include "inefficient" input bundles. Note that the input requirement set is well-defined in both the single-output and multiple-output cases. For the earlier example of the Cobb–Douglas production possibility set $\mathbf{Y} = \{(v_1,v_2,y_1) | v_1,v_2 \geqq 0 \,\&\, v_1^{1/2} v_2^{1/2} = y_1\}$, the producible output set is the non-negative real line and the input requirement sets are the isoquants $\mathbf{V}(y_1) = \{(v_1,v_2) | v_1,v_2 \geqq 0,\ v_1^{1/2} v_2^{1/2} = y_1\}$.

A production possibility set $\mathbf{Y}$ will be termed *input-regular* if (1) the

[1]A set is closed if it contains its boundaries; i.e., if the limit of each convergent sequence of points from the set is also contained in the set. Closedness does not rule out the possibility of lumpy (integer-valued) commodities. For example, the set $\mathbf{Y} = \{(v_1,y_1) | v_1 = 0,1,\ldots \,\&\, v_1 \geqq y_1 \geqq 0\}$ is closed, as is the set $\mathbf{Y} = \{(v_1,y_1) | v_1 \geqq 0 \,\&\, [v_1] \geqq y_1 \geqq 0\}$, where $[v]$ denotes the largest integer less than or equal to v.

set of producible outputs $\mathbf{Y}^*$ is non-empty, and (2) for each $\mathbf{y}$ in the set of producible outputs, the input requirement set $\mathbf{V}(\mathbf{y})$ is closed, and for a non-zero output bundle does not contain the zero input bundle. Clearly, if a production possibility set is regular, then it is also input-regular.

In the conventional theory of the firm, marginal products of inputs are assumed to be non-negative, and marginal rates of substitution between inputs are assumed to be non-increasing. Stated in terms of the input requirement sets, these conditions become:

> *Assumption A.* There is free disposal of inputs; i.e., if an input bundle $\mathbf{v}$ can produce an output bundle $\mathbf{y}$, and a second input bundle $\mathbf{v}'$ is at least as large as $\mathbf{v}$ in every component, then $\mathbf{v}'$ can also produce $\mathbf{y}$.

> *Assumption B.* The input requirement sets are convex from below; i.e., if two input bundles $\mathbf{v}$ and $\mathbf{v}'$ are in an input requirement set $\mathbf{V}(\mathbf{y})$, then for any weighted combination of $\mathbf{v}$ and $\mathbf{v}'$, say $\mathbf{v}'' = \theta\mathbf{v} + (1-\theta)\mathbf{v}'$ with θ a scalar, $0 < \theta < 1$, there exists an input bundle $\mathbf{v}^*$ in the input requirement set such that $\mathbf{v}''$ is at least as large as $\mathbf{v}^*$ in every component.

In set notation, Assumption A is sometimes written $\mathbf{V}(\mathbf{y}) + \mathbf{E}_+^N \subseteq \mathbf{V}(\mathbf{y})$, where $\mathbf{E}_+^N$ is the non-negative orthant of the N-dimensional input commodity space, and the algebraic sum of sets is defined by $\mathbf{V}(\mathbf{y}) + \mathbf{E}_+^N = \{\mathbf{v} + \mathbf{v}' | \mathbf{v} \in \mathbf{V}(\mathbf{y}) \;\&\; \mathbf{v}' \in \mathbf{E}_+^N\}$. Geometrically, $\mathbf{V}(\mathbf{y}) + \mathbf{E}_+^N$ is the set formed from $\mathbf{V}(\mathbf{y})$ by adding all points northeast of each point in $\mathbf{V}(\mathbf{y})$; $\mathbf{V}(\mathbf{y}) + \mathbf{E}_+^N$ is called the *free disposal hull* of $\mathbf{V}(\mathbf{y})$. The assumption is then that $\mathbf{V}(\mathbf{y})$ contains its free disposal hull. A set is said to be *convex* if it contains the line segment connecting any two of its elements. Assumption B can be restated as requiring that the free disposal hull $\mathbf{V}(\mathbf{y}) + \mathbf{E}_+^N$ be convex.

Justifications for these assumptions appear in most textbooks. Free disposal holds if firms can stockpile or refuse delivery of inputs, or if the technology is such that application of an additional unit of input always yields some non-negative amount of additional output and outputs can be disposed freely if necessary. Convexity from below holds if the technology is such that substitution of one input combination for a second, keeping output constant, results in a diminishing marginal reduction in the second input combination, or if production activities can be operated side by side (or sequentially) without interfering with each other. However, the importance of Assumptions A and B in traditional production analysis lies in their analytic convenience rather than in their economic realism; they provide the groundwork for application of calculus tools to the firm's cost minimization problem. One of the useful observations resulting from the analysis of cost functions is that the standard qualitative implications for supply and demand by the competitive firm can be obtained *without* imposing these conditions. Observed input demand functions for a cost minimizing firm facing positive input prices can be treated *as if* they come from input requirement sets satisfying Assumptions A and B even if these conditions fail to hold for the true technology.

Figure 1 illustrates Assumptions A and B. In (a), the input requirement set contains all the points northeast of any point in the set, thus satisfying Assumption A. In (b), the bundle **v** is in the set while the larger bundle **v**′

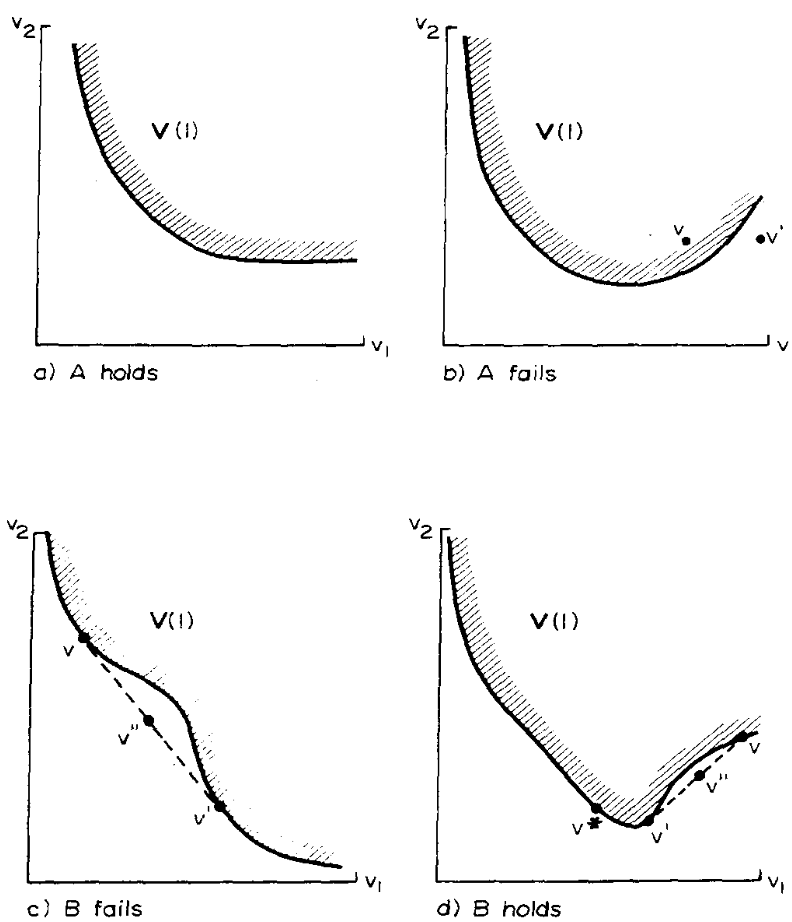


FIGURE 1

is not, and Assumption A fails. Assumption B fails in (c), where $\mathbf{v}''$ is an average of two points $\mathbf{v}$ and $\mathbf{v}'$ in the set, but is not itself northeast of any point in the set. In (d), on the other hand, Assumption B holds. Even though the weighted average $\mathbf{v}''$ of $\mathbf{v}$ and $\mathbf{v}'$ is not in the set, it lies northeast of $\mathbf{v}^*$ and the definition of convexity from below is satisfied.

A regular production possibility set satisfying Assumptions A and B will be termed *conventional.* Thus, in summary, a conventional production possibility set is non-empty and closed, with non-zero outputs requiring non-zero inputs, and has input requirement sets satisfying free disposal and convexity from below. An input-regular production possibility set satisfying Assumptions A and B will be termed *input-conventional.*

4. The Cost Function

Suppose that a firm has an input-regular production possibility set with a producible output set $\mathbf{Y}^*$ and input requirement sets $\mathbf{V}(\mathbf{y})$ for $\mathbf{y}$ in $\mathbf{Y}^*$. Suppose that the firm faces competitive input markets with strictly positive prices $\mathbf{r} = (r_1,\ldots,r_N)$, and chooses an input bundle $\mathbf{v}$ to minimize the cost $c = \mathbf{r}\cdot\mathbf{v} = r_1v_1 + \cdots + r_Nv_N$ of producing a given producible output bundle $\mathbf{y} = (y_1,\ldots,y_M)$. The *cost function* is then defined by

$$c = C(\mathbf{y},\mathbf{r}) = \mathrm{Min}\{\mathbf{r}\cdot\mathbf{v} \mid \mathbf{v} \in \mathbf{V}(\mathbf{y})\}, \tag{1}$$

and specifies the least cost of producing $\mathbf{y}$ with input prices $\mathbf{r}$.

We first verify that the cost function exists for all $\mathbf{y}$ in the producible output set and all strictly positive $\mathbf{r}$, using a mathematical theorem that a continuous function on a non-empty, closed, bounded set achieves a minimum in the set. The linear function $\mathbf{r}\cdot\mathbf{v}$ is continuous in $\mathbf{v}$. Since $\mathbf{V}(\mathbf{y})$ is non-empty, it contains at least one input bundle $\mathbf{v}'$, and the search for a minimizing bundle can be confined to the points in $\mathbf{V}(\mathbf{y})$ satisfying $\mathbf{r}\cdot\mathbf{v} \leqq \mathbf{r}\cdot\mathbf{v}'$. But this set is closed and bounded since $\mathbf{r}$ is strictly positive (see Figure 2), and the mathematical theorem above implies that $\mathbf{r}\cdot\mathbf{v}$ achieves a minimum on this set (at $\mathbf{v}''$ in the figure).

Since $\mathbf{v}$ and $\mathbf{r}$ are non-negative, the cost function is clearly non-negative. Further, if the output bundle $\mathbf{y}$ is non-zero, then every input bundle $\mathbf{v}$ which can produce $\mathbf{y}$ is non-zero. Since $\mathbf{r}$ is strictly positive, this implies that the cost function is strictly positive for non-zero output bundles.

We next show that for a fixed producible output bundle $\mathbf{y}$, the cost function is non-decreasing in input prices. Consider any strictly positive

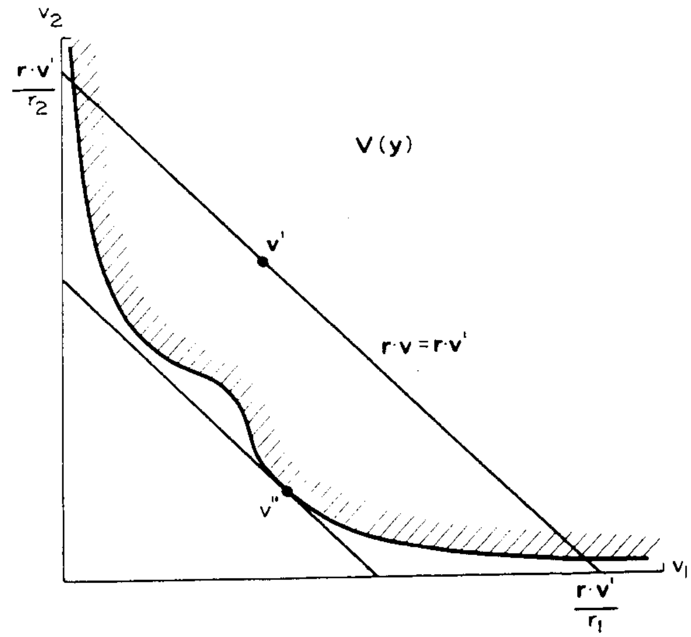


FIGURE 2

input price vector **r** and a second price vector **r**′ which is at least as large in every component. Suppose for the price vector **r**′, cost is minimized at some input bundle **v**′. Then, minimum cost at the input price vector **r** can be no higher than **r**·**v**′, which in turn can be no higher than **r**′·**v**′, which is the minimum cost at the input price vector **r**′.

Note that if an input bundle **v** is cost minimizing at a strictly positive price vector **r**, and if all prices are multiplied by a positive scalar θ, then **v** remains a cost minimizing bundle and the level of minimum cost is multiplied by θ. A function with this property is termed *positively linear homogeneous.*[2]

[2]A function $C(\mathbf{r})$ is said to be homogeneous of degree k in $\mathbf{r}$ if $C(\lambda\mathbf{r}) = \lambda^k C(\mathbf{r})$ for all $\lambda > 0$, and linear homogeneous if $k = 1$. If C is differentiable in r, then C is homogeneous of degree k if and only if $r_1(\partial C/\partial r_1) + \cdots + r_n(\partial C/\partial r_n) \equiv kC$ for all $\mathbf{r}$. This is *Euler's law*. To demonstrate its validity, first differentiate the identity $C(\lambda\mathbf{r}) = \lambda^k C(\mathbf{r})$ with respect to λ, obtaining $r_1C_1(\lambda\mathbf{r}) + \cdots + r_nC_n(\lambda\mathbf{r}) = k\lambda^{k-1}C(\mathbf{r})$, and set $\gamma = 1$. [We let $C_i(\mathbf{r}) = \partial C/\partial r_i$.] Second, evaluate the formula $r_1C_1(\mathbf{r}) + \cdots + r_nC_n(\mathbf{r}) = kC(\mathbf{r})$ at $\lambda\mathbf{r}$ for a fixed vector $\mathbf{r}$, obtaining $\lambda[r_1C_1(\lambda r) + \cdots + r_nC_n(\lambda r)] = kC(\lambda r)$. Treating C as a function of λ, the term in brackets is just $dC/d\lambda$, and we have $(1/C)(dC/d\lambda) = (k/\lambda)$. This differential equation has the solution $C(\lambda\mathbf{r}) = \lambda^k A$, where A is a term independent of λ but depending in general on $\mathbf{r}$. Setting $\lambda = 1$ implies $A = C(\mathbf{r})$, and hence $C(\lambda\mathbf{r}) = \lambda^k C(\mathbf{r})$.

An implication of homogeneity is that if $C(\mathbf{r})$ is homogeneous of degree k, then its derivatives $C_i(\mathbf{r})$ are homogeneous of degree $k-1$, and second derivatives $C_{ij}(\mathbf{r}) = \partial^2 C/\partial r_i \partial r_j$ are homogeneous of degree $k-2$.

A function is concave if it has the curvature of an overturned bowl.[3] We next show the cost function to be concave in input prices for each fixed output level. Consider any pair of strictly positive input price vectors $\mathbf{r}^0$ and $\mathbf{r}'$, and a weighted average of these vectors, $\mathbf{r}^* = \theta\mathbf{r}^0 + (1-\theta)\mathbf{r}'$, with $0 < \theta < 1$. Let $\mathbf{v}^0$, $\mathbf{v}'$, and $\mathbf{v}^*$ be cost minimizing input bundles corresponding to $\mathbf{r}^0$, $\mathbf{r}'$, and $\mathbf{r}^*$, respectively. Then, $\mathbf{r}^0\cdot\mathbf{v}^* \geqq C(\mathbf{y},\mathbf{r}^0)$ and $\mathbf{r}'\cdot\mathbf{v}^* \geqq C(\mathbf{y},\mathbf{r}')$, implying $C(\mathbf{y},\mathbf{r}^*) = \mathbf{r}^*\cdot\mathbf{v}^* = \theta(\mathbf{r}^0\cdot\mathbf{v}^*) + (1-\theta)(\mathbf{r}'\cdot\mathbf{v}^*) \geqq \theta C(\mathbf{y},\mathbf{r}^0) + (1-\theta)C(\mathbf{y},\mathbf{r}')$. This inequality is just the algebraic definition of a concave function, requiring that the chord between any two points in the graph of the function is no higher than the graph itself. Hence, the cost function is concave in input prices.

It is possible to obtain a further result that the cost function is continuous in input prices for fixed output, as a mathematical consequence of the concavity of the function.[4]

The property that the cost function is positively linear homogeneous in prices is one form of the old adage that only relative prices enter the economic calculus. The concavity of the cost function in prices is less intuitive economically, despite the almost trivial argument by which it was demonstrated. The reader's intuition may be helped by the following example: if the price of an input, say input 1, is raised by one infinitesimal unit, the cost of production is raised by v_1 units, where v_1 is the quantity of this input used. (One might expect an offsetting effect due to compensating adjustments in the input mix. However, this effect turns out to be a higher order infinitesimal which can be neglected.) At a

[3]A real-valued function f on $\mathbf{E}^n$ is *concave* if for every pair of points $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbf{E}^n$ and every scalar θ satisfying $0 < \theta < 1$, $f(\theta\mathbf{x} + (1-\theta)\mathbf{x}') \geqq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{x}')$. Geometrically, this requires that the chord between any two points in the graph of the function be no higher than the graph itself. f is *quasi-concave* if $f(\theta\mathbf{x} + (1-\theta)\mathbf{x}') \geqq \min\{f(\mathbf{x}),f(\mathbf{x}')\}$ for $0 < \theta < 1$. Geometrically, this requires that upper contour sets, $\{\mathbf{x} \in \mathbf{E}^n | f(\mathbf{x}) \geqq \alpha\}$, be convex for all real α. A function f is (quasi-) convex if $-f$ is (quasi-) concave.

[4]See Fenchel (1949, p. 75) or Rockafellar (1970, p. 82). We can also give a direct argument for this result. Suppose a sequence of strictly positive prices $\mathbf{r}^i$ converges to a strictly positive price $\mathbf{r}^0$. Then, there exist strictly positive price vectors $\mathbf{r}'$ and $\mathbf{r}''$ bounding the $\mathbf{r}^i$, i.e., $\mathbf{r}' \geqq \mathbf{r}^i \geqq \mathbf{r}''$ for each i. Let $\mathbf{v}^i$ and $\mathbf{v}^0$ be cost minimizing bundles for $\mathbf{r}^i$ and $\mathbf{r}^0$, respectively. Since $\mathbf{r}''\cdot\mathbf{v}^i \leqq \mathbf{r}^i\cdot\mathbf{v}^i \leq \mathbf{r}^i\cdot\mathbf{v}^0$, the set of minimizing bundles $\mathbf{v}^i$ lie in the closed and bounded set of non-negative $\mathbf{v}$ satisfying $\mathbf{r}''\cdot\mathbf{v} \leqq \mathbf{r}'\cdot\mathbf{v}^0$. Hence, the sequence of $\mathbf{v}^i$ will have a subsequence converging to $\mathbf{v}^*$ in the input requirement set. Retaining the notation $\mathbf{v}^i$ for *any* convergent subsequence, we then have the inequalities $C(\mathbf{y},\mathbf{r}^i) \leqq \mathbf{r}^i\cdot\mathbf{v}^0$ and $C(\mathbf{y},\mathbf{r}^0) \leqq \mathbf{r}^0\cdot\mathbf{v}^*$ and the limits $\mathbf{r}^i\cdot\mathbf{v}^0 \to \mathbf{r}^0\cdot\mathbf{v}^0 = C(\mathbf{y},\mathbf{r}^0)$ and $\mathbf{r}^i\cdot\mathbf{v}^i \to \mathbf{r}^0\cdot\mathbf{v}^*$. The first inequality and limit imply $\lim C(\mathbf{y},\mathbf{r}^i) \leqq C(\mathbf{y},\mathbf{r}^0)$, while the second inequality and limit imply $\lim C(\mathbf{y},\mathbf{r}^i) = \mathbf{r}^0\cdot\mathbf{v}^* \geqq C(\mathbf{y},\mathbf{r}^0)$. Since these inequalities hold for *every* limit point $\mathbf{v}^*$ of the original sequence, the result $\lim_{\mathbf{r}^i\to\mathbf{r}^0} C(\mathbf{y},\mathbf{r}^i) = C(\mathbf{y},\mathbf{r}^0)$ is established.

higher price of input 1, a lower quantity of the input will be used at the cost minimum, and the effect on cost of an infinitesimal unit increase in the price will be less than previously. This declining marginal effect is a classical characterization of the concavity property.

Thus far, the cost function has been defined only for strictly positive input prices. We can extend the definition (1) to the case in which some prices are zero, provided we relax the requirement that a minimum cost input bundle actually be achievable. This is done for a non-negative price vector **r** by defining

$$C(\mathbf{y},\mathbf{r}) = \text{Inf}\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \mathbf{V}(\mathbf{y})\}, \tag{1a}$$

where "Inf" denotes the infinum, or greatest lower bound, of the numbers in the set. For positive **r**, this definition coincides with (1). For non-negative **r** with some zero components, if a cost minimizing input bundle exists, the definition (1a) will yield a cost equal to the value of this input bundle. Alternately, no cost minimizing input bundle may exist (this is the case, for example, in the Cobb–Douglas input requirement sets illustrated in Section 3), and the cost $C(\mathbf{y},\mathbf{r})$ in (1a) is approached by the values of an unbounded sequence of input bundles. With minor variations, the arguments we gave earlier that the cost function is positively linear homogeneous and concave in positive input prices for a fixed output bundle can be applied to the extended definition (1a) to establish these properties for all non-negative prices. A more difficult argument [see Rockafellar (1970, p. 85) or Appendix A.3, Section 12.7] establishes that the extended cost function is continuous in all non-negative input prices for a fixed output bundle.

The basic properties of the cost function demonstrated in this section are summarized in the following result.

> *Lemma* 1. Suppose that a firm has an input-regular production possibility set with a producible output set $\mathbf{Y}^*$ and input requirement sets $\mathbf{V}(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$. Suppose that the firm faces competitive input markets with a non-negative input price vector **r**. Then, the cost function defined by (1) exists for all $\mathbf{y} \in \mathbf{Y}^*$ and all strictly positive **r**, and coincides with the extended cost function defined by (1a), which exists for all $\mathbf{y} \in \mathbf{Y}^*$ and all non-negative **r**. Further, for each $\mathbf{y} \in \mathbf{Y}^*$, the (extended) cost function as a function of **r** is non-negative, positive when **r** is strictly positive and **y** is non-zero, non-decreasing, positively linear homogeneous, concave, and continuous.

5. The Derivative Property

The cost function is related to the cost minimizing input demand functions through its partial derivatives with respect to input prices. Again consider a firm with an input-regular production possibility set, and let $c = C(\mathbf{y},\mathbf{r})$ denote its cost function, with $\mathbf{r} = (r_1,\dots,r_N)$ a vector of positive input prices. When the partial derivative of the cost function with respect to an input price r_n exists at an argument $(\mathbf{y},\mathbf{r})$, it will be denoted by $C_n(\mathbf{y},\mathbf{r}) = \partial C/\partial r_n$. We now establish the following result: If $C_n(\mathbf{y},\mathbf{r})$ exists, then it equals the unique cost minimizing input of good n at the argument $(\mathbf{y},\mathbf{r})$; and if there is a unique cost minimizing input of good n at the argument $(\mathbf{y},\mathbf{r})$, then $C_n(\mathbf{y},\mathbf{r})$ exists. This property, known as *Shephard's lemma*, was first noted by Hotelling (1932) and established formally by Shephard (1953). The demonstration given below was first used by McKenzie (1957).

Suppose $\mathbf{y}$ is a producible output bundle and $\mathbf{r}^0$ is a strictly positive input price vector, and suppose $\mathbf{v}^0$ is a corresponding cost minimizing input bundle. Consider any vector of input price increments $\Delta\mathbf{r} = (\Delta r_1,\dots,\Delta r_N)$. For any scalar θ which is sufficiently small to make $\mathbf{r}^0 + \theta\Delta\mathbf{r}$ strictly positive, the definition of the cost function implies the inequality $C(\mathbf{y},\mathbf{r}^0 + \theta\Delta\mathbf{r}) \leqq (\mathbf{r}^0 + \theta\Delta\mathbf{r})\cdot\mathbf{v}^0$. Since $\mathbf{r}^0\cdot\mathbf{v}^0 = C(\mathbf{y},\mathbf{r}^0)$, this inequality can be rewritten as

$$C(\mathbf{y},\mathbf{r}^0 + \theta\Delta\mathbf{r}) - C(\mathbf{y},\mathbf{r}^0) \leqq \theta(\Delta\mathbf{r})\cdot\mathbf{v}^0. \tag{2}$$

Single out one commodity, say the first, and define $\Delta r_1 = 1$ and $\Delta r_2 = \cdots = \Delta r_N = 0$. Define the ratio

$$g(\theta) = [C(\mathbf{y},r_1^0 + \theta,r_2^0,\dots,r_N^0) - C(\mathbf{y},r_1^0,r_2^0,\dots,r_N^0)]/\theta,$$

for $\theta \neq 0$. If θ is positive, (2) can be written

$$g(\theta) \leqq v_1^0. \tag{3a}$$

If θ is negative, the inequality reverses to give

$$g(\theta) \geqq v_1^0. \tag{3b}$$

If the partial derivative $C_1(\mathbf{y},\mathbf{r}^0)$ exists, then by its definition $g(\theta)$ has a limiting value, as θ approaches zero from above or below, equal to $C_1(\mathbf{y},\mathbf{r}^0)$. The inequalities then imply $C_1(\mathbf{y},\mathbf{r}^0) = v_1^0$. Since this equality must hold for any cost minimizing input vector, the cost minimizing input of good 1 is unique. This proves the first half of the lemma, and shows that differentiability of the cost function in input prices rules out the existence of flat segments in isoquants where multiple minima can occur.

The second half of Shephard's lemma requires a more advanced mathematical argument; see Appendix A.3, Lemma 13.8, or Rockafellar (1970, p. 265(e)).

A second justification of the derivative property of cost functions can be given using classical calculus arguments provided we add some facilitating assumptions on the technology. The following argument is due to Samuelson (1938). Suppose for a given producible output bundle **y**, the input requirement set is defined by the input bundles **v** satisfying $F(\mathbf{y},\mathbf{v}) \geqq 1$, where F is a transformation function which is twice continuously differentiable in **v**. The problem of cost minimization can then be restated as a classical constrained minimization problem: Minimize $\mathbf{r}\cdot\mathbf{v}$ subject to $F(\mathbf{y},\mathbf{v}) \geqq 1$. Form the Lagrangian $L = \mathbf{r}\cdot\mathbf{v} - \lambda(F(\mathbf{y},\mathbf{v}) - 1)$. Ignoring for simplicity the possibility of a corner solution or non-binding constraint, the first-order conditions for a mimimum are given by equating to zero the partial derivatives of the Lagrangian with respect to **v** and λ. (See Appendix A.2.) This procedure yields $N+1$ equations, the constraint $F(\mathbf{y},\mathbf{v}) = 1$ plus the marginal conditions $r_n = \lambda \partial F/\partial v_n$, $n = 1,\ldots,N$. Suppose this system has a unique solution for **v** and λ as a function of $(\mathbf{y},\mathbf{r})$, and let $v_n = h^n(\mathbf{y},\mathbf{r})$ denote the solution for v_n. Assume the h^n are continuously differentiable in **r**. From the definition $C(\mathbf{y},\mathbf{r}) = \sum_{n=1}^{N} r_n h^n(\mathbf{y},\mathbf{r})$, we obtain the condition $C_1(\mathbf{y},\mathbf{r}) = h^1(\mathbf{y},\mathbf{r}) + \sum_{n=1}^{N} r_n \partial h^n/\partial r_1$. But $r_n = \lambda \partial F/\partial v_n$ and $F(\mathbf{y},h^1(\mathbf{y},\mathbf{r}),\ldots,h^N(\mathbf{y},\mathbf{r})) = 1$ imply, by differentiation,

$$\frac{1}{\lambda}\sum_{n=1}^{N} r_n(\partial h^n/\partial r_1) = \sum_{n=1}^{N} (\partial F/\partial v_n)(\partial h^n/\partial r_1) = 0, \tag{4}$$

and hence $C_1(\mathbf{y},\mathbf{r}) = h^1(\mathbf{y},\mathbf{r})$.

Several stronger derivative properties of the cost function can be obtained as corollaries of the mathematical theory of convex functions. For each producible output bundle, the cost function can be shown to possess first and second differentials for almost all strictly positive input price vectors (i.e., for all positive input price vectors except those in a set of Lebesgue measure zero). This implies that for almost all input price vectors there is a unique input bundle demanded under cost minimization. Further, the second partial derivatives of the cost function with respect to input prices are found to be independent of the order of differentiation whenever the second differential exists. Since these second differentials are the first partial derivatives of the cost minimizing input demands, this result implies a production analogue of the symmetry of the Slutsky substitution effects in consumer theory. It should be noted that these properties hold without any assumptions on the structure of the technology beyond the condition that it be regular. In particular, they hold even if the underlying technology exhibits non-convexities, indivisible inputs or outputs, or failures of free disposal. Lemma 12.1 in Appendix A.3 states these results formally.

In many economic applications, particularly comparative statics, it is convenient to know that the cost minimizing input demands are unique for all positive input prices (Shephard's lemma then implies that the cost function possesses a first differential in input prices for all positive values of these prices). A stronger version of Assumption B on the convexity of the input requirement sets from below is necessary and sufficient to give this property. Define a *plane* (or *hyperplane*) in input space to be a set of "isocost" points; i.e., a set **H** of points **v** satisfying $\mathbf{r}\cdot\mathbf{v} = r_1v_1 + \cdots + r_Nv_N = r_0$ for some fixed non-zero vector **r** and some scalar r^0. The vector **r** gives the direction numbers of the plane, and is termed a *normal* to the plane. A plane **H** *bounds* a set **V** if the set is contained in one of the closed half-spaces defined by the plane; i.e., if $\mathbf{r}\cdot\mathbf{v} = r_0$ for **v** in **H**, and $\mathbf{r}\cdot\mathbf{v} \geqq r_0$ for $\mathbf{v} \in \mathbf{V}$, then **H** bounds **V**. A plane **H** *supports* a set **V** if it bounds **V**, and **H** and **V** meet.

> *Assumption B*-2. The input requirement sets are strictly convex from below; i.e., if **H** is any plane with a strictly positive normal which supports **V(y)** from below,[5] then **H** meets **V(y)** at exactly one point.

This assumption states that if **v** and **v**′ are in **V(y)**, with $\mathbf{v} \neq \mathbf{v}'$, and $\mathbf{v}'' = \theta\mathbf{v} + (1-\theta)\mathbf{v}'$, $0 < \theta < 1$, then there exists $\mathbf{v}^*$ in **V(y)** such that $\mathbf{v}^* \leqq \mathbf{v}''$ and either (i) $\mathbf{v}^* \neq \mathbf{v}''$ or (ii) there exists no plane **H** with strictly positive normal which contains **v** and **v**′ and which supports **V(y)** from below. If **V(y)** or its free disposal hull is a strictly convex set, then Assumption B-2 holds, and condition (i) above is always satisfied.

Figure 3 illustrates this assumption. In (a), the weighted average $\mathbf{v}''$ of two points **v** and **v**′ lies northeast of $\mathbf{v}^*$ in the set. The points $\mathbf{v}^3$ and $\mathbf{v}^4$ satisfy condition (ii) above since the only plane through them is parallel to the $\mathbf{v}_2$ axis, and hence has a zero direction number. In (b), the assumption fails because the isoquant contains a flat segment. A three-

[5]The plane **H** bounds (or supports) **V(y)** *from below* if $\mathbf{r}\cdot\mathbf{v} = r_0$ for $\mathbf{v} \in \mathbf{H}$, and $\mathbf{r}\cdot\mathbf{v} \geqq r_0$ for $\mathbf{v} \in \mathbf{V(y)}$.

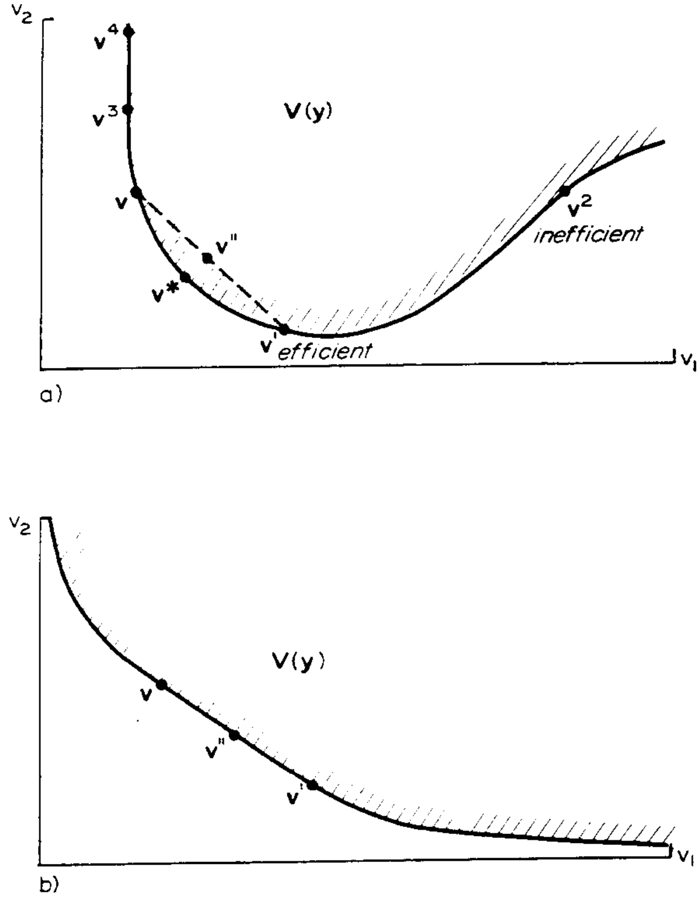


FIGURE 3. (a) Assumption B-2 holds. (b) Assumption B-2 fails.

input requirement set is illustrated in (c). The points **v** and **v**′ both lie in a plane, identified by the rectangle ABCD, parallel to the v_3 coordinate axis and bounding **V(y)** from below. This plane has a zero direction number in the direction v_3. Every other plane containing **v** and **v**′ cuts through the input requirement set rather than bounding it. Hence, condition (ii) above holds for **v** and **v**′. For distinct pairs of points such as $\mathbf{v}^3$ and $\mathbf{v}^4$, the input requirement set contains a point $\mathbf{v}^6$ no greater than and unequal to a linear combination $\mathbf{v}^5 = \theta\mathbf{v}^3 + (1-\theta)\mathbf{v}^4$, $0 < \theta < 1$. Hence, $\mathbf{v}^3$ and $\mathbf{v}^4$ satisfy condition (i) above.

It is clear that this condition implies that minimum cost is achieved by a unique input bundle for any strictly positive input price vector: if two distinct input bundles simultaneously minimized cost, then a weighted average of them would also have this minimum cost, and Assumption B-2 would imply the existence of another bundle in the input require-

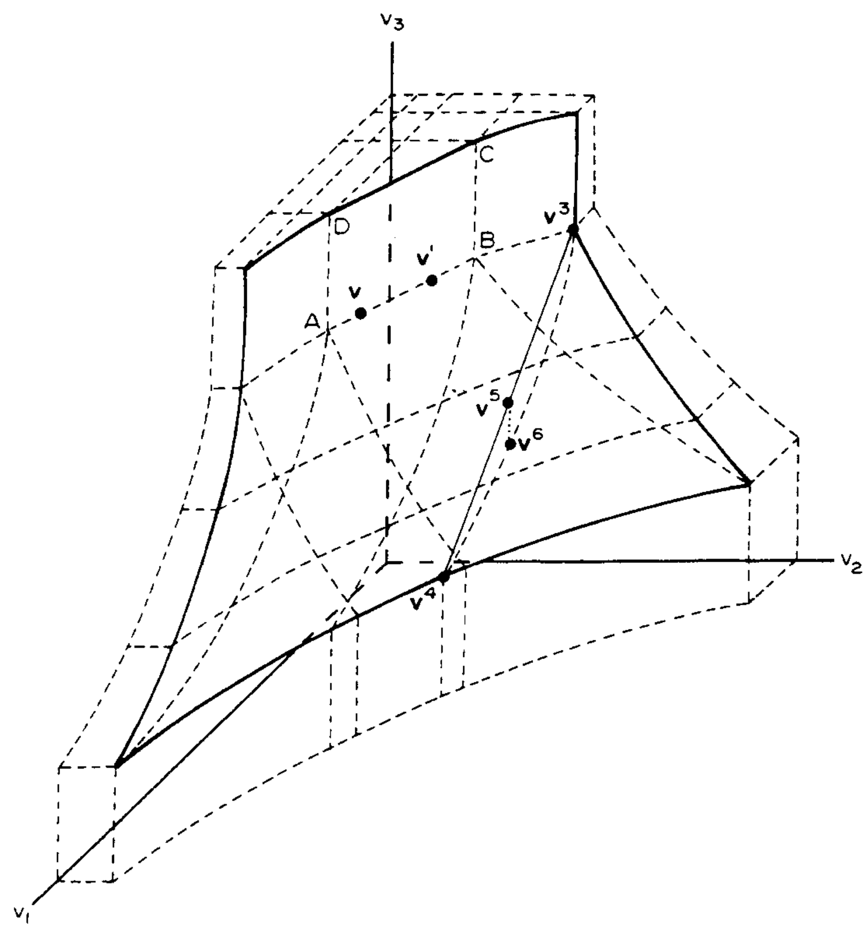


FIGURE 3(c)

ment set costing less, and thus contradict the initial supposition. As noted earlier, this uniqueness of the cost minimizing input demand bundle guarantees that the cost function has a differential in input prices for *all* positive input price vectors. The mathematical properties of convex functions then imply that the cost function is *continuously* differentiable in input prices.

Then, Assumption B-2 and Shephard's lemma imply that (1) unique cost minimizing input demands exist for all positive input prices and are given by the price derivatives of the cost function, and (2) the input demands vary continuously with input prices.

Returning to the case of input-regular production possibility sets without added assumptions on structure, it is possible to generalize the concept of a vector of partial derivatives of the cost function in a mathematically meaningful way so that (1) this generalized derivative, called the *sub-differential*, always exists and is a *set* of N-dimensional vectors, (2) the vectors in the sub-differential correspond to the cost minimizing input bundles in a sense elaborated below, and (3) in the case where there is a unique cost minimizing input bundle, the sub-differential contains exactly the ordinary vector of partial derivatives. This concept is developed formally in Appendix A.3, Sections 13.7–13.9. Expanding informally on the conclusions of this construction, the sub-differential will contain a single vector if and only if there is a unique cost minimizing input bundle, in which case the definition of the sub-differential reduces to the ordinary definition of a vector of partial derivatives and the vector in the sub-differential coincides with the cost minimizing input bundle. More generally, the sub-differential will contain all the cost minimizing input bundles. If the input requirement sets satisfy Assumption B of convexity from below, then the sub-differential equals the set of cost minimizing bundles at any $(\mathbf{y},\mathbf{r})$ argument. When no convexity assumptions are imposed on the input requirement sets, then the sub-differential may contain, in addition to the true cost minimizing input bundles, some input bundles which lie outside the input requirement set. However, all these latter bundles can be written as weighted averages of a finite number of true cost minimizing input vectors.

6. Duality

We have established that corresponding to every input-regular production possibility set is a cost function with the properties summarized in Lemma 1. We now pose the converse question: given a function with the properties specified in Lemma 1, does there exist an input-regular production possibility set such that this function is its minimum cost function? A duality between input-conventional production possibility sets and cost functions first proved by Shephard (1953) and Uzawa (1962) provides an affirmative answer. This theoretical result is of considerable practical importance. It allows the economist to write down cost functions and their input demand systems and verify their consistency with the cost minimization hypothesis without difficult constructive arguments. Further, it establishes that the cost function contains all the information necessary to reconstruct the structure of production possibilities. It is in a sense a "sufficient statistic" for the technology. Thus, corresponding to every hypothesis the economist might impose on the structure of a conventional production possibility set is a hypothesis on the form of the cost function.

We begin the discussion of duality with several definitions. An *input-conventional cost structure* is defined by (1) a non-empty set of non-negative M-dimensional vectors, denoted by $\mathbf{Y}^*$ and interpreted as a producible output set, and (2) a real-valued function $c = C(\mathbf{y},\mathbf{r})$, defined on the domain consisting of $\mathbf{y} \in \mathbf{Y}^*$ and strictly positive N-dimensional price vectors $\mathbf{r}$, this function being non-negative, non-decreasing, positively linear homogeneous, and concave in $\mathbf{r}$ for each fixed $\mathbf{y} \in \mathbf{Y}^*$, and positive for non-zero $\mathbf{y}$.

Consider an input-conventional cost structure $C(\mathbf{y},\mathbf{r})$ defined for $\mathbf{y}$ in a set $\mathbf{Y}^*$. For each $\mathbf{y} \in \mathbf{Y}^*$, define an *implicit input requirement set*

$$\mathbf{V}^*(\mathbf{y}) = \{\mathbf{v} \in \mathbf{E}^N | \mathbf{v} \geqq 0,\ \mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r}) \quad \text{for all strictly positive } \mathbf{r}\}. \quad (5)$$

The implicit input requirement sets will be shown to be non-empty, allowing the definition of an *implicit production possibility set*

$$\mathbf{Y} = \{(\mathbf{y},\mathbf{v}) \in \mathbf{E}^{M+N} | \mathbf{y} \in \mathbf{Y}^*,\ \mathbf{v} \in \mathbf{V}^*(\mathbf{y})\}. \quad (6)$$

The first duality result establishes that each input-conventional cost function determines an implicit production possibility set which is input-conventional (i.e., is input-regular and satisfies Assumptions A and B).

> *Lemma* 2. If $C(\mathbf{y},\mathbf{r})$ is an input-conventional cost function defined for $\mathbf{y}$ in a set $\mathbf{Y}^*$, then the implicit input requirement sets $\mathbf{V}^*(\mathbf{y})$ are non-empty for each $\mathbf{y} \in \mathbf{Y}^*$, and the implicit production possibility set $\mathbf{Y}$ is input-conventional.

Proof: The lemma will be proved in three steps. First, the implicit input requirement sets are shown to be non-empty for each $\mathbf{y} \in \mathbf{Y}^*$. This allows the implicit production possibility set (6) to be defined. Second, this production possibility set is shown to be input-regular. Third, Assumptions A and B are shown to hold.

Step 1. By hypothesis, $\mathbf{Y}^*$ is non-empty. Consider any $\mathbf{y} \in \mathbf{Y}^*$. Let $\mathbf{r}^0 = (1,1,\ldots,1)$ be an N-vector of ones, and define an input bundle $\mathbf{v}^0 = c\mathbf{r}^0$ with $c = C(\mathbf{y},\mathbf{r}^0)$. Let $|\mathbf{r}| = \sum_{n=1}^{N}|r_n|$ denote the norm of an N-vector. Since the function $C(\mathbf{y},\mathbf{r})$ is non-decreasing and positively linear homogeneous in $\mathbf{r}$, we have for any strictly positive $\mathbf{r}$ the inequality

$$C(\mathbf{y},\mathbf{r}) = C(\mathbf{y},\mathbf{r}/|\mathbf{r}|)\cdot|\mathbf{r}| \leqq C(\mathbf{y},\mathbf{r}^0)\cdot|\mathbf{r}| = C(\mathbf{y},\mathbf{r}^0)(\mathbf{r}\cdot\mathbf{r}^0) = \mathbf{r}\cdot\mathbf{v}^0.$$

Then by (5), $\mathbf{v}^0$ is contained in the implicit input requirement set, which is thus non-empty.

Step 2. To show that the implicit production possibility set is input-regular, we must show that each implicit input requirement set is closed and does not contain the zero input bundle when the output bundle is non-zero. Consider any $\mathbf{y} \in \mathbf{Y}^*$. To show that $\mathbf{V}^*(\mathbf{y})$ is closed, consider any sequence $\mathbf{v}^k \in \mathbf{V}^*(\mathbf{y})$ converging to a bundle $\mathbf{v}^0$. For any positive $\mathbf{r}$, the $\mathbf{v}^k$ satisfy $\mathbf{r}\cdot\mathbf{v}^k \geqq C(\mathbf{y},\mathbf{r})$ by (5). Then this inequality must hold also in the limit, $\mathbf{r}\cdot\mathbf{v}^0 \geqq C(\mathbf{y},\mathbf{r})$. But (5) then implies $\mathbf{v}^0 \in \mathbf{V}^*(\mathbf{y})$. Hence, $\mathbf{V}^*(\mathbf{y})$ is closed. If the zero input bundle is in $\mathbf{V}^*(\mathbf{y})$, then by (5), $0 = C(\mathbf{y},\mathbf{r})$, implying $\mathbf{y} = \mathbf{0}$ by hypothesis.

Step 3. We first establish that $\mathbf{Y}$ satisfies Assumption A, free disposal of inputs. If a bundle $\mathbf{v}$ is in $\mathbf{V}^*(\mathbf{y})$, and a second bundle $\mathbf{v}'$ is at least as large in every component, then for any positive $\mathbf{r}$, $\mathbf{r}\cdot\mathbf{v}' \geqq \mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r})$, implying $\mathbf{v}' \in \mathbf{V}^*(\mathbf{y})$. Hence, Assumption A holds. We next establish Assumption B, convexity from below of $\mathbf{V}^*(\mathbf{y})$. If $\mathbf{v}$, $\mathbf{v}'$ are input bundles in $\mathbf{V}^*(\mathbf{y})$ and for a scalar θ, $0 < \theta < 1$, $\mathbf{v}'' = \theta\mathbf{v} + (1-\theta)\mathbf{v}'$ is a weighted combination of these bundles, then for any positive $\mathbf{r}$ the inequalities $\mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r})$ and $\mathbf{r}\cdot\mathbf{v}' \geqq C(\mathbf{y},\mathbf{r})$ imply $\mathbf{r}\cdot\mathbf{v}'' \geqq C(\mathbf{y},\mathbf{r})$. Hence, $\mathbf{v}'' \in \mathbf{V}^*(\mathbf{y})$, and $\mathbf{V}^*(\mathbf{y})$ is convex. Q.E.D.

The next result, called the Shephard–Uzawa duality theorem [Shephard (1970), Uzawa (1962)], establishes a one-to-one relationship between input-conventional production possibility sets and input-conventional cost structures. Let us call the procedure (1) which obtains a minimum cost function from a production possibility set the *cost mapping*, and the procedure (5) which obtains an implicit production possibility set from a cost function the *technology mapping*. Lemma 1 establishes that the cost mapping is a function from the class of input-conventional (actually, more generally, input-regular) production possibility sets into the class of input-conventional cost structures. Lemma 2 establishes that the technology mapping is a function from the class of input-conventional cost structures into the class of input-conventional production possibility sets. The duality theorem establishes that on the two input-conventional classes above, the cost mapping and technology mapping are mutual inverses; i.e., applying the cost mapping to an input-conventional production possibility set yields a cost function, and applying the technology mapping to this cost function yields the initial production possibility set; and similarly, applying the technology mapping to an input-conventional cost structure yields a production possibility set, and applying the cost mapping to this production possibility set yields the initial cost function. Consequently, all structural features of the production possibilities are embodied in the functional specification of the cost function and are recovered by the technology mapping. As a corollary, distinct input-conventional technologies yield distinct input-conventional cost functions, and vice versa.

It should be noted that the one-to-one link between the input-conventional classes described above does not hold between input-conventional cost structures and input-regular production possibility sets. Distinct input-regular production possibility sets may yield the same input-conventional cost function. However, while going from the production possibility set to the cost function can entail a real loss of technological information in this case, the information lost is precisely that which is superfluous to the determination of observed competitive cost minimizing behavior. Figure 4 illustrates input-regular technologies which yield the same cost structure. In this example, under cost minimization the portions of the isoquant labeled "alternative 1" and "alternative 2" are never utilized, and hence cannot be distinguished on the basis of the behavior of the firm.

Lemma 3. Application of the cost mapping (1) to an input-conventional production possibility set yields an input-conventional cost structure. Application of the technology mapping (5) to this cost structure yields the initial production possibility set. Conversely, application of the technology mapping (5) to an input-

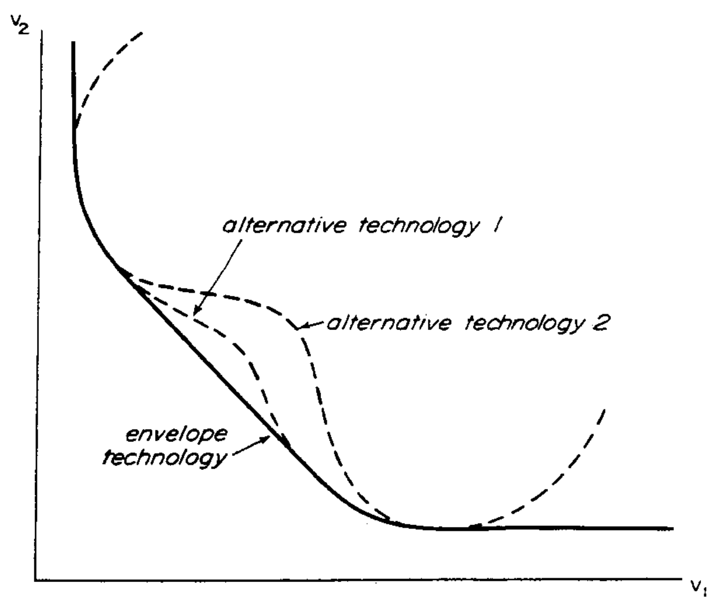


FIGURE 4

conventional cost structure yields an input-conventional production possibility set. Application of the cost mapping (1) to this production possibility set yields the initial cost structure.

Proof: Consider an input-conventional production possibility set defined by a producible output set $\mathbf{Y}^*$ and input requirement sets $\mathbf{V}(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$. The cost mapping yields a cost function $C(\mathbf{y},\mathbf{r})$, and the technology mapping applied to this cost function yields implicit input requirement sets $\mathbf{V}^*(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$. By Lemmas 1 and 2, the $\mathbf{V}^*(\mathbf{y})$ are input-conventional. We now show that $\mathbf{V}(\mathbf{y}) = \mathbf{V}^*(\mathbf{y})$.

If $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}^0 \in \mathbf{V}(\mathbf{y})$, then $\mathbf{r}\cdot\mathbf{v}^0 \geqq C(\mathbf{y},\mathbf{r})$ for all positive $\mathbf{r}$ by (1), and (5) then implies $\mathbf{v}^0 \in \mathbf{V}^*(\mathbf{y})$. Alternately, suppose $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}^0 \notin \mathbf{V}(\mathbf{y})$. We can apply a strict separating hyperplane theorem (Appendix A.3, 10.13) to establish the existence of a non-zero N-vector $\mathbf{r}$ and a positive scalar θ such that $\mathbf{r}\cdot\mathbf{v}^0 + \theta \leqq \mathbf{r}\cdot\mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}(\mathbf{y})$. Since $\mathbf{V}(\mathbf{y})$ satisfies the free disposal Assumption A, this inequality implies that $\mathbf{r}$ is non-negative. Choose $\mathbf{r}^0$ larger than $\mathbf{r}$ in every component and sufficiently close to $\mathbf{r}$ to satisfy $|\mathbf{r}\cdot\mathbf{v}^0 - \mathbf{r}^0\cdot\mathbf{v}^0| < \theta/2$. Then, $\mathbf{r}^0\cdot\mathbf{v}^0 + \theta/2 \leqq \mathbf{r}\cdot\mathbf{v} \leqq \mathbf{r}^0\cdot\mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}(\mathbf{y})$, implying $\mathbf{r}^0\cdot\mathbf{v}^0 < C(\mathbf{y},\mathbf{r}^0)$. By (5), $\mathbf{v}^0 \notin \mathbf{V}^*(\mathbf{y})$. This establishes $\mathbf{V}(\mathbf{y}) = \mathbf{V}^*(\mathbf{y})$.

To prove the second half of the lemma, consider an input-conventional cost structure given by a function $C(\mathbf{y},\mathbf{r})$ defined on a set $\mathbf{y} \in \mathbf{Y}^*$. The technology mapping yields implicit input requirement sets $\mathbf{V}^*(\mathbf{y})$, and the cost mapping applied to these input requirement sets yields a cost function $C^*(\mathbf{y},\mathbf{r})$ for $\mathbf{y} \in \mathbf{Y}^*$. By Lemmas 1 and 2, $C^*(\mathbf{y},\mathbf{r})$ is input-conventional. We now show that $C(\mathbf{y},\mathbf{r}) = C^*(\mathbf{y},\mathbf{r})$ for $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{r}$ positive.

Since $\mathbf{v} \in \mathbf{V}^*(\mathbf{y})$ implies $\mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r})$, we have immediately the inequality $C^*(\mathbf{y},\mathbf{r}) \geqq C(\mathbf{y},\mathbf{r})$. The proof is completed by supposing that $C^*(\mathbf{y},\mathbf{r}^0) > C(\mathbf{y},\mathbf{r}^0)$ for some $\mathbf{y} \in \mathbf{Y}^*$ and positive $\mathbf{r}^0$, and showing a contradiction results. Define the set $\mathbf{B} = \{(\mathbf{r},\xi) \in \mathbf{E}^{N+1} | \mathbf{r} \text{ positive}, \ \xi \geqq -C(\mathbf{y},\mathbf{r})\}$. Since C is concave and positively linear homogeneous in $\mathbf{r}$, the set $\mathbf{B}$ is a non-empty, convex cone. The point $(\mathbf{r}^0,\xi^0)$ with $\xi^0 = -C^*(\mathbf{y},\mathbf{r}^0)$ is by supposition not contained in $\mathbf{B}$. Further, by the continuity of C established in Lemma 1, $(\mathbf{r}^0,\xi^0)$ is not contained in the closure of $\mathbf{B}$. Then, the strict separating hyperplane theorem (Appendix A.3, 10.13) establishes the existence of a non-zero vector $(\mathbf{v}^0,\lambda) \in \mathbf{E}^{N+1}$ and a positive scalar θ such that $(\mathbf{v}^0,\lambda)\cdot(\mathbf{r}^0,\xi^0) + \theta \leqq (\mathbf{v}^0,\lambda)\cdot(\mathbf{r},\xi)$ for all $(\mathbf{r},\xi) \in \mathbf{B}$. Since $\mathbf{B}$ satisfies "free disposal", this inequality implies $\mathbf{v}^0$ and λ non-negative. If λ were zero, then the inequality would be violated by a point

$(\mathbf{r}^0,\xi) \in B$. Hence, we can assume without loss that $\lambda = 1$. Since $\mathbf{B}$ is a cone, the inequality can be written

$$\mathbf{r}^0 \cdot \mathbf{v}^0 - C^*(\mathbf{y},\mathbf{r}^0) + \theta \leqq 0 \leqq \mathbf{r} \cdot \mathbf{v}^0 - C(\mathbf{y},\mathbf{r}), \tag{7}$$

for all positive $\mathbf{r}$. By (5), $\mathbf{r} \cdot \mathbf{v}^0 \geqq C(\mathbf{y},\mathbf{r})$ for all positive $\mathbf{r}$ implies $\mathbf{v}^0 \in \mathbf{V}^*(\mathbf{y})$, and hence $\mathbf{r}^0 \cdot \mathbf{v}^0 \geqq C^*(\mathbf{y},\mathbf{r}^0)$. But this contradicts (7). Hence, $C(\mathbf{y},\mathbf{r}) = C^*(\mathbf{y},\mathbf{r})$. Q.E.D.

7. Distance Functions and Economic Transformation Functions

Frequently economists characterize production possibility sets implicitly using transformation functions or, in the one-output case, production functions. We will now give a straightforward restatement of the basic duality theorem of Section 6 in terms of the cost function and a form of a transformation function known as the distance function. The concept of a distance function comes from the mathematical theory of convex sets, and was introduced into economics by Shephard (1970). While the reformulation of duality in terms of distance functions is potentially useful in applications, its primary appeal comes from the fact that it allows us to establish a full, formal mathematical duality between transformation and cost functions, in the sense that both can be thought of as drawn from the same class of functions and having the same properties. We can exploit this formal duality to get "double our money" in further investigations of production and cost structures: if we can prove that a property "P" on a transformation function implies a property "Q" on a cost function, we can conclude by duality that property "P" on a cost function implies property "Q" on a transformation function. Hanoch's Chapter I.2 in this volume develops and applies this formal duality to functional forms in production theory.

Consider an input-conventional production possibility set characterized by a producible output set $\mathbf{Y}^*$ and input requirement sets $\mathbf{V}(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$. For this technology, define the *distance function*

$$F(\mathbf{y},\mathbf{v}) = \mathrm{Max}\left\{\lambda > 0 \,\middle|\, \frac{1}{\lambda}\mathbf{v} \in \mathbf{V}(\mathbf{y})\right\}, \tag{8}$$

for $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}$ strictly positive. In Lemma 4 below, we show that this formula defines a unique function which is finite valued for non-zero

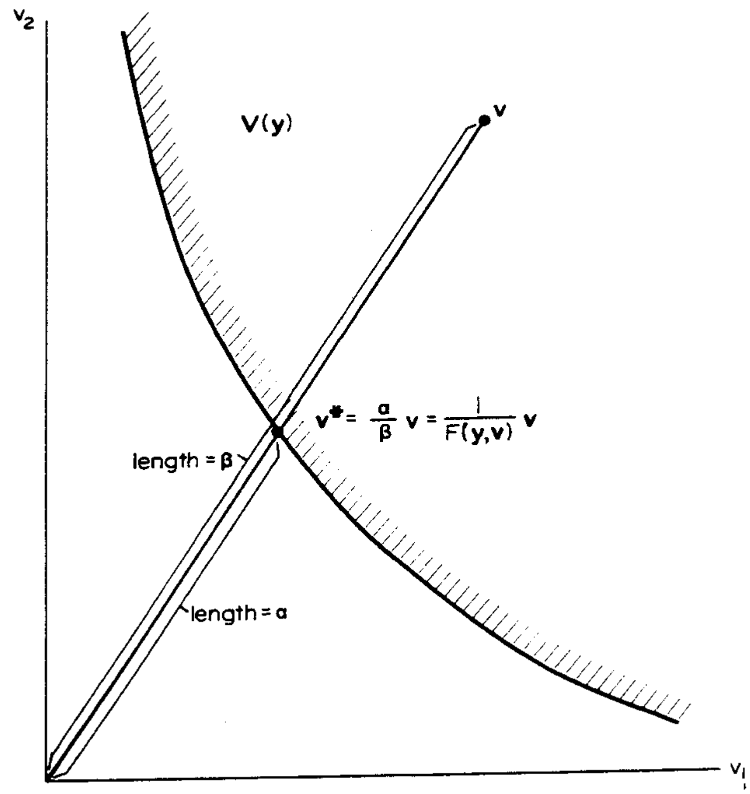


FIGURE 5

$\mathbf{y} \in \mathbf{Y}^*$. For $\mathbf{y} = \mathbf{0}$, the vector $\mathbf{0}$ may be in $\mathbf{V}(\mathbf{0})$, in which case $F(\mathbf{y},\mathbf{v})$ is defined to take the extended value $+\infty$. As illustrated in Figure 5, the value of $F(\mathbf{y},\mathbf{v})$ is given by the ratio of the length of the vector $\mathbf{v}$ to the length of a vector $\mathbf{v}^*$ defined by the intersection of the "y-isoquant" and the ray through $\mathbf{v}$.

For $\mathbf{y} \in \mathbf{Y}^*$, the strictly positive vectors $\mathbf{v}$ in the input requirement set $\mathbf{V}(\mathbf{y})$ are exactly those satisfying $F(\mathbf{y},\mathbf{v}) \geqq 1$. From the definition of the distance function, $\mathbf{v}/F(\mathbf{y},\mathbf{v})$ is contained in $\mathbf{V}(\mathbf{y})$, but no point southwest of it is in $\mathbf{V}(\mathbf{y})$. If $F(\mathbf{y},\mathbf{v}) \geqq 1$, then $\mathbf{v} \geqq \mathbf{v}/F(\mathbf{y},\mathbf{v})$, and $\mathbf{v} \in \mathbf{V}(\mathbf{y})$ by free disposal. If $F(\mathbf{y},\mathbf{v}) < 1$, then $\mathbf{v} < \mathbf{v}/F(\mathbf{y},\mathbf{v})$ is not in $\mathbf{V}(\mathbf{y})$.

Suppose the technology has a single output, and is defined by a production function $y = f(\mathbf{v})$. Then, at any point $(y,\mathbf{v})$, the distance function $F(y,\mathbf{v})$ takes on the value necessary to satisfy $y = f(\mathbf{v}/F(y,\mathbf{v}))$. This formula has a particularly simple form when the production function f is homothetic; i.e., $f(\mathbf{v}) = \phi(h(\mathbf{v}))$, where $h(\mathbf{v})$ is a linear homogeneous function and ϕ is a strictly monotone increasing function with

$\phi(0) = 0$. Then,

$$y = \phi\left[h\left(\frac{\mathbf{v}}{F(y,\mathbf{v})}\right)\right] = \phi\left[\frac{1}{F(y,\mathbf{v})}h(\mathbf{v})\right] \quad \text{or} \quad F(y,\mathbf{v}) = \frac{h(\mathbf{v})}{\phi^{-1}(y)},$$

where ϕ^{-1} is the inverse function of ϕ.

In the case of multiple outputs and a technology described by a tranformation function $G(\mathbf{y},\mathbf{v}) = 0$, the distance function is defined for $(\mathbf{y},\mathbf{v})$ by the value necessary to make $G(\mathbf{y}, \mathbf{v}/F(\mathbf{y},\mathbf{v})) = 0$. The distance function is then itself one representation of the transformation function for the technology, $F(\mathbf{y},\mathbf{v}) = 1$.

A distance function $F(\mathbf{y},\mathbf{v})$, defined for $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}$ positive, will be termed *input-conventional* if for each $\mathbf{y} \in \mathbf{Y}^*$, F as a function of $\mathbf{v}$ is positive, non-decreasing, positively linear homogeneous, concave, and continuous and if $F(\mathbf{y},\mathbf{v}) = +\infty$ implies $\mathbf{y} = 0$. Generally, we expect a cost function $C(\mathbf{y},\mathbf{r})$ to be increasing in the output bundle $\mathbf{y}$ and a distance function $F(\mathbf{y},\mathbf{v})$ to be decreasing in the output bundle $\mathbf{y}$. However, input-conventional cost structures and distance functions are defined to have identical mathematical properties with respect to their second arguments, input prices or inputs respectively. It is this formal duality that proves useful in obtaining further results. We first establish the relation between input-conventional production possibilities and input-conventional distance functions.

Lemma 4. Suppose a producible output set $\mathbf{Y}^*$ and input requirement sets $\mathbf{V}(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$ define an input-conventional technology. Then, the distance function $F(\mathbf{y},\mathbf{v})$ defined by (8) exists and is input-conventional. Conversely, given a non-empty set $\mathbf{Y}^*$ and an input-conventional distance function $F(\mathbf{y},\mathbf{v})$ defined for $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}$ positive, the relation

$$\mathbf{V}^*(\mathbf{y}) = \text{Closure } \{\mathbf{v} | \mathbf{v} \text{ positive}, F(\mathbf{y},\mathbf{v}) \geqq 1\} \tag{9}$$

defines indirect input requirement sets for $\mathbf{y} \in \mathbf{Y}^*$ which are input-conventional. If F is the distance function of an input-conventional technology with input requirement sets $\mathbf{V}(\mathbf{y})$, then $\mathbf{V}^*(\mathbf{y}) = \mathbf{V}(\mathbf{y})$ for $\mathbf{y} \in \mathbf{Y}^*$.

Proof: The first two steps of the proof verify that F exists and is input-conventional. Step 3 verifies that $\mathbf{V}^*(\mathbf{y})$ defined by (9) is input-conventional. Step 4 verifies the last result of the lemma, $\mathbf{V}^*(\mathbf{y}) = \mathbf{V}(\mathbf{y})$.

Step 1. We first show that $F(\mathbf{y},\mathbf{v})$ exists. Since $\mathbf{V}(\mathbf{y})$ is non-empty for

$\mathbf{y} \in \mathbf{Y}^*$ and free disposal holds, there is for each positive $\mathbf{v}$ some positive scalar λ' such that $(1/\lambda')\mathbf{v}$ is at least as large in every component as some fixed vector in $\mathbf{V}(\mathbf{y})$. Then, $(1/\lambda')\mathbf{v} \in \mathbf{V}(\mathbf{y})$. If $\mathbf{0} \in \mathbf{V}(\mathbf{y})$ in the case $\mathbf{y} = \mathbf{0}$, then $F(\mathbf{y},\mathbf{v}) = +\infty$ by definition. Suppose $\mathbf{0} \notin \mathbf{V}(\mathbf{y})$. Since $\mathbf{V}(\mathbf{y})$ is closed by hypothesis, there is an upper bound on the set of λ satisfying $(1/\lambda)\mathbf{v} \in \mathbf{V}(\mathbf{y})$, and the maximum in (8) is attained.

Step 2. That F is positive and positively linear homogeneous in positive $\mathbf{v}$ for each $\mathbf{y} \in \mathbf{Y}^*$ follows directly from (8). To show that F is non-decreasing in $\mathbf{v}$, note that if $\mathbf{v}^0,\mathbf{v}^1$ are positive input bundles with $\mathbf{v}^1 \geqq \mathbf{v}^0$, then $\mathbf{v}^1/F(\mathbf{y},\mathbf{v}^0) \geqq \mathbf{v}^0/F(\mathbf{y},\mathbf{v}^0) \in \mathbf{V}(\mathbf{y})$, implying $\mathbf{v}^1/F(\mathbf{y},\mathbf{v}^0) \in \mathbf{V}(\mathbf{y})$ by free disposal. Hence, $F(\mathbf{y},\mathbf{v}^1)/F(\mathbf{y},\mathbf{v}^0)) \geqq 1$, and $F(\mathbf{y},\mathbf{v}^1) \geqq F(\mathbf{y},\mathbf{v}^0)$ by the positive linear homogeneity of F. To show F concave in $\mathbf{v}$, it is sufficient (because of linear homogeneity) to show for any positive $\mathbf{v}^0$ and $\mathbf{v}^1$ that $F(\mathbf{y},\mathbf{v}^0+\mathbf{v}^1) \geqq F(\mathbf{y},\mathbf{v}^0) + F(\mathbf{y},\mathbf{v}^1)$. Since $\mathbf{v}^i/F(\mathbf{y},\mathbf{v}^i) \in \mathbf{V}(\mathbf{y})$ for $i = 0,1$, the convexity of $\mathbf{V}(\mathbf{y})$ implies

$$\alpha \frac{\mathbf{v}^0}{F(\mathbf{y},\mathbf{v}^0)} + (1-\alpha)\frac{\mathbf{v}^1}{F(\mathbf{y},\mathbf{v}^1)} \in \mathbf{V}(\mathbf{y}),$$

for any α satisfying $0 \leqq \alpha \leqq 1$. In particular, for $\alpha = F(\mathbf{y},\mathbf{v}^0)/[F(\mathbf{y},\mathbf{v}^0) + F(\mathbf{y},\mathbf{v}^1)]$, one obtains

$$\frac{\mathbf{v}^0+\mathbf{v}^1}{F(\mathbf{y},\mathbf{v}^0)+F(\mathbf{y},\mathbf{v}^1)} \in \mathbf{V}(\mathbf{y}),$$

implying

$$F\left[\mathbf{y}, \frac{\mathbf{v}^0+\mathbf{v}^1}{F(\mathbf{y},\mathbf{v}^0)+F(\mathbf{y},\mathbf{v}^1)}\right] \geqq 1.$$

By linear homogeneity, $F(\mathbf{y},\mathbf{v}^0+\mathbf{v}^1) \geqq F(\mathbf{y},\mathbf{v}^0) + F(\mathbf{y},\mathbf{v}^1)$. The continuity of F in positive $\mathbf{v}$ is an implication of concavity. This verifies that F is input-conventional.

Step 3. Suppose $F(\mathbf{y},\mathbf{v})$ defined for $\mathbf{y} \in \mathbf{Y}^*$ and $\mathbf{v}$ positive is input-conventional. Consider the indirect input requirement sets $\mathbf{V}^*(\mathbf{y})$ defined by (9). If $F(\mathbf{y},\mathbf{v}) = +\infty$, then $\mathbf{y}=\mathbf{0}$ and $\mathbf{V}^*(\mathbf{0})$ is the non-negative orthant. Consider $F(\mathbf{y},\mathbf{v}) < +\infty$. From (9), the $\mathbf{V}^*(\mathbf{y})$ are closed. By the positive linear homogeneity of F, $\mathbf{0} \notin \mathbf{V}^*(\mathbf{y})$. Since F is concave, the contour set $\mathbf{V}^*(\mathbf{y})$ is convex. Since F is non-decreasing in $\mathbf{v}$, the free disposal condition is satisfied by $\mathbf{V}^*(\mathbf{y})$. Hence, the indirectly defined technology is input-conventional.

Step 4. By (9), if $\mathbf{v}$ is positive, then $\mathbf{v} \in \mathbf{V}(\mathbf{y})$ if and only if $F(\mathbf{y},\mathbf{v}) \geqq 1$, and hence if and only if $\mathbf{v} \in \mathbf{V}^*(\mathbf{y})$. Since $\mathbf{V}(\mathbf{y})$ and $\mathbf{V}^*(\mathbf{y})$ are closed and the convexity of $\mathbf{V}(\mathbf{y})$ implies that it equals the closure of its interior, the equality $\mathbf{V}^*(\mathbf{y}) = \mathbf{V}(\mathbf{y})$ follows. Q.E.D.

It is sometimes useful to extend the definition of the distance function to all non-negative input bundles $\mathbf{v}$ by applying the formula (8) provided $\mathbf{v}/\lambda$ is in $\mathbf{V}(\mathbf{y})$ for some positive scalar λ, and setting $F(\mathbf{y},\mathbf{v}) = 0$ otherwise. Appealing to the arguments used to establish Lemma 1, one can show that this extended distance function is a positively linear homogeneous, non-decreasing, concave, continuous function of non-negative $\mathbf{v}$ for each $\mathbf{y} \in \mathbf{Y}^*$ when the hypotheses of Lemma 4 hold. In applications, it is sometimes useful to employ this extended definition of the distance function.

We can now restate the duality conditions of Lemmas 2 and 3 in terms of the distance function. This form of the duality theorem is due to Shephard (1970), who has made an exhaustive examination of the implications of the resulting formal mathematical duality.

Lemma 5. Consider (a) the family of input-conventional cost structures and (b) the family of input-conventional distance functions. For a cost structure $C(\mathbf{y},\mathbf{r})$, $\mathbf{y} \in \mathbf{Y}^*$, in family (a), define a technology mapping

$$F(\mathbf{y},\mathbf{v}) = \text{Max}\{\lambda > 0 | \mathbf{r}\cdot\mathbf{v} \geqq \lambda C(\mathbf{y},\mathbf{r}) \text{ for all } \mathbf{r} \text{ positive}\}. \tag{10}$$

For a distance function $F(\mathbf{y},\mathbf{v})$, $\mathbf{y} \in \mathbf{Y}^*$, in the family (b), define a cost mapping $C(\mathbf{y},\mathbf{r}) = 0$ if $\mathbf{y} = 0$, and for $\mathbf{y} \neq 0$,

$$C(\mathbf{y},\mathbf{r}) = \text{Max}\{\lambda > 0 | \mathbf{r}\cdot\mathbf{v} \geqq \lambda F(\mathbf{y},\mathbf{v}) \text{ for all } \mathbf{v} \text{ positive}\}. \tag{11}$$

Then, the function $F(\mathbf{y},\mathbf{v})$ defined by (10) is in family (b), and the function $C(\mathbf{y},\mathbf{r})$ defined by (11) is in family (a). The technology mapping (10) is equivalent to application of the mapping (5) to obtain implicit input requirement sets, and application of the mapping (8) to these sets to obtain a distance function. The cost mapping (11) is equivalent to application of the mapping (9) to obtain indirect input requirement sets, and application of the mapping (1) to these sets to obtain a cost function. Hence, the technology and cost mappings (10) and (11) are mutual inverses on the families (a) and (b).

Corollary. For all positive $\mathbf{r}$ and $\mathbf{v}$,

$$F(\mathbf{y},\mathbf{v})C(\mathbf{y},\mathbf{r}) \leqq \mathbf{r}\cdot\mathbf{v},$$

with equality if and only if **v** is a cost minimizing input vector for the argument (**y**,**r**).

Proof: The first step of the proof shows that the mapping (10) is the composition of the mappings (5) and (8). The second step shows that mapping (11) is the composition of the mappings (9) and (1). Then, Lemmas 2–4 will establish the implications of this lemma.

Step 1. Suppose an input-conventional cost structure $C(\mathbf{y},\mathbf{r})$, $\mathbf{y} \in \mathbf{Y}^*$, is given. The mapping (5) defines implicit input requirement sets $\mathbf{V}^*(\mathbf{y})$ with $\mathbf{v} \in \mathbf{V}^*(\mathbf{y})$ if and only if $\mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r})$ for all positive **r**. The mapping (8) defines an implicit distance function

$$\begin{aligned} F^*(\mathbf{y},\mathbf{v}) &= \operatorname{Max}\left\{\lambda > 0 \,\middle|\, \frac{1}{\lambda}\mathbf{v} \in \mathbf{V}^*(\mathbf{y})\right\} \\ &= \operatorname{Max}\left\{\lambda > 0 \,\middle|\, \mathbf{r}\cdot\left(\frac{1}{\lambda}\mathbf{v}\right) \geqq C(\mathbf{y},\mathbf{r}) \text{ for all } \mathbf{r} \text{ positive}\right\}. \end{aligned}$$

But this is the technology mapping (10), and $F^*(\mathbf{y},\mathbf{v}) = F(\mathbf{y},\mathbf{v})$.

Step 2. Given an input-conventional distance function $F(\mathbf{y},\mathbf{v})$, $\mathbf{y} \in \mathbf{Y}^*$, the mapping (9) defines indirect input requirement sets $\mathbf{V}^*(\mathbf{y})$, and the mapping (1) defines a minimum cost function $C^*(\mathbf{y},\mathbf{r})$ for these indirect input requirement sets. We need consider only $\mathbf{y} \neq \mathbf{0}$.

$$\begin{aligned} C^*(\mathbf{y},\mathbf{r}) &= \operatorname{Min}\{\mathbf{r}\cdot\mathbf{v} \mid \mathbf{v} \in \mathbf{V}^*(\mathbf{y})\} \\ &= \operatorname{Max}\{\lambda > 0 \mid \mathbf{r}\cdot\mathbf{v} \geqq \lambda \text{ for all } \mathbf{v} \in \mathbf{V}^*(\mathbf{y})\} \\ &= \operatorname{Max}\{\lambda > 0 \mid \mathbf{r}\cdot\mathbf{v} \geqq \lambda \text{ for all } \mathbf{v} \in \mathbf{V}^*(\mathbf{y}),\ \mathbf{v} \text{ positive}\}. \end{aligned}$$

Now, for all positive **v**, $\mathbf{v}/F(\mathbf{y},\mathbf{v}) \in \mathbf{V}^*(\mathbf{y})$. Further, $\mathbf{v} \in \mathbf{V}^*(\mathbf{y})$ implies $F(\mathbf{y},\mathbf{v}) \geqq 1$, and hence $\mathbf{r}\cdot\mathbf{v} \geqq \mathbf{r}\cdot\mathbf{v}/F(\mathbf{y},\mathbf{v})$. Therefore,

$$C^*(\mathbf{y},\mathbf{r}) = \operatorname{Max}\{\lambda > 0 \mid \mathbf{r}\cdot\mathbf{v}/F(\mathbf{y},\mathbf{v}) \geqq \lambda \text{ for all } \mathbf{v} \text{ positive}\}.$$

But this is the cost mapping (11), and $C^*(\mathbf{y},\mathbf{r}) = C(\mathbf{y},\mathbf{r})$. Q.E.D.

8. Extensions of Duality

The duality theorem established in Section 6 provides a basis for relating structural properties of production possibilities to structural properties of the cost function. In applications, it is useful to have a large family of duality relationships of the form: "the production possibility set has property 'P' if and only if the cost function has property 'Q'." Using the formal duality of cost and distance functions derived in the preceding section, we will be able to establish also the validity of such propositions with the properties "P" and "Q" interchanged. Through the remainder of this section, we shall assume that production possibility sets are described by distance functions, and that all cost structures and distance functions are input-conventional. We begin with a series of definitions.

A positive input bundle $\mathbf{v}$ is *efficient* for an output bundle $\mathbf{y}$ and distance function F if $F(\mathbf{y},\mathbf{v}) = 1$ and any distinct positive input bundle $\mathbf{v}'$ with $\mathbf{v}' \leqq \mathbf{v}$ has $F(\mathbf{y},\mathbf{v}') < 1$. Alternately, define an input bundle $\mathbf{v}$ to be *efficient* for an input requirement set $\mathbf{V}(\mathbf{y})$ if any distinct input bundle $\mathbf{v}'$ with $\mathbf{v}' \leqq \mathbf{v}$ has $\mathbf{v}' \notin \mathbf{V}(\mathbf{y})$. The reader can verify that for positive input bundles, these definitions of efficient input bundles are equivalent. In (a) of Figure 3, the points $\mathbf{v}^*$ and $\mathbf{v}^3$ are efficient, while $\mathbf{v}^2$ and $\mathbf{v}^4$ are not.

Recall that the distance function F is concave in $\mathbf{v}$, by (12) and linear homogeneity. Define F to be *strictly quasi-concave from below* if its upper contour sets $\{\mathbf{v} \in \mathbf{E}_+^N | F(\mathbf{y},\mathbf{v}) \geqq 1\}$ are strictly convex from below (see Assumption B-2) for all $\mathbf{y} \in \mathbf{Y}^*$. This property can be restated as requiring, for any positive, distinct points $\mathbf{v}^0$ and $\mathbf{v}^1$ and output $\mathbf{y} \in \mathbf{Y}^*$, that either (i) every plane which contains $\mathbf{v}^0/F(\mathbf{y},\mathbf{v}^0)$ and $\mathbf{v}^1/F(\mathbf{y},\mathbf{v}^1)$ and bounds $\{\mathbf{v} \in \mathbf{E}_+^N | F(\mathbf{y},\mathbf{v}) \geqq 1\}$ from below is parallel to a coordinate axis, or else (ii) for every weighted average $\mathbf{v}'' = \theta\mathbf{v}^0 + (1-\theta)\mathbf{v}^1$, with $0 < \theta < 1$.

$$F(\mathbf{y},\mathbf{v}'') > \mathrm{Min}\{F(\mathbf{y},\mathbf{v}^0),\ F(\mathbf{y},\mathbf{v}^1)\}. \tag{12}$$

Figure 3 illustrates the geometry of this condition, which guarantees that the "efficient" boundary of each input requirement set is rotund, containing no "flat segments".

A stronger version of strict quasi-concavity from below will also be used. When the transformation function $F(\mathbf{y},\mathbf{v})$ is differentiable in the inputs, let $F_v(\mathbf{y},\mathbf{v})$ denote the vector of partial derivatives $F_n(\mathbf{y},\mathbf{v}) \equiv \partial F/\partial v_n$, $n = 1,\ldots,N$, evaluated at $(\mathbf{y},\mathbf{v})$. This vector is termed the *gradient* of F. Let $F_{vv}(\mathbf{y},\mathbf{v})$ denote the N-dimensional matrix of second partial derivatives $\partial^2 F/\partial v_n \partial v_m$, $n,m = 1,\ldots,N$, evaluated at $(\mathbf{y},\mathbf{v})$. This array is termed the *Hessian matrix* of F. The transformation function F is *strictly differentiably quasi-concave from below* in positive $\mathbf{v}$ if for any positive efficient $\mathbf{v}^0,\mathbf{v}^1$ and weighted average $\mathbf{v}'' = \theta\mathbf{v}^0 + (1-\theta)\mathbf{v}^1$, $0 < \theta < 1$, it follows that the Hessian matrix $F_{vv}(\mathbf{v}'')$ is negative semi-definite of rank $N-1$.

A remark on the relation of these definitions is in order. The conditions that F is concave and positively linear homogeneous in $\mathbf{v}$ imply that when the Hessian of F exists, it is symmetric, negative semi-definite, and singular, with a zero characteristic root corresponding to the characteristic vector $\mathbf{v}''$. Hence, strict differential quasi-concavity from below requires that the quadratic form,

$$Q(\mathbf{v},F_{vv}(\mathbf{v}'')) = \sum_{n=1}^{N} \sum_{m=1}^{N} v_n v_m F_{v_n v_m}(\mathbf{y},\mathbf{v}''), \tag{13}$$

be negative for any non-zero vector $\mathbf{v}$ not proportional to $\mathbf{v}''$. It is shown in Appendix A.3 that strict differential quasi-concavity from below implies strict quasi-concavity from below. As a partial converse it is shown that continuous second-order differentiability plus strict quasi-concavity from below implies that the condition of strict differential quasi-concavity from below holds on a subset of $\mathbf{v}$ which is open and dense[6] relative to the set of efficient $\mathbf{v}$.

The distance function $F(\mathbf{y},\mathbf{v})$ is *non-increasing* in the output bundle $\mathbf{y}$ if for any $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$, it follows that $F(\mathbf{y}^0,\mathbf{v}) \geqq F(\mathbf{y}^1,\mathbf{v})$. This property is equivalent to the condition on input requirement sets that $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$, $\mathbf{y}^0 \leqq \mathbf{y}^1$ implies that $\mathbf{V}(\mathbf{y}^1)$ is contained in $\mathbf{V}(\mathbf{y}^0)$. Similarly, the cost function $C(\mathbf{y},\mathbf{r})$ is non-decreasing in $\mathbf{y}$ if for any $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$, it follows that $C(\mathbf{y}^0,\mathbf{r}) \leqq C(\mathbf{y}^1,\mathbf{r})$.

The distance function $F(\mathbf{y},\mathbf{v})$ is *uniformly decreasing* in the output bundle $\mathbf{y}$ if for any distinct $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$, there exists a small positive scalar α such that $F(\mathbf{y}^0,\mathbf{v})/F(\mathbf{y}^1,\mathbf{v}) \geqq 1+\alpha$ for all positive $\mathbf{v}$. In terms of the input requirement sets, this condition is equivalent to the property that distinct $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$ implies $\mathbf{V}(\mathbf{y}^1)$ a proper subset of $\mathbf{V}(\mathbf{y}^0)$, with each input bundle in $\mathbf{V}(\mathbf{y}^1)$ at least as large as a $(1+\alpha)$-multiple of an input bundle in $\mathbf{V}(\mathbf{y}^0)$. When the set of efficient input bundles in $\mathbf{V}(\mathbf{y}^0)$ is bounded, this condition reduces to the requirement that $\mathbf{V}(\mathbf{y}^1)$ not contain the efficient bundles in $\mathbf{V}(\mathbf{y}^0)$.

The cost function $C(\mathbf{y},\mathbf{r})$ is *uniformly increasing* in the output bundle $\mathbf{y}$ if for any distinct $\mathbf{y}^0$, $\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$, there exists a small positive scalar α such that $C(\mathbf{y}^1,\mathbf{r})/C(\mathbf{y}^0,\mathbf{r}) > 1+\alpha$ for all positive $\mathbf{r}$.

The distance function $F(\mathbf{y},\mathbf{v})$ is *strongly upper semicontinuous* in $(\mathbf{y},\mathbf{v})$ if for any sequence $(\mathbf{y}^i,\mathbf{v}^i)$ with $\mathbf{y}^i \in \mathbf{Y}^*$ and $\mathbf{v}^i$ positive which converges to a point $(\mathbf{y}^0,\mathbf{v}^0)$, two properties hold: (a) If $F(\mathbf{y}^i,\mathbf{v}^*)$ is bounded away from zero for some positive $\mathbf{v}^*$, then $\mathbf{y}^0 \in \mathbf{Y}^*$. (b) If $\mathbf{y}^0 \in \mathbf{Y}^*$ and $\mathbf{v}^0$ is positive, then $F(\mathbf{y}^0,\mathbf{v}^0) \geqq \limsup_i F(\mathbf{y}^i,\mathbf{v}^i)$. The cost function $C(\mathbf{y},\mathbf{r})$ is *strongly lower semicontinuous* in $(\mathbf{y},\mathbf{r})$ if for any sequence $(\mathbf{y}^i,\mathbf{r}^i)$ with $\mathbf{y}^i \in \mathbf{Y}^*$ and $\mathbf{r}^i$ positive which converges to a point $(\mathbf{y}^0,\mathbf{r}^0)$, two properties

[6]A set is *open* if it contains a neighborhood of each point in the set, and is *dense* if every neighborhood contains some point of the set.

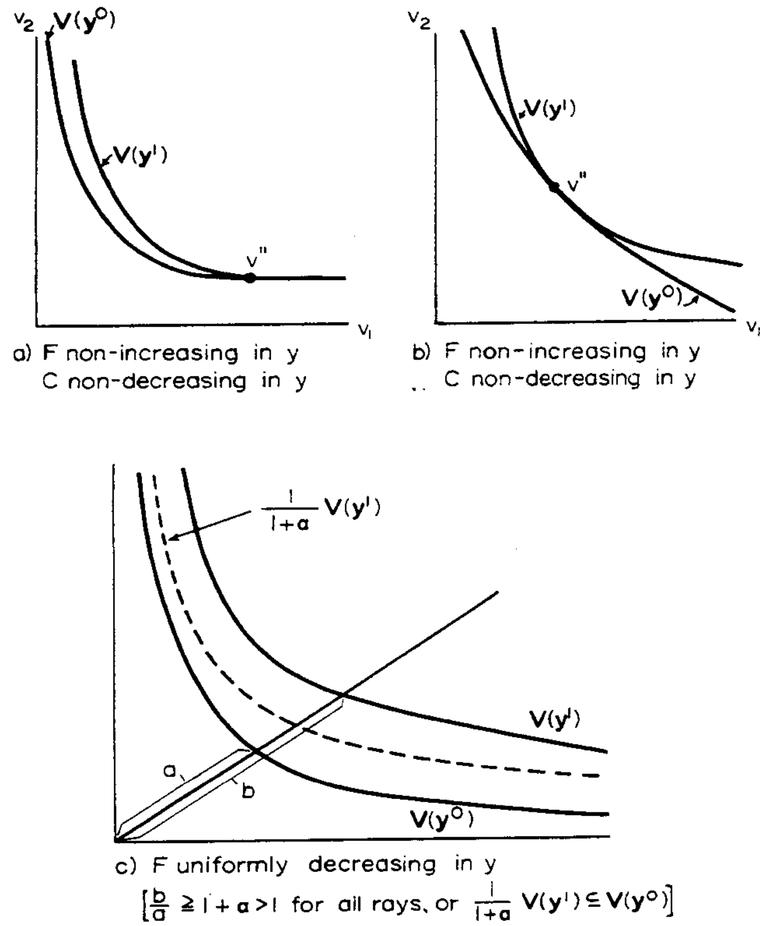


FIGURE 6

hold: (a) If $C(\mathbf{y}^i,\mathbf{r}^*)$ is bounded for some positive $\mathbf{r}^*$, then $\mathbf{y}^0 \in \mathbf{Y}^*$. (b) If $\mathbf{y}^0 \in \mathbf{Y}^*$ and $\mathbf{r}^0$ is positive, then $C(\mathbf{y}^0,\mathbf{r}^0) \leqq \liminf_i C(\mathbf{y}^i,\mathbf{r}^i)$.

Figures 6 and 7 illustrate these concepts. In (a) of Figure 6 the cost of producing $\mathbf{y}^1$ exceeds the cost of producing $\mathbf{y}^0$ at any strictly positive prices. However, at $\mathbf{v}''$ one has $F(\mathbf{y}^0,\mathbf{v}'') = F(\mathbf{y}^1,\mathbf{v}'')$ and F is not strictly decreasing in $\mathbf{y}$. In (b), F is again not strictly decreasing in $\mathbf{y}$ at $(\mathbf{y}^0,\mathbf{v}'')$. At the price vector $\mathbf{r}''$ at which $\mathbf{v}''$ is optimal, C is not strictly increasing in $\mathbf{y}$. Both (a) and (b) of Figure 6 correspond to pathological technologies which are unlikely to arise in practice. (c) illustrates the assumption of uniform monotonicity. This condition requires that isoquants not converge (when the distance between them is measured along rays). In Figure 7, (a) illustrates upper semicontinuity of a function F. At the argument $\mathbf{y}^0$, the function takes the largest of the limiting values. In this

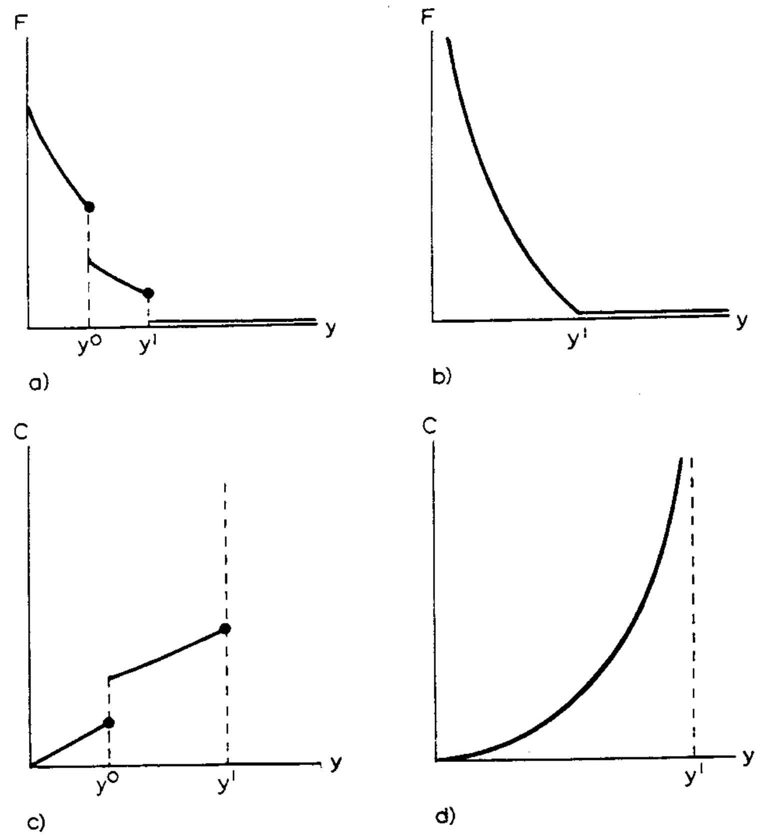


FIGURE 7

graph, F is bounded away from zero for y in the closed interval $[0,y^1]$; hence strong upper semicontinuity implies $\mathbf{Y}^* = [0,y^1]$. In (b), F approaches zero as y approaches y^1, implying $\mathbf{Y}^* = [0,y^1]$. [In general, $\mathbf{Y}^* = \{\mathbf{y}|F(\mathbf{y},\mathbf{v}) > 0$ for some $\mathbf{v} \gg 0\}$.] (c) of Figure 7 illustrates lower semicontinuity of a function C. At the argument y^0, C takes the smallest of the limiting values. At y^1, C is bounded, implying by strong lower semicontinuity that $\mathbf{Y}^* = [0,y^1]$. In (d), C is unbounded as y approaches y^1, implying $\mathbf{Y}^* = [0,y^1)$. The next result relates the strong upper semicontinuity of the distance function to a property of the production possibility set.

Lemma 6. Consider an input-conventional production possibility set $\mathbf{Y}$, and let $F(\mathbf{y},\mathbf{v})$ be its distance function, so that (8) and (9) hold. Then, the set $\mathbf{Y}$ is closed if and only if F is strongly upper semicontinuous in $(\mathbf{y},\mathbf{v})$.

Proof: First, suppose F is strongly upper semicontinuous in $(\mathbf{y},\mathbf{v})$. Consider a sequence $(\mathbf{y}^i,\mathbf{v}^i)\in\mathbf{Y}$ with $(\mathbf{y}^i,\mathbf{v}^i)\to(\mathbf{y}^0,\mathbf{v}^0)$. Choose $\mathbf{v}^*$ strictly larger than $\mathbf{v}^0$. Then, for i large, $\mathbf{v}^i\leqq\mathbf{v}^*$, implying $F(\mathbf{y}^i,\mathbf{v}^*)\geqq 1$. This implies $\mathbf{y}^0\in\mathbf{Y}^*$. Let $\mathbf{w}$ be an arbitrarily small positive vector. Then, $(\mathbf{y}^i,\mathbf{v}^i+\mathbf{w})\in\mathbf{Y}$ and $(\mathbf{y}^i,\mathbf{v}^i+\mathbf{w})\to(\mathbf{y}^0,\mathbf{v}^0+\mathbf{w})$, implying, since $F(\mathbf{y}^i,\mathbf{v}^i+\mathbf{w})\geqq 1$, that $F(\mathbf{y}^0,\mathbf{v}^0+\mathbf{w})\geqq 1$. Letting $\mathbf{w}\to 0$, (9) implies $\mathbf{v}^0\in\mathbf{V}(\mathbf{y}^0)$, and hence $(\mathbf{y}^0,\mathbf{v}^0)\in\mathbf{Y}$.

Next, suppose $\mathbf{Y}$ is closed. Consider a sequence $(\mathbf{y}^i,\mathbf{v}^i)$ with $\mathbf{y}^i\in\mathbf{Y}^*$ and $\mathbf{v}^i$ positive which converges to a point $(\mathbf{y}^0,\mathbf{v}^0)$. Then, $(\mathbf{y}^i,\mathbf{v}^i/F(\mathbf{y}^i,\mathbf{v}^i))\in\mathbf{Y}$. If $F(\mathbf{y}^i,\mathbf{v}^i)$ is unbounded, then the closedness of $\mathbf{Y}$ implies $(\mathbf{y}^0,\mathbf{0})\in\mathbf{Y}$, implying $\mathbf{y}^0=\mathbf{0}$ and $F(\mathbf{y}^0,\mathbf{v}^0)=+\infty\geqq\lim_i F(\mathbf{y}^i,\mathbf{v}^i)$. Alternately, assume $F(\mathbf{y}^i,\mathbf{v}^i)$ bounded. Then $F(\mathbf{y}^i,\mathbf{v}^i)$ has a subsequence (retain notation) converging to a scalar α. If α is positive, it follows that $(\mathbf{y}^0,\mathbf{v}^0/\alpha)\in\mathbf{Y}$, implying $\mathbf{y}^0\in\mathbf{Y}^*$ and, if $\mathbf{v}^0$ is positive, $F(\mathbf{y}^0,\mathbf{v}^0)\geqq\alpha$. If α is zero, but $\mathbf{y}^0\in\mathbf{Y}^*$ and $\mathbf{v}^0$ positive, then $F(\mathbf{y}^0,\mathbf{v}^0)>0=\lim_i F(\mathbf{y}^i,\mathbf{v}^i)$ for the subsequence. In either case, the condition for strong upper semicontinuity of F is met. Q.E.D.

The following result relates properties of the distance function and the cost function.

> *Lemma* 7. Consider (a) the family of input-conventional cost structures $C(\mathbf{y},\mathbf{r})$, $\mathbf{y}\in\mathbf{Y}^*$, and (b) the family of input-conventional distance functions $F(\mathbf{y},\mathbf{v})$, $\mathbf{y}\in\mathbf{Y}^*$. Suppose these families are related by the mutually inverse technology and cost mappings (10) and (11). Then, in Table 1, the distance function has property "P" if and only if the cost structure has the corresponding property "Q".

Proof: A detailed proof of this lemma is tedious and of minimal inherent interest. Hence, only outlines of proofs will be given, and mathematically difficult points will be deferred to Appendix A.3. The steps of this proof correspond to the eight results in Table 1. In each step, we first show that "P" implies "Q", and then show that "Q" implies "P".

Step 1. Suppose F is non-increasing in $\mathbf{y}$, so that $\mathbf{y}^0,\mathbf{y}^1\in\mathbf{Y}^*$ and $\mathbf{y}^0\leqq\mathbf{y}^1$ imply $F(\mathbf{y}^1,\mathbf{v})\leqq F(\mathbf{y}^0,\mathbf{v})$. By (11), for any positive price vector $\mathbf{r}$ and any $\epsilon>0$, there exists a positive vector $\mathbf{v}$ such that $C(\mathbf{y}^1,\mathbf{r})F(\mathbf{y}^1,\mathbf{v})\geqq(\mathbf{r}\cdot\mathbf{v})/(1+\epsilon)$. Further, $C(\mathbf{y}^0,\mathbf{r})F(\mathbf{y}^0,\mathbf{v})\leqq\mathbf{r}\cdot\mathbf{v}$. Hence, $C(\mathbf{y}^0,\mathbf{r})F(\mathbf{y}^0,\mathbf{v})\leqq$

TABLE 1

Property "P" holds for an input-conventional transformation function, $F(\mathbf{y},\mathbf{v})$, if and only if property "Q" holds for its input-conventional cost function, $C(\mathbf{y},\mathbf{r})$.[a]

	"P" on $F(\mathbf{y},\mathbf{v})$	"Q" on $C(\mathbf{y},\mathbf{r})$
1.	Non-increasing in **y**	Non-decreasing in **y**
2.	Uniformly decreasing in **y**	Uniformly increasing in **y**
3.[b]	Strongly upper semicontinuous in (**y**,**v**)	Strongly lower semicontinuous in (**y**,**r**)
4.[c]	Strongly lower semicontinuous in (**y**,**v**)	Strongly upper semicontinuous in (**y**,**r**)
5.[d]	Strongly continuous in (**y**,**v**)	Strongly continuous in (**y**,**r**)
6.[e]	Strictly quasi-concave from below in **v**	Continuously differentiable in positive **r**
7.[f]	Continuously differentiable in positive **v**	Strictly quasi-concave from below in **r**
8.[g]	Twice continuously differentiable and strictly differentiably quasi-concave from below in **v**	Twice continuously differentiable and strictly differentiably quasi-concave from below in **r**

[a]By the formal duality of cost and transformation functions, the implications of this table continue to hold when properties "P" and "Q" are reversed; i.e., "P" holds for the cost function and "Q" holds for the transformation function.

[b]Recall that this property is equivalent to the condition that the production possibility set be a closed set.

[c]Input requirement sets $\mathbf{V}(\mathbf{y})$ form a ***strongly lower hemicontinuous correspondence*** if two properties hold: (a) If $\mathbf{y}^i \in \mathbf{Y}^*$, $\mathbf{y}^i \to \mathbf{y}^0 \notin \mathbf{Y}^*$ and $\mathbf{A}$ is any bounded set in $\mathbf{E}^N$, then for sufficiently large i, $\mathbf{V}(\mathbf{y}^i)$ does not meet $\mathbf{A}$. (b) If $\mathbf{y}^0 \in \mathbf{Y}^*$, $\mathbf{v}^0 \in \mathbf{V}(\mathbf{y}^0)$, and $\mathbf{y}^i \in \mathbf{Y}^*$ $\mathbf{y}^i \to \mathbf{y}^0$, then there exist $\mathbf{v}^i \in \mathbf{V}(\mathbf{y}^i)$ such that $\mathbf{v}^i \to \mathbf{v}^0$. This condition implies that the cost function is strongly upper semicontinuous in $(\mathbf{y},\mathbf{r})$. To show this, note first that $\mathbf{y}^i \in \mathbf{Y}^*$, $\mathbf{y}^i \to \mathbf{y}^0 \notin \mathbf{Y}^*$ implies $C(\mathbf{y}^i,\mathbf{r}^*) \to +\infty$ for $\mathbf{r}^*$ positive. Hence, $C(\mathbf{y}^i,\mathbf{r}^*)$ bounded implies $\mathbf{y}^0 \in \mathbf{Y}^*$. Next, note that if $(\mathbf{y}^i,\mathbf{r}^i) \to (\mathbf{y}^0,\mathbf{r}^0)$ with $\mathbf{y}^i,\mathbf{y}^0 \in \mathbf{Y}^*$, and $\mathbf{r}^i,\mathbf{r}^0$ positive, there exists $\mathbf{v}^0 \in \mathbf{V}(\mathbf{y}^0)$ such that $C(\mathbf{y}^0,\mathbf{r}^0) = \mathbf{r}^0 \cdot \mathbf{v}^0$ and there exist $\mathbf{v}^i \in \mathbf{V}(\mathbf{y}^i)$ such that $\mathbf{v}^i \to \mathbf{v}^0$. Then $C(\mathbf{y}^i,\mathbf{r}^i) \leqq \mathbf{r}^i \cdot \mathbf{v}^i \to \mathbf{r}^0 \cdot \mathbf{v}^0$ implies $\limsup_i C(\mathbf{y}^i,\mathbf{r}^i) \leqq C(\mathbf{y}^0,\mathbf{r}^0)$. A more difficult argument, given in Appendix A.3. 15.5, establishes the converse implication from C to $\mathbf{V}$, and consequently the equivalence of the condition that the distance function be strongly lower semicontinuous and the condition that the input requirement sets define a strongly lower hemicontinuous correspondence.

[d]A function is strongly continuous if it is strongly upper and strongly lower semicontinuous. This property is equivalent to a requirement that the input requirement sets $\mathbf{V}(\mathbf{y})$ define a strongly continuous correspondence (Appendix A.3, 13.2).

[e]This property guarantees that isoquants are rotund, with no flat segments.

[f]This property guarantees that isoquants have no "kinks".

[g]An input-conventional transformation function with these properties is termed *neoclassical*. This result then provides a formal duality theorem for neoclassical distance functions and neoclassical cost functions.

$(1+\epsilon)C(\mathbf{y}^1,\mathbf{r})F(\mathbf{y}^1,\mathbf{v})$, or

$$\frac{C(\mathbf{y}^0,\mathbf{r})}{C(\mathbf{y}^1,\mathbf{r})} \leqq (1+\epsilon)\frac{F(\mathbf{y}^1,\mathbf{v})}{F(\mathbf{y}^0,\mathbf{v})} \leqq 1+\epsilon, \tag{14}$$

implying $C(\mathbf{y}^0,\mathbf{r}) \leqq C(\mathbf{y}^1,\mathbf{r})$.

Next suppose C is non-decreasing in $\mathbf{y}$, so that $\mathbf{y}^0,\mathbf{y}^1 \in \mathbf{Y}^*$ and $\mathbf{y}^0 \leqq \mathbf{y}^1$ imply $C(\mathbf{y}^0,\mathbf{r}) \leqq C(\mathbf{y}^1,\mathbf{r})$. Analogously to the preceding argument, (10) implies, for any positive $\mathbf{v}$ and any $\epsilon > 0$, the existence of $\mathbf{r}$ such that $C(\mathbf{y}^0,\mathbf{r})F(\mathbf{y}^0,\mathbf{v}) \geqq (\mathbf{r}\cdot\mathbf{v})/(1+\epsilon)$. Since $C(\mathbf{y}^1,\mathbf{r})F(\mathbf{y}^1,\mathbf{v}) \leqq \mathbf{r}\cdot\mathbf{v}$,

$$\frac{F(\mathbf{y}^1,\mathbf{v})}{F(\mathbf{y}^0,\mathbf{v})} \leqq (1+\epsilon)\frac{C(\mathbf{y}^0,\mathbf{r})}{C(\mathbf{y}^1,\mathbf{r})} \leqq 1+\epsilon, \tag{15}$$

implying $F(\mathbf{y}^0,\mathbf{v}) \geqq F(\mathbf{y}^1,\mathbf{v})$.

Step 2. Suppose F is uniformly decreasing in $\mathbf{y}$, so that distinct $\mathbf{y}^0,\mathbf{y}^1 \in \mathbf{Y}^*$ with $\mathbf{y}^0 \leqq \mathbf{y}^1$ imply $F(\mathbf{y}^0,\mathbf{v})/F(\mathbf{y}^1,\mathbf{v}) \geqq 1+\alpha$ for some positive scalar α, uniformly in $\mathbf{v}$. In (14), this implies $C(\mathbf{y}^1,\mathbf{r})/C(\mathbf{y}^0,\mathbf{r}) \geqq 1+\alpha$ uniformly in $\mathbf{r}$. Conversely, suppose C is uniformly increasing in $\mathbf{y}$. Then, a similar argument applied to (15) yields the result that F is uniformly decreasing in $\mathbf{y}$.

Step 3. Suppose F is strongly upper semicontinuous. By Lemmas 4 and 6, the indirectly defined production possibility set $\mathbf{Y}$ is closed. Consider a sequence $(\mathbf{y}^i,\mathbf{r}^i)$ with $\mathbf{y}^i \in \mathbf{Y}^*$, $\mathbf{r}^i$ positive which converges to a point $(\mathbf{y}^0,\mathbf{r}^0)$. Then there exist $\mathbf{v}^i \in \mathbf{V}(\mathbf{y}^i)$ such that $C(\mathbf{y}^i,\mathbf{r}^i) = \mathbf{r}^i\cdot\mathbf{v}^i$. If $C(\mathbf{y}^i,\mathbf{r}^i)$ is bounded and $\mathbf{r}^0$ is positive, this equality implies that $\mathbf{v}^i$ is bounded and has at least one limit point $\mathbf{v}^0$. The closedness of $\mathbf{Y}$ implies $(\mathbf{y}^0,\mathbf{v}^0) \in \mathbf{Y}$. Hence, $\mathbf{y}^0 \in \mathbf{Y}^*$ and $C(\mathbf{y}^0,\mathbf{r}^0) \leqq \mathbf{r}^0\cdot\mathbf{v}^0$. Since this inequality holds for each limit point, $C(\mathbf{y}^0,\mathbf{r}^0) \leqq \liminf_i C(\mathbf{y}^i,\mathbf{r}^i)$. This establishes condition (a) for C to be lower semicontinuous, and condition (b) in the case that $C(\mathbf{y}^i,\mathbf{r}^i)$ has a bounded subsequence. Finally, if $C(\mathbf{y}^i,\mathbf{r}^i)$ has no bounded subsequence, but $\mathbf{y}^0 \in \mathbf{Y}^*$ and $\mathbf{r}^0$ is positive, then obviously $C(\mathbf{y}^0,\mathbf{r}^0) \leqq \liminf_i C(\mathbf{y}^i,\mathbf{r}^i)$. Hence, C is strongly lower semicontinuous.

Next, suppose C is strongly lower semicontinuous. By (5), we have $\mathbf{V}(\mathbf{y}) = \{\mathbf{v} \geqq 0 | \mathbf{r}\cdot\mathbf{v} \geqq C(\mathbf{y},\mathbf{r})$ for all positive $\mathbf{r}\}$ for $\mathbf{y} \in \mathbf{Y}^*$ and by (6) a production possibility set $\mathbf{Y}$. Consider a sequence $(\mathbf{y}^i,\mathbf{v}^i) \in \mathbf{Y}$ converging to a point $(\mathbf{y}^0,\mathbf{v}^0)$. For each positive $\mathbf{r}$, $\mathbf{r}\cdot\mathbf{v}^0 = \lim_i \mathbf{r}\cdot\mathbf{v}^i \geqq \lim_i C(\mathbf{y}^i,\mathbf{r})$, implying $\mathbf{y}^0 \in \mathbf{Y}^*$ and $\lim_i C(\mathbf{y}^i,\mathbf{r}) \geqq C(\mathbf{y}^0,\mathbf{r})$ by strong lower semicontinuity. This implies $\mathbf{v}^0 \in \mathbf{V}(\mathbf{y}^0)$, and hence $(\mathbf{y}^0,\mathbf{v}^0) \in \mathbf{Y}$. Therefore, $\mathbf{Y}$ is closed, and Lemma 6 implies that F is strongly upper semicontinuous.

Step 4. Utilizing the formal duality of F and C, properties "P" and "Q" in Step 3 can be reversed to yield result 4.

Step 5. This result is implied by the results 3 and 4.

Step 6. Note that a concave function which is differentiable on an open set is continuously differentiable on that set, and that the negative of a concave function is a convex function. Then, a lengthy argument given in the Appendix A.3, 16.7(7) and 16.7(10), yields this result.

Step 7. This result is implied by result 6 using the formal duality of C and F.

Step 8. This result is established in the Appendix A.3, 16.7(11). Q.E.D.

One implication of the duality theory developed above is that the input requirement sets have image sets in the space of input prices, defined for $\mathbf{y} \in \mathbf{Y}^*$ by

$$\begin{aligned} \mathbf{R}(\mathbf{y}) &= \{\mathbf{r} \geqq \mathbf{0} | \mathbf{r}\cdot\mathbf{v} \geqq F(\mathbf{y},\mathbf{v}) \text{ for all positive } \mathbf{v}\} \\ &= \text{Closure}\,\{\mathbf{r}|\mathbf{r} \text{ positive}, C(\mathbf{y},\mathbf{r}) \geqq 1\}. \end{aligned} \tag{16}$$

This set is termed the *factor price requirement set*, and its boundary is termed the *factor price frontier*, for the output bundle $\mathbf{y}$. This concept has been employed in applications by Samuelson (1953–54), Bruno (1968), and others.

The properties of the cost function – concavity, monotonicity, linear homogeneity, and continuity – imply that the factor price requirement set $\mathbf{R}(\mathbf{y})$ is closed, is non-empty for $\mathbf{y} \neq \mathbf{0}$, and satisfies the free disposal and convexity assumptions A and B. Therefore, there is a formal mathematical duality between input requirement sets $\mathbf{V}(\mathbf{y})$ and factor price requirement sets $\mathbf{R}(\mathbf{y})$; they are termed *polar reciprocal sets*, and can be characterized directly by the relationship $\mathbf{r}\cdot\mathbf{v} \geqq 1$ for all $\mathbf{r} \in \mathbf{R}(\mathbf{y})$ and $\mathbf{v} \in \mathbf{V}(\mathbf{y})$.

The factor price frontier is a solution $r_1 = c(r_2,\ldots,r_n,\mathbf{y})$, of the equation $C(\mathbf{y},\mathbf{r}) = 1$. The frontier c is a convex, non-increasing function of $(r_2,\ldots,r_n)$, and a non-increasing function of $\mathbf{y}$. In the case of a single output, the factor price frontier is usually defined for unit output, $r_1 = c(r_2,\ldots,r_n,1)$. When the technology exhibits constant returns to scale, it is completely determined once the input requirement set for unit output is specified. Then duality implies that an input-conventional constant returns technology is completely characterized by the factor price frontier $r_1 = c(r_2,\ldots,r_n,1)$.

9. Cobb–Douglas and C.E.S. Cost Functions

In econometric applications of production theory, one normally works with parametric families of transformation or distance functions. Cobb–Douglas and C.E.S. (or, Arrow–Chenery–Minhas–Solow) production functions are widely used cases. Cost functions are derived in this section for these two families. Dual functions for other parametric families are derived elsewhere in this volume (Diewert, Chapter III.2; Hanoch, Chapter II.3; Lau, Chapter I.3).

Consider a technology with N inputs, $\mathbf{v} = (v_1,...,v_N)$, producing a single output y. The technology is of the Cobb–Douglas form if it has the distance function

$$F(y,\mathbf{v}) = Dv_1^{\theta_1}v_2^{\theta_2}\cdots v_N^{\theta_N}/\gamma(y), \tag{17}$$

where D is a positive efficiency parameter, the θ_i are positive distribution parameters satisfying $\theta_1 + \theta_2 + \cdots + \theta_N = 1$, and γ is a function from a subset $\mathbf{Y}^*$ of the non-negative real line onto the non-negative real line. In case $\gamma(y)$ has the special form $\gamma(y) = y^{1/\mu}$, production possibilities exhibit returns to scale of degree μ. The cost function obtained by applying (1) to the technology defined by (17) has the functional form

$$C(y,\mathbf{r}) = D^*\gamma(y)r_1^{\theta_1}r_2^{\theta_2}\cdots r_N^{\theta_N}, \tag{18}$$

where $D^* = D^{-1}\theta_1^{-\theta_1}\theta_2^{-\theta_2}\cdots\theta_N^{-\theta_N}$, and is called the *Cobb–Douglas cost function.*

The technology is of the C.E.S. form if it has the distance function

$$F(y,\mathbf{v}) = [(v_1/D_1(y))^{1-1/\sigma} + (v_2/D_2(y))^{1-1/\sigma} + \cdots + (v_N/D_N(y))^{1-1/\sigma}]^{1/(1-1/\sigma)}, \tag{19}$$

where σ is a positive elasticity of substitution parameter, $\sigma \neq 1$, and the $D_i(y)$ are positive (non-decreasing) functions of positive y. The cost function obtained by applying (1) to the technology defined by (19) has the functional form

$$C(y,\mathbf{r}) = [(r_1D_1(y))^{1-\sigma} + (r_2D_2(y))^{1-\sigma} + \cdots + (r_ND_N(y))^{1-\sigma}]^{1/(1-\sigma)}, \tag{20}$$

and is called the C.E.S. cost function.

Two limiting cases of the C.E.S. transformation function are most easily treated separately. In the limit $\sigma \to 0$, one obtains the Leontief transformation function

$$F(y,\mathbf{v}) = \text{Min}\{(v_1/D_1(y)), (v_2/D_2(y)),...,(v_N/D_N(y))\}, \tag{21}$$

which has the corresponding cost function

$$C(y,\mathbf{r}) = r_1D_1(y) + r_2D_2(y) + \cdots + r_ND_N(y). \tag{22}$$

Alternately, in the limit $\sigma \to +\infty$, one obtains the perfect substitute transformation function

$$F(y,\mathbf{v}) = (v_1/D_1(y)) + (v_2/D_2(y)) + \cdots + (v_N/D_N(y)), \tag{23}$$

which has the corresponding cost function

$$C(y,\mathbf{r}) = \text{Min}\{(r_1D_1(y)), (r_2D_2(y)),\ldots,(r_ND_N(y))\}. \tag{24}$$

These formulae can be verified by indirect methods (Lau, Chapter I.3), or by direct computation of the minimizing input bundle. For the C.E.S. case, the steps in the direct computation are the following: (1) obtain as a first-order condition for cost minimization the expression $r_i/r_j = (v_i/v_j)^{-1/\sigma}(D_i(y)/D_j(y))^{1/\sigma-1}$; (2) reverse this expression to obtain the expression $r_iv_i/r_jv_j = (r_iD_i(y)/r_jD_j(y))^{1-\sigma}$; (3) sum this expression over i to obtain $r_jv_j/C(y,\mathbf{r}) = (r_jD_j(y))^{1-\sigma}/[(r_1D_1(y))^{1-\sigma} + \cdots + (r_ND_N(y))^{1-\sigma}]$; (4) solve this expression for v_j, substitute the result into (19) with $F(y,\mathbf{v}) = 1$, and simplify to obtain (20).

10. The Geometry of Two-Input Cost Functions

Dual distance and cost functions have a geometric structure which can be used to establish qualitative relationships between these functions. Consider the case of two inputs $\mathbf{v} = (v_1,v_2)$, and suppose production possibilities are defined by input-conventional input requirement sets $\mathbf{V}(\mathbf{y})$, $\mathbf{y} \in \mathbf{Y}^*$. Let $F(\mathbf{y},\mathbf{v})$ and $C(\mathbf{y},\mathbf{r})$ denote the transformation and cost functions, respectively, for this technology, and let $\mathbf{R}(\mathbf{y}) = \{\mathbf{r} \geqq \mathbf{0} | C(\mathbf{y},\mathbf{r}) \geqq 1\}$ denote the factor price requirement set.

Figure 8 illustrates a typical input requirement set $\mathbf{V}(\mathbf{y})$ and corresponding factor price requirement set $\mathbf{R}(\mathbf{y})$. Hereafter, we shall refer to the boundaries of these sets as the isoquant and the factor price frontier, respectively. Let $\mathbf{v}^0$ denote an input bundle in the isoquant, and let $\mathbf{r}^0$ be a normal to a plane tangent to $\mathbf{V}(\mathbf{y})$ at $\mathbf{v}^0$. Choose the magnitude of $\mathbf{r}^0$ to make $\mathbf{r}^0 \cdot \mathbf{v}^0 = 1$. Using Lemma 5 and the derivative property of the cost function, one can conclude that $\mathbf{r}^0$ is in the factor price frontier, and that $\mathbf{v}^0$ is a normal to a plane tangent to $\mathbf{R}(\mathbf{y})$ at $\mathbf{r}^0$. Furthermore, this geometric relationship is completely dual: starting from $\mathbf{r}^0$ in the factor price frontier, one can proceed in the opposite direction to locate $\mathbf{v}^0$ in

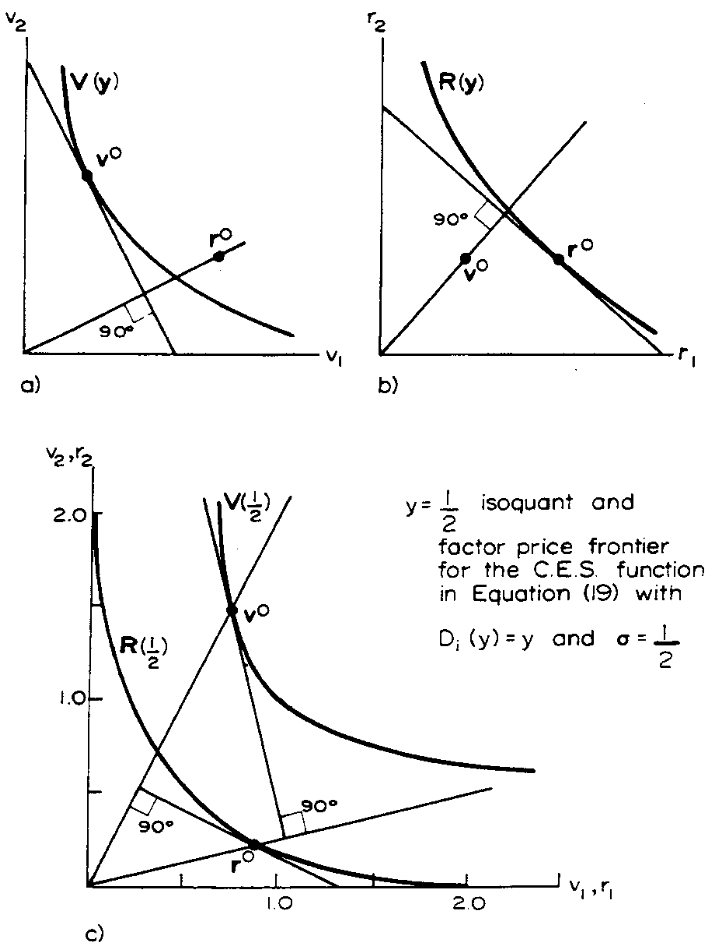


FIGURE 8

the isoquant. In Figure 8, the mapping between points in the isoquant and factor price frontier is one to one, and as $\mathbf{v}^0$ moves from northwest to southeast along the isoquant, its image $\mathbf{r}^0$ moves from southeast to northwest along the factor price frontier.

These movements correspond to a rise in the relative intensity of use of factor 1 and a rise in the marginal rate of substitution of factor 1 per unit of factor 2 (i.e., a rise in the relative price of factor 2). Thus, the value of a factor rises as its relative scarcity rises.

Employing this geometric mapping rule, we can establish a simple inverse relationship between the degree of curvature of the isoquant and the degree of curvature of the factor price frontier, as illustrated in Figure 9. Curves A, B, C, D denote dual isoquants and factor price frontiers. A straight line isoquant (A) maps into a rectangular factor

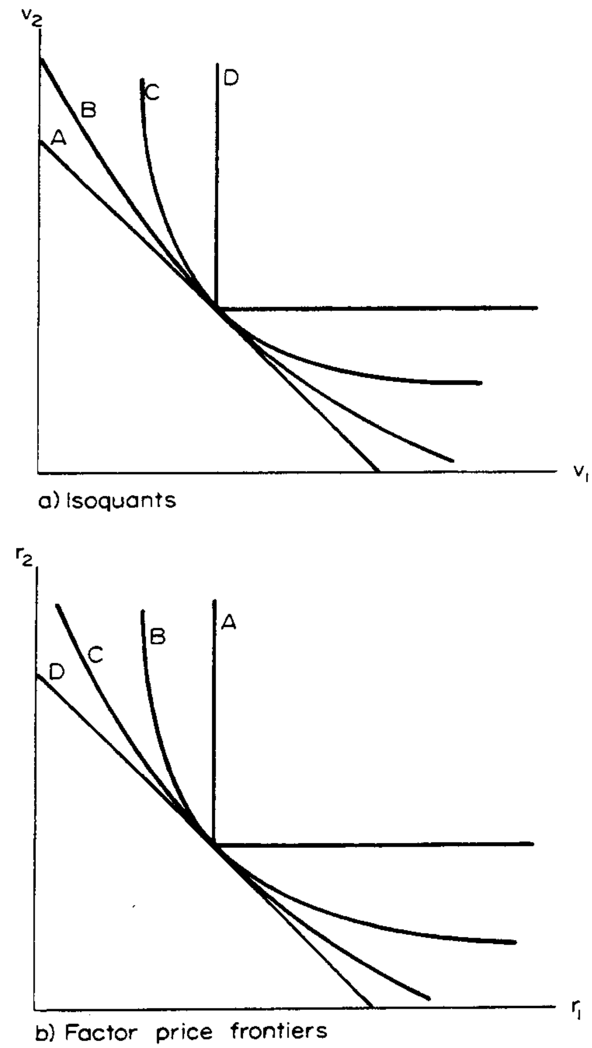


FIGURE 9

price frontier, and a rectangular isoquant (D) maps into a linear factor price frontier. Isoquant (C), with a sharper curvature than isoquant (B), maps into the factor price frontier with less sharp curvature. Using the elasticity of factor substitution as an index of curvature, this inverse curvature relationship can be made quantitative. Assume the distance function to be twice continuously differentiable and strictly differentiably quasi-concave from below. Then, the cost function also has these properties, by Lemma 7, and the dual points $\mathbf{v}^0$, $\mathbf{r}^0$ in Figure 8 satisfy

$$r_1^0/r_2^0 = F_1(\mathbf{y},\mathbf{v}^0)/F_2(\mathbf{y},\mathbf{v}^0), \tag{25}$$

and

$$v_1^0/v_2^0 = C_1(\mathbf{y},\mathbf{r}^0)/C_2(\mathbf{y},\mathbf{r}^0), \tag{26}$$

where $F_i(\mathbf{y},\mathbf{r}^0)$ denotes the partial derivative $\partial F(\mathbf{y},\mathbf{v}^0)/\partial v_i$, and $C_i(\mathbf{y},\mathbf{v}^0)$ denotes $\partial C(\mathbf{y},\mathbf{r}^0)/\partial r_i$. Define the elasticity of input substitution at $(\mathbf{y},\mathbf{v}^0)$,

$$\sigma(\mathbf{y},\mathbf{v}^0) = -\frac{\mathrm{d}\log(v_1^0/v_2^0)}{\mathrm{d}\log(r_1^0/r_2^0)}\bigg|_{\mathbf{y}\text{ fixed and }F(\mathbf{y},\mathbf{v}^0)=1} \tag{27}$$

From (a) in Figure 8, (r_1^0/r_2^0) falls as (v_1^0/v_2^0) rises, at a rate which increases in magnitude as the curvature of the isoquant rises. Then, $\sigma(\mathbf{y},\mathbf{v}^0)$ is positive, is near zero if the isoquant has high curvature and is nearly rectangular, and is near infinity if the isoquant has low curvature and is nearly linear. A formula for the elasticity can be obtained by logarithmic differentiation of (25):

$$\begin{aligned}\mathrm{d}\ln\frac{r_1}{r_2} &= \left[\frac{F_{11}}{F_1}-\frac{F_{21}}{F_2}\right]\mathrm{d}v_1+\left[\frac{F_{12}}{F_1}-\frac{F_{22}}{F_2}\right]\mathrm{d}v_2\\ &= \left[-\frac{v_2F_{12}}{v_1F_1}-\frac{F_{21}}{F_2}\right]\mathrm{d}v_1+\left[\frac{F_{12}}{F_1}+\frac{v_1F_{12}}{v_2F_2}\right]\mathrm{d}v_2\\ &= -\frac{F_{12}}{F_1F_2}\left[\frac{F}{v_1}\mathrm{d}v_1-\frac{F}{v_2}\mathrm{d}v_2\right] = -\frac{FF_{12}}{F_1F_2}\mathrm{d}\ln\frac{v_1}{v_2}.\end{aligned}$$

The second equation uses the homogeneity conditions $v_1F_{11}+v_2F_{12}=0$ and $v_1F_{12}+v_2F_{22}=0$, while the third uses the condition $F=v_1F_1+v_2F_2$. Substituting this formula in (27) yields

$$\sigma(\mathbf{y},\mathbf{v}^0) = \frac{F_1(\mathbf{y},\mathbf{v}^0)F_2(\mathbf{y},\mathbf{v}^0)}{F(\mathbf{y},\mathbf{v}^0)F_{12}(\mathbf{y},\mathbf{v}^0)}.$$

Alternately, logarithmic differentiation of (26) yields

$$\begin{aligned}\mathrm{d}\ln\frac{v_1}{v_2} &= \left[\frac{C_{11}}{C_1}-\frac{C_{21}}{C_2}\right]\mathrm{d}r_1+\left[\frac{C_{12}}{C_1}-\frac{C_{22}}{C_2}\right]\mathrm{d}r_2\\ &= \left[-\frac{r_2C_{12}}{r_1C_1}-\frac{C_{21}}{C_2}\right]\mathrm{d}r_1+\left[\frac{C_{12}}{C_1}+\frac{r_1C_{21}}{r_2C_2}\right]\mathrm{d}r_2\\ &= -\frac{CC_{12}}{C_1C_2}\mathrm{d}\ln\frac{r_1}{r_2},\end{aligned}$$

where the same homogeneity arguments are used as in the preceding derivation. Then,

$$\sigma(\mathbf{y},\mathbf{v}^0) = \frac{C(\mathbf{y},\mathbf{r}^0)C_{12}(\mathbf{y},\mathbf{r}^0)}{C_1(\mathbf{y},\mathbf{r}^0)C_2(\mathbf{y},\mathbf{r}^0)},$$

where $\mathbf{r}^0$ is the vector dual to $\mathbf{v}^0$ [i.e., $\mathbf{r}^0 = F_v(\mathbf{y},\mathbf{v}^0)/F(\mathbf{y},\mathbf{v}^0)$].

Define a similar curvature index for the factor price frontier at $(\mathbf{y},\mathbf{r}^0)$,

$$\rho(\mathbf{y},\mathbf{r}^0) = -\left.\frac{\mathrm{d}\log(r_1^0/r_2^0)}{\mathrm{d}\log(v_1^0/v_2^0)}\right|_{\mathbf{y}\text{ fixed and } C(\mathbf{y},\mathbf{r}^0)=1} \tag{28}$$

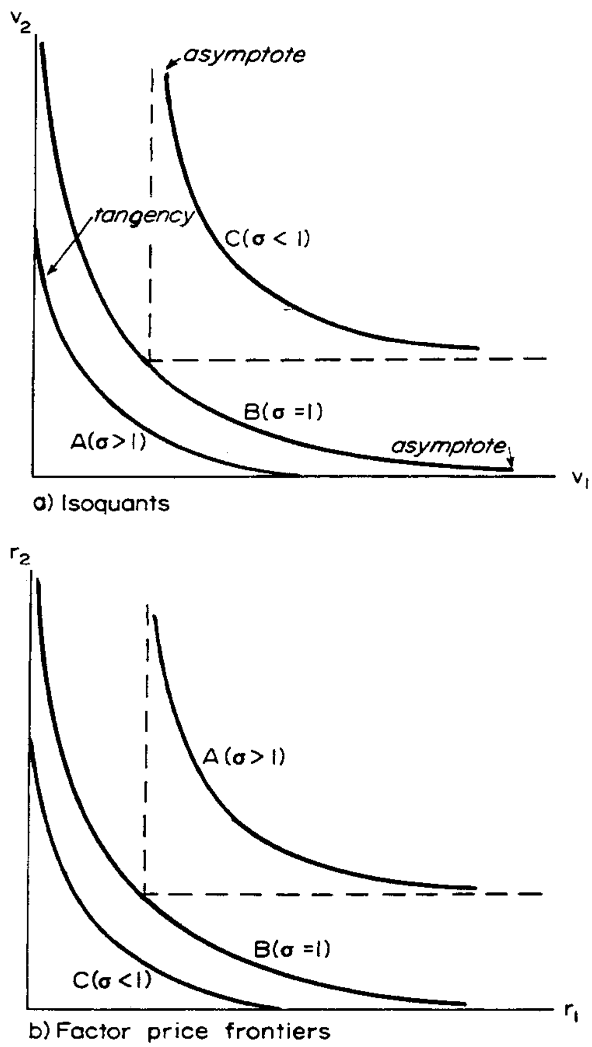


FIGURE 10

Then we obtain in the same manner as above the formula

$$\rho(\mathbf{y},\mathbf{r}^0) = C_1(\mathbf{y},\mathbf{r}^0)C_2(\mathbf{y},\mathbf{r}^0)/C(\mathbf{y},\mathbf{r}^0)C_{12}(\mathbf{y},\mathbf{r}^0).$$

Comparing the formulae for $\sigma(\mathbf{y},\mathbf{v}^0)$ and $\rho(\mathbf{y},\mathbf{r}^0)$, we obtain the condition $\rho(\mathbf{y},\mathbf{r}^0) = 1/\sigma(\mathbf{y},\mathbf{v}^0)$. Thus, an isoquant with an elasticity of substitution equal to one is dual to a factor price frontier with a curvature index $\rho(\mathbf{y},\mathbf{r}^0)$ equal to one, and an isoquant with an elasticity of substitution less (greater) than one has a factor price frontier with a curvature index greater (less) than one. Figure 10 illustrates this relationship for C.E.S. isoquants in (a) with an elasticity greater than one (A) and an elasticity less than one (C), and a Cobb–Douglas isoquant with an elasticity equal to one (B). The corresponding factor price frontiers are given in (b) of Figure 10.

Figure 11 illustrates the mapping of Figure 8 when there is a "kink" in the isoquant at $\mathbf{v}^0$. The image of this point is a line segment in the factor price frontier from $\mathbf{r}^0$ to $\mathbf{r}^1$. Any vector $\mathbf{r}$ in this line segment is a normal to a plane "supporting" the input requirement set at $\mathbf{v}^0$. Then, $\mathbf{r}^0$ and $\mathbf{r}^1$ are normals to the extreme supporting planes, as illustrated. Proceeding in the opposite direction, we note that each $\mathbf{r}$ in the line segment $\mathbf{r}^0$ to $\mathbf{r}^1$ in the factor price frontier has the same normal vector $\mathbf{v}^0$, and hence maps into the "kink" $\mathbf{v}^0$. Since, by duality, we can interchange $\mathbf{r}$ and $\mathbf{v}$ in this figure, we can show that flat segments in the isoquant map into "kinks" in the factor price frontier. Thus, we can conclude generally that "kinks" (or, lack of differentiability) in one function map into "flats" (or, lack of strict quasi-concavity) in the dual function, and vice versa. In the special case of an activity analysis model[7] of the technology, this duality is complete, with each "kink" ("flat") in an isoquant mapping into a "flat" ("kink") in the factor price frontier.

Our discussion of the geometry of two-factor cost functions will be concluded with an examination of the behavior of isoquants and factor price frontiers near the boundaries of the non-negative orthant. Five classes of boundary behavior can be distinguished:

A. The curve is asymptotic to an axis.
B. The curve is asymptotic to a line parallel to an axis.
C. The curve is tangent to an axis.
D. The curve meets an axis, but is not tangent to the axis.

[7]An input requirement set $\mathbf{V}(\mathbf{y})$ comes from an activity analysis model if it can be obtained from a *finite* set of input vectors by forming convex combinations and/or using free disposal of inputs.

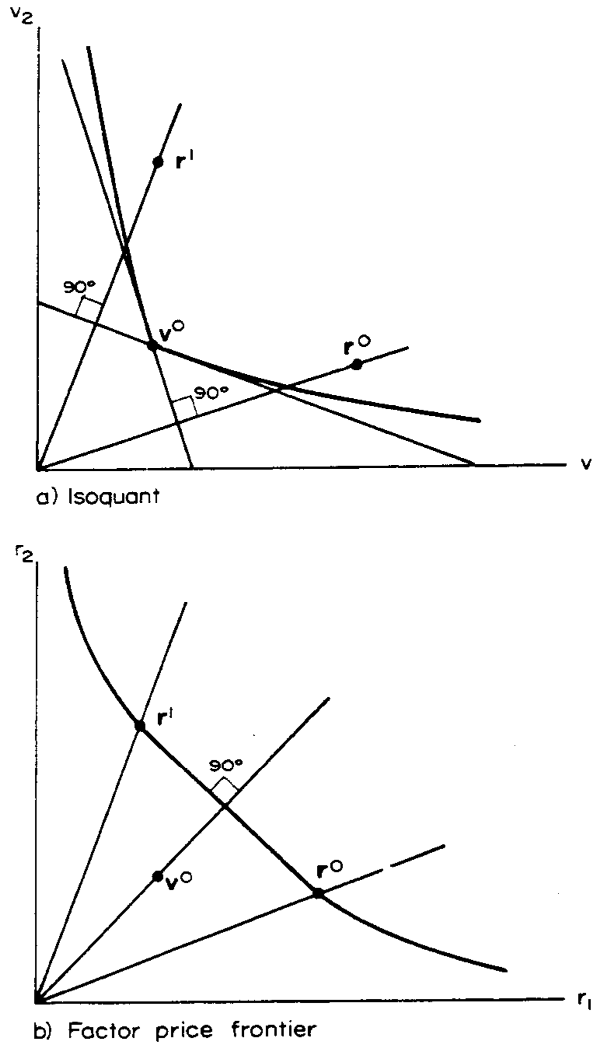


FIGURE 11

E. The curve meets and does not extend beyond a line parallel to an axis.

Figures 12 and 13 illustrate these classes of behavior, and the following geometric duality relationships between then:

1. A curve satisfies A on one axis if and only if the dual curve satisfies A on the other axis.
2. A curve satisfies B on one axis if and only if the dual curve satisfies C on the other axis.
3. A curve satisfies D on one axis if and only if the dual curve satisfies E on the other axis.

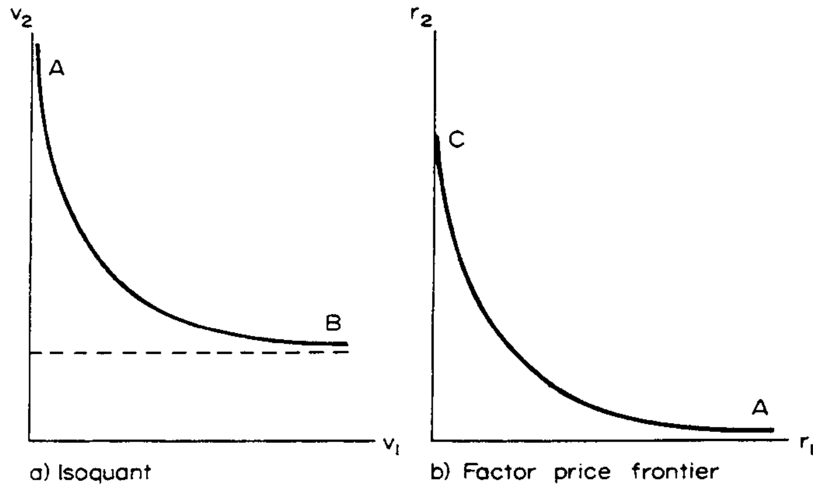


FIGURE 12

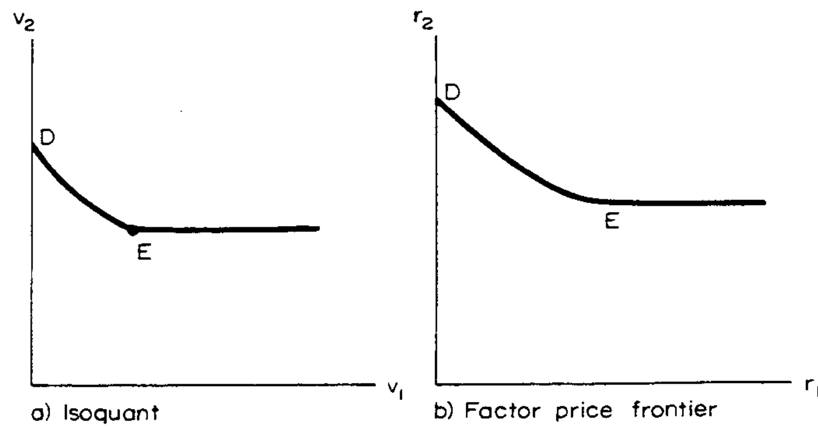


FIGURE 13

11. Comparative Statics for the Cost Minimizing Firm

The basic qualitative questions in the theory of the cost minimizing firm, as formulated by Samuelson (1947, p. 59) are the effects on an input demand of a change in its own price, in the price of another input, or in the output bundle, and the effects on total cost and marginal costs of changes in input prices or the output bundle.

We have noted in Section 5 on the derivative property of the cost function that for an input-regular production possibility set, the cost function has first and second derivatives with respect to input prices for almost all positive input prices. Since these first derivatives equal the cost minimizing demands when they exist, concavity of the cost function implies that an input demand function is non-increasing in its own price, and that the matrix of partial derivatives of inputs with respect to input prices is negative semi-definite and symmetric. It should be emphasized that these results hold with only the weak input-regular conditions imposed on production possibilities. In particular, some inputs may be non-divisible, or "isoquants" may fail to be convex, without altering this conclusion. This observation was first noticed by Samuelson (1953, p. 359), and first deduced formally in an economic application by McKenzie (1957).

For further comparative statics results, we shall for the remainder of this section impose classical assumptions on production possibilities: the technology is input-conventional and can be represented by a distance function $F(\mathbf{y},\mathbf{v})$, which is strongly continuous in $(\mathbf{y},\mathbf{v})$, twice continuously differentiable in $(\mathbf{y},\mathbf{v})$, uniformly decreasing in $\mathbf{y}$, and strictly differentiably quasi-concave from below in $\mathbf{v}$. (We term a technology satisfying these conditions *input-classical.*) Lemma 7 then implies that the cost function $C(\mathbf{y},\mathbf{r})$ is strongly continuous in $(\mathbf{y},\mathbf{r})$, twice continuously differentiable in $\mathbf{r}$, uniformly increasing in $\mathbf{y}$, and strictly differentiably quasi-concave from below in $\mathbf{r}$. A classical calculus argument using the implicit function theorem establishes that $C(\mathbf{y},\mathbf{r})$ is continuously differentiable in $\mathbf{y}$.[8] Under these conditions the input demands $v_i = D^i(\mathbf{y},\mathbf{r}) = C_i(\mathbf{y},\mathbf{r})$ are continuously differentiable in $(\mathbf{y},\mathbf{r})$, with a negative own price effect

$$\partial v_i/\partial r_i = D_i^i(\mathbf{y},\mathbf{r}) = C_{ii}(\mathbf{y},\mathbf{r}) < 0, \tag{29}$$

and symmetric cross-price effects

$$\partial v_i/\partial r_j = C_{ij}(\mathbf{y},\mathbf{r}) = C_{ji}(\mathbf{y},\mathbf{r}) = \partial v_j/\partial r_i. \tag{30}$$

[8]The cost function satisfies $C(\mathbf{y},\mathbf{r}) = \min_{\mathbf{v}} \mathbf{r}\cdot\mathbf{v}$ subject to $F(\mathbf{y},\mathbf{v}) = 1$. For $\mathbf{r}$ such that the minimum is achieved at strictly positive $\mathbf{v}$, the first-order conditions for minimization are $\lambda F_v(\mathbf{y},\mathbf{v}) = \mathbf{r}$ and $F(\mathbf{y},\mathbf{v}) = 1$, where λ is a Lagrangian multiplier. From the assumptions on F, these equations have a total differential which is continuous in $\mathbf{y}$.

$$\begin{bmatrix} \lambda F_{vv}(\mathbf{y},\mathbf{v}) & F_v(\mathbf{y},\mathbf{v}) \\ F_v(\mathbf{y},\mathbf{v}) & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\mathbf{v} \\ \mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} -\lambda F_{vy} \\ -F_y \end{bmatrix} \mathrm{d}\mathbf{y}.$$

The left-hand-side matrix is non-singular by the assumption of strict differential quasi-concavity of F in $\mathbf{v}$. Therefore, $\mathrm{d}v/\mathrm{d}y$ exists and is continuous in $(\mathbf{y},\mathbf{r})$, implying $C(\mathbf{y},\mathbf{r}) = \mathbf{r}\cdot\mathbf{v}$ continuously differentiable in $\mathbf{y}$.

The matrix of price effects $[\partial v_i/\partial r_j] = [C_{ij}(\mathbf{y},\mathbf{r})]$ is symmetric, negative semi-definite, and of rank $N-1$, with

$$r_1\partial v_i/\partial r_1 + r_2\partial v_i/\partial r_2 + \cdots + r_N\partial v_i/\partial r_N = 0. \tag{31}$$

Inputs i and j are termed *substitutes* if $\partial v_i/\partial r_j > 0$, and *complements* if $\partial v_i/\partial r_j < 0$.

The effect on input i of an increase in output k is given by

$$\partial v_i/\partial y_k = C_{iy_k}(\mathbf{y},\mathbf{r}) = C_{y_k i}(\mathbf{y},\mathbf{r}) = \partial m_k/\partial r_i, \tag{32}$$

where $m_k = M^k(\mathbf{y},\mathbf{r}) = C_{y_k}(\mathbf{y},\mathbf{r})$ is the marginal cost of producing output k. Input i is termed *normal* for output k at $(\mathbf{y},\mathbf{r})$ if $\partial v_i/\partial y_k$ is positive, and is termed *regressive* for output k otherwise. Equation (32) shows that the marginal cost of output k rises when the price of a normal input rises, but falls when the price of a regressive input rises.

Since the cost function is uniformly increasing in $\mathbf{y}$, the marginal cost of output k, $M^k(\mathbf{y},\mathbf{r})$, is non-negative, and is positive for almost all y_k, given any values for the remaining arguments. The effect on total cost of an increase in input price i is non-negative, and is positive when the demand for input i is positive, since $C_i(\mathbf{y},\mathbf{r}) = v_i$.

Next, we examine the effects of output changes on marginal costs, $\partial m_k/\partial y_l = C_{y_k y_l}(\mathbf{y},\mathbf{r})$. Outputs k and l are termed *substitutes* if $\partial m_k/\partial y_l > 0$ and *complements* if $\partial m_k/\partial y_l < 0$. A production possibility set $\mathbf{Y}$ is said to exhibit *generally non-increasing returns* if $\mathbf{Y}$ is a convex set. We say that $\mathbf{Y}$ exhibits *eventually diminishing returns to scale* if $(\lambda\mathbf{y},\lambda\mathbf{v}) \in \mathbf{Y}$ for all $\lambda > 0$ implies $\mathbf{y} = 0$.

A cost function $C(\mathbf{y},\mathbf{r})$ is said to exhibit *generally non-decreasing costs* if C is a convex function of $\mathbf{y}$ for each positive $\mathbf{r}$. We say $C(\mathbf{y},\mathbf{r})$ exhibits *eventually increasing costs* if $\lim_{\lambda\to\infty} C(\lambda\mathbf{y},\mathbf{r})/\lambda = +\infty$ for all positive $\mathbf{r}$ and all $\mathbf{y} \neq \mathbf{0}$. [A competitive profit maximum exists for all strictly positive output prices if and only if $C(\mathbf{y},\mathbf{r})$ exhibits eventually increasing costs.]

Lemma 8. Assume the production possibility set $\mathbf{Y}$ to be input-conventional. Then the following implications hold:

(i) $\mathbf{Y}$ exhibits generally non-increasing returns if and only if $C(\mathbf{y},\mathbf{r})$ exhibits generally non-decreasing costs.

(ii) $\mathbf{Y}$ exhibits eventually diminishing returns to scale if and only if $C(\mathbf{y},\mathbf{r})$ exhibits eventually increasing costs.

Proof: (i) If $\mathbf{Y}$ is convex, and costs are minimized for $(\mathbf{y}^i,\mathbf{r}^0)$ at a bundle $\mathbf{v}^i$ with $(\mathbf{y}^i,\mathbf{v}^i)\in\mathbf{Y}$, then for $0<\theta<1$ and $(\mathbf{y}^0,\mathbf{v}^0)=\theta(\mathbf{y}^1,\mathbf{v}^1)+(1-\theta)(\mathbf{y}^2,\mathbf{v}^2)\in\mathbf{Y}$, we have

$$C(\mathbf{y}^0,\mathbf{r}^0)\leqq \mathbf{r}^0\cdot\mathbf{v}^0=\theta\mathbf{r}^0\cdot\mathbf{v}^1+(1-\theta)\mathbf{r}^0\cdot\mathbf{v}^2=\theta C(\mathbf{y}^1,\mathbf{r}^0)+(1-\theta)C(\mathbf{y}^2,\mathbf{r}^0).$$

Hence C is a convex function of $\mathbf{y}$.

Alternately, suppose $C(\mathbf{y},\mathbf{r})$ convex in $\mathbf{y}$ for fixed $\mathbf{r}$. Given $(\mathbf{y}^i,\mathbf{v}^i)\in\mathbf{Y}$ and $(\mathbf{y}^0,\mathbf{v}^0)=\theta(\mathbf{y}^1,\mathbf{v}^1)+(1-\theta)(\mathbf{y}^2,\mathbf{v}^2)$ for $0<\theta<1$, we have for any positive $\mathbf{r}$,

$$C(\mathbf{y}^0,\mathbf{r})\leqq\theta C(\mathbf{y}^1,\mathbf{r})+(1-\theta)C(\mathbf{y}^2,\mathbf{r})\leqq\theta\,\mathbf{r}\cdot\mathbf{v}^1+(1-\theta)\mathbf{r}\cdot\mathbf{v}^2=\mathbf{r}\cdot\mathbf{v}^0,$$

implying by Lemma 3 that $(\mathbf{y}^0,\mathbf{v}^0)\in\mathbf{Y}$. Hence $\mathbf{Y}$ is convex.

(ii) Suppose that for some $\mathbf{y}\neq\mathbf{0}$ and positive $\mathbf{r}$, $C(\lambda\mathbf{y},\mathbf{r})/\lambda$ fails to converge to $+\infty$ as $\lambda\to+\infty$. Then there exists a sequence $\lambda_i\to\infty$ such that $\{C(\lambda_i\mathbf{y},\mathbf{r})/\lambda_i\}$ is bounded. Let $\mathbf{v}^i$ be such that $C(\lambda_i\mathbf{y},\mathbf{r})=\mathbf{r}\cdot\mathbf{v}^i$. Then $\{\mathbf{v}^i/\lambda_i\}$ is bounded, and we can choose $\mathbf{v}'$ such that $(\mathbf{v}_i/\lambda_i)\leqq\mathbf{v}'$ for all i. Then, $(\lambda_i\mathbf{y},\mathbf{v}^i)\in\mathbf{Y}$ implies $(\lambda_i\mathbf{y},\lambda_i\mathbf{v}')\in\mathbf{Y}$, and the production possibility set fails to exhibit eventually diminishing returns to scale.

Alternately, suppose there exists $(\mathbf{y},\mathbf{v})\in\mathbf{Y}$, $\mathbf{y}\neq\mathbf{0}$, and $\lambda_i\to+\infty$ such that $(\lambda_i\mathbf{y},\lambda_i\mathbf{v})\in\mathbf{Y}$. Then $C(\lambda_i\mathbf{y},\mathbf{r})/\lambda_i\leqq\mathbf{r}\cdot(\lambda_i\mathbf{v})/\lambda_i=\mathbf{r}\cdot\mathbf{v}$, and C fails to exhibit eventually increasing cost. Q.E.D.

12. Composition of Distance and Cost Functions

For some simple parametric families of distance and cost functions, such as the Cobb–Douglas and C.E.S. cases analyzed in Section 9, it is possible to perform the cost and technology mappings constructively. However, many applications require more complex parametric specifications. One method of forming such functions is to build them up from simple functions for which the duality mappings are known. The primary result of this section gives a series of rules for the composition of these functions and the implications for their duals.

Theorem 9. Consider a producible output set $\mathbf{Y}^*$, and input-conventional input requirement sets $\mathbf{V}^j(\mathbf{y})\subseteq\mathbf{E}_+^N$, defined for $\mathbf{y}\in\mathbf{Y}^*$ and $j=1,\ldots,J$. Also, let $\mathbf{V}^*(\mathbf{y})\subseteq\mathbf{E}_+^J$ be an input-conventional input requirement set for $\mathbf{y}\in\mathbf{Y}^*$. Let $F^j(\mathbf{y},\mathbf{v})$ and $F^*(\mathbf{y},\mathbf{z})$ be the distance functions, and $C^j(\mathbf{y},\mathbf{r})$ and $C^*(\mathbf{y},\mathbf{q})$ be the cost functions, for $\mathbf{V}^j(\mathbf{y})$

TABLE 2

Composition rules for distance functions (property P), cost functions (property Q), input requirement sets (property S), and factor price requirement sets (property T).

1. *Neutral Scaling*[a]

For an arbitrary positive real-valued function $\alpha(\mathbf{y})$ defined in $\mathbf{Y}^*$,

P: $F^0(\mathbf{y},\mathbf{v}) = F^1(\mathbf{y},\mathbf{v})/\alpha(\mathbf{y}) = F^1(\mathbf{y},\mathbf{v}/\alpha(\mathbf{y}))$

Q: $C^0(\mathbf{y},\mathbf{r}) = \alpha(\mathbf{y})C^1(\mathbf{y},\mathbf{r}) = C^1(\mathbf{y},\alpha(\mathbf{y})\mathbf{r})$

S: $\mathbf{V}^0(\mathbf{y}) = \alpha(\mathbf{y})\mathbf{V}^1(\mathbf{y}) = \{\alpha(\mathbf{y})\mathbf{v}|\mathbf{v} \in \mathbf{V}^1(\mathbf{y})\}$

T: $\mathbf{R}^0(\mathbf{y}) = \mathbf{R}^1(\mathbf{y})/\alpha(\mathbf{y}) = \{\mathbf{r}/\alpha(\mathbf{y})|\mathbf{r} \in \mathbf{R}^1(\mathbf{y})\}$

2. *Non-neutral Scaling*[b]

For an arbitrary diagonal N-dimensional matrix $\mathbf{A}(\mathbf{y})$, where the diagonal elements of $\mathbf{A}(\mathbf{y})$ are positive real-valued functions defined in $\mathbf{Y}^*$,

P: $F^0(\mathbf{y},\mathbf{v}) = F^1(\mathbf{y},\mathbf{A}(\mathbf{y})^{-1}\mathbf{v})$

Q: $C^0(\mathbf{y},\mathbf{r}) = C^1(\mathbf{y},\mathbf{A}(\mathbf{y})\mathbf{r})$

S: $\mathbf{V}^0(\mathbf{y}) = \{\mathbf{A}(\mathbf{y})\mathbf{v}|\mathbf{v} \in \mathbf{V}^1(\mathbf{y})\}$

T: $\mathbf{R}^0(\mathbf{y}) = \{\mathbf{A}(\mathbf{y})^{-1}\mathbf{r}|\mathbf{r} \in \mathbf{R}^1(\mathbf{y})\}$

3. *Union of Input Requirement Sets*

P: $F^0(\mathbf{y},\mathbf{v}) = \text{Sup}\left\{\sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{v}^j)|\mathbf{v}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}\right\}$

Q: $C^0(\mathbf{y},\mathbf{r}) = \text{Min}_{j=1,\dots,J} C^j(\mathbf{y},\mathbf{r})$

S: $\mathbf{V}^0(\mathbf{y}) = \text{Convex hull of } \bigcup_{j=1}^{J} \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \bigcap_{j=1}^{J} \mathbf{R}^j(\mathbf{y})$

4. *Intersection of Input Requirement Sets*

P: $F^0(\mathbf{y},\mathbf{v}) = \text{Min}_{j=1,\dots,J} F^j(\mathbf{y},\mathbf{v})$

Q: $C^0(\mathbf{y},\mathbf{r}) = \text{Sup}\left\{\sum_{j=1}^{J} C^j(\mathbf{y},\mathbf{r}^j)|\mathbf{r}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{r}^j = \mathbf{r}\right\}$

S: $\mathbf{V}^0(\mathbf{y}) = \bigcap_{j=1}^{J} \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \text{Convex hull of } \bigcup_{j=1}^{J} \mathbf{R}^j(\mathbf{y})$

[a]The function $\alpha(\mathbf{y})$ may depend upon exogenous factors such as technical change, and may be independent of $\mathbf{y}$. It is convenient to include the value $\alpha(\mathbf{y}) = 0$ in this rule by defining $\mathbf{V}^0(\mathbf{y}) = \mathbf{E}_+^N$, $F^0(\mathbf{y},\mathbf{v}) = +\infty$, $C^0(\mathbf{y},\mathbf{r}) = 0$, and $\mathbf{R}^0(\mathbf{y}) = \emptyset$. Then Rule 1 holds in the limit as $\alpha(\mathbf{y}) \to 0$. Note that for $\mathbf{V}^0(\mathbf{y})$ to be input-conventional in this case, one must have $\mathbf{y} = \mathbf{0}$.

[b]The matrix $\mathbf{A}(\mathbf{y})$ may depend upon exogenous variables such as technical change.

TABLE 2 (*continued*)

5. *Summation of Input Requirement Sets*[c]

P: $F^0(\mathbf{y},\mathbf{v}) = \text{Sup}\left\{\text{Min}_{j=1,\dots,J} F^j(\mathbf{y},\mathbf{v}^j) | \mathbf{v}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}\right\}$

Q: $C^0(\mathbf{y},\mathbf{r}) = \sum_{j=1}^{J} C^j(\mathbf{y},\mathbf{r})$

S: $\mathbf{V}^0(\mathbf{y}) = \sum_{j=1}^{J} \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \bigcup_{\substack{z_j \geq 0 \\ \sum_{j=1}^{J} z_j = 1}} \bigcap_{j=1}^{J} z_j \mathbf{R}^j(\mathbf{y})$

6. *Convolution of Input Requirement Sets*[d]

P: $F^0(\mathbf{y},\mathbf{v}) = \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{v})$

Q: $C^0(\mathbf{y},\mathbf{r}) = \text{Sup}\left\{\text{Min}_{j=1,\dots,J} C^j(\mathbf{y},\mathbf{r}^j) | \mathbf{r}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{r}^j = \mathbf{r}\right\}$

S: $\mathbf{V}^0(\mathbf{y}) = \bigcup_{\substack{z_j \geq 0 \\ \sum_{j=1}^{J} z_j = 1}} \bigcap_{j=1}^{J} z_j \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \sum_{j=1}^{J} \mathbf{R}^j(\mathbf{y})$

7. *General Concave Composition of Distance Functions*[e]

P: $F^0(\mathbf{y},\mathbf{v}) = F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}),\dots,F^J(\mathbf{y},\mathbf{v}))$

Q: $C^0(\mathbf{y},\mathbf{r}) = \text{Sup}\left\{C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}^1),\dots,C^J(\mathbf{y},\mathbf{r}^J)) | \mathbf{r}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{r}^j = \mathbf{r}\right\}$

S: $\mathbf{V}^0(\mathbf{y}) = \text{Closure} \bigcup_{\mathbf{z} \in \mathbf{V}^*(\mathbf{y})} \bigcap_{j=1}^{J} z_j \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \bigcup_{\mathbf{q} \in \mathbf{R}^*(\mathbf{y})} \sum_{j=1}^{J} q_j \mathbf{R}^j(\mathbf{y})$

8. *General Concave Composition of Cost Functions*[e]

P: $F^0(\mathbf{y},\mathbf{v}) = \text{Sup}\left\{F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}^1),\dots,F^J(\mathbf{y},\mathbf{v}^J)) | \mathbf{v}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}\right\}$

Q: $C^0(\mathbf{y},\mathbf{r}) = C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}),\dots,C^J(\mathbf{y},\mathbf{r}))$

S: $\mathbf{V}^0(\mathbf{y}) = \bigcup_{\mathbf{z} \in \mathbf{V}^*(\mathbf{y})} \sum_{j=1}^{J} z_j \mathbf{V}^j(\mathbf{y})$

T: $\mathbf{R}^0(\mathbf{y}) = \text{Closure} \bigcup_{\mathbf{q} \in \mathbf{R}^*(\mathbf{y})} \bigcap_{j=1}^{J} q_j \mathbf{R}^j(\mathbf{y})$

[c]By convention, for $z_j = 0$ we define $z_j \mathbf{R}^j(\mathbf{y}) = \mathbf{E}_+^N$, even if $\mathbf{R}^j(\mathbf{y})$ is empty.
[d]By convention, for $z_j = 0$ we define $z_j \mathbf{V}^j(\mathbf{y}) = \mathbf{E}_+^N$.
[e]Any of the functions F^* or F^j may, as a special case, be independent of $\mathbf{y}$.

and $\mathbf{V}^*(\mathbf{y})$, respectively. Let $\mathbf{R}^j(\mathbf{y})$ and $\mathbf{R}^*(\mathbf{y})$ be the factor price requirement sets for $\mathbf{V}^j(\mathbf{y})$ and $\mathbf{V}^*(\mathbf{y})$, respectively. Then, the composition rules in Table 2 hold, defining dual input-conventional input requirement sets $\mathbf{V}^0(\mathbf{y})$, factor price requirement sets $\mathbf{R}^0(\mathbf{y})$, distance functions $F^0(\mathbf{y},\mathbf{v})$, and cost functions $C^0(\mathbf{y},\mathbf{r})$.

Proof: *Rules* 1 *and* 2 – Given the positive diagonal matrix $\mathbf{A}(\mathbf{y})$, the set $\mathbf{V}^0(\mathbf{y}) = \{\mathbf{A}(\mathbf{y})\mathbf{v} | \mathbf{v} \in \mathbf{V}^1(\mathbf{y})\}$ is obviously input-conventional. From equation (8),

$$F^0(\mathbf{y},\mathbf{v}) = \mathrm{Max}\left\{\lambda > 0 \,\middle|\, \frac{1}{\lambda}\mathbf{v} \in \mathbf{V}^0(\mathbf{y})\right\} = \mathrm{Max}\left\{\lambda > 0 \,\middle|\, \frac{1}{\lambda}\mathbf{A}(\mathbf{y})^{-1}\mathbf{v} \in \mathbf{V}^1(\mathbf{y})\right\}$$
$$= F^1(\mathbf{y},\mathbf{A}(\mathbf{y})^{-1}\mathbf{v}).$$

From equation (1),

$$C^0(\mathbf{y},\mathbf{r}) = \mathrm{Min}\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \mathbf{V}^0(\mathbf{y})\} = \mathrm{Min}\{\mathbf{r}\mathbf{A}(\mathbf{y})\mathbf{v} | \mathbf{v} \in \mathbf{V}^1(\mathbf{y})\} = C^1(\mathbf{y},\mathbf{r}\mathbf{A}(\mathbf{y})).$$

Finally,

$$\mathbf{R}^0(\mathbf{y}) = \{\mathbf{r} | C^0(\mathbf{y},\mathbf{r}) \geqq 1\} = \{\mathbf{r} | C^1(\mathbf{y},\mathbf{r}\mathbf{A}(\mathbf{y})) \geqq 1\}$$
$$= \{\mathbf{r}\mathbf{A}(\mathbf{y})^{-1} | C^1(\mathbf{y},\mathbf{r}) \geqq 1\} = \{\mathbf{r}\mathbf{A}(\mathbf{y})^{-1} | \mathbf{r} \in \mathbf{R}^1(\mathbf{y})\}.$$

Duality then implies that each of the composition rules P, Q, S, and T holds for Rule 2. Taking all the diagonal elements of $A(\mathbf{y})$ to be the scalar function $\alpha(\mathbf{y})$ implies Rule 1.

Rules 3 *and* 4 – Consider Rule 3, and suppose that S holds, defining $\mathbf{V}^0(\mathbf{y})$ as the convex hull of the union of the $\mathbf{V}^j(\mathbf{y})$. The minimum of a linear function on a convex hull of a closed set can always be attained at some point in the original set. Hence,

$$C^0(\mathbf{y},\mathbf{r}) = \mathrm{Min}\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \mathbf{V}^0(\mathbf{y})\} = \mathrm{Min}\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \mathbf{V}^j(\mathbf{y}), \text{ some } j\}$$
$$= \mathrm{Min}_{j=1,\ldots,J} C^j(\mathbf{y},\mathbf{r}).$$

Using duality, this establishes the equivalence of Q and S.

For a positive $\mathbf{v}$, one has $\mathbf{v}/F^0(\mathbf{y},\mathbf{v}) \in \mathbf{V}^0(\mathbf{y})$, implying the existence of scalars $z_j \geqq 0$, $\sum_{j=1}^J z_j = 1$ and points $\mathbf{v}^j/F^0(\mathbf{y},\mathbf{v}) \in \mathbf{V}^j(\mathbf{y})$ such that $\sum_{j=1}^J z_j\mathbf{v}^j = \mathbf{v}$. But this implies $F^j(\mathbf{y},\mathbf{v}^j) \geqq F^0(\mathbf{y},\mathbf{v})$, and hence, using the linear homogeneity of F^j in $\mathbf{v}$, $\sum_{j=1}^J F^j(\mathbf{y},z_j\mathbf{v}^j) \geqq F^0(\mathbf{y},\mathbf{v})$.

Alternately, consider the relation $F^0(\mathbf{y},\mathbf{v}) = \mathrm{Max}\{\lambda | \mathbf{r}\cdot\mathbf{v} \geqq \lambda C^0(\mathbf{y},\mathbf{r})$ for all positive $\mathbf{r}\}$. Take any positive $\mathbf{w}^j$ with $\sum_{j=1}^J \mathbf{w}^j = \mathbf{v}$. By Lemma 5, $C^j(\mathbf{y},\mathbf{r})F^j(\mathbf{y},\mathbf{w}^j) \leqq \mathbf{r}\cdot\mathbf{w}^j$. For $\lambda = \sum_{j=1}^J F^j(\mathbf{y},\mathbf{w}^j)$, one has

$$\lambda C^0(\mathbf{y},\mathbf{r}) = \lambda \ \mathrm{Min}_{j=1,\dots,J} C^j(\mathbf{y},\mathbf{r}) = \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{w}^j)\ \mathrm{Min}_{i=1,\dots,J} C^i(\mathbf{y},\mathbf{r})$$

$$\leqq \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{w}^j) C^j(\mathbf{y},\mathbf{r}) \leqq \sum_{j=1}^{J} \mathbf{r}\cdot\mathbf{w}^j = \mathbf{r}\cdot\mathbf{v}.$$

Hence,

$$F^0(\mathbf{y},\mathbf{v}) \geqq \lambda = \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{w}^j),$$

for all $\mathbf{w}^j$ with $\sum_{j=1}^{J} \mathbf{w}^j = \mathbf{v}$. With the inequality in the preceding paragraph, this establishes P. Then P and S are equivalent by duality.

Given $F^0(\mathbf{y},\mathbf{v})$ from P, note that

$$\mathbf{R}^0(\mathbf{y}) = \{\mathbf{r} | \mathbf{r}\cdot\mathbf{v} \geqq F^0(\mathbf{y},\mathbf{v}) \text{ for all positive } \mathbf{v}\}$$

$$= \left\{\mathbf{r} | \mathbf{r}\cdot\mathbf{v} \geqq \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{v}^j) \text{ for all positive } \mathbf{v}^j, \sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}\right\}$$

$$= \left\{\mathbf{r} \left| \sum_{j=1}^{J} \mathbf{r}\cdot\mathbf{v}^j \geqq \sum_{j=1}^{J} F^j(\mathbf{y},\mathbf{v}^j) \text{ for all positive } \mathbf{v}^j\right.\right\}$$

$$= \{\mathbf{r} | \mathbf{r}\cdot\mathbf{v}^j \geqq F^j(\mathbf{y},\mathbf{v}^j), \text{ all } j \text{ and all positive } \mathbf{v}^j\}$$

$$= \bigcap_{j=1}^{J} \mathbf{R}^j(\mathbf{y}).$$

Hence, P and T are equivalent. This establishes Rule 3.

Rule 4 can be deduced from Rule 3 using the formal duality of C and F.

Rules 5 and 6 – Consider Rule 5: Given $\mathbf{V}^0(\mathbf{y}) = \sum_{j=1}^{J} \mathbf{V}^j(\mathbf{y})$, we see that $\mathbf{V}^0(\mathbf{y})$ is input-conventional, and that equation (1) implies $C^0(\mathbf{y},\mathbf{r}) = \sum_{j=1}^{J} C^j(\mathbf{y},\mathbf{r})$. Then Q and S are equivalent by duality. Next consider

$$F^0(\mathbf{y},\mathbf{v}) = \mathrm{Max}\left\{\lambda \left| \frac{1}{\lambda}\mathbf{v} \in \mathbf{V}^0(\mathbf{y})\right.\right\} = \mathrm{Max}\left\{\lambda \left| \frac{1}{\lambda}\mathbf{v} \in \sum_{j=1}^{J} \mathbf{V}^j(\mathbf{y})\right.\right\}$$

$$= \mathrm{Max}\left\{\lambda \left| \frac{1}{\lambda}\mathbf{v}^j \in \mathbf{V}^j(\mathbf{y}) \text{ for some } \mathbf{v}^j \text{ with } \sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}\right.\right\}.$$

Given a small positive scalar α, there exist positive $\mathbf{v}^j$ with $\sum_{j=1}^{J} \mathbf{v}^j = \mathbf{v}$ such that $(\mathbf{v}^j/(F^0(\mathbf{y},\mathbf{v}) - \alpha)) \in \mathbf{V}^j(\mathbf{y})$, and hence $F^j(\mathbf{y},\mathbf{v}^j) \geqq F^0(\mathbf{y},\mathbf{v}) - \alpha$. Conversely, for any positive $\mathbf{w}^j$ with $\sum_{j=1}^{J} \mathbf{w}^j = \mathbf{v}$ and $\lambda = \mathrm{Min}_{j=1,\dots,J} F^j(\mathbf{y},\mathbf{w}^j)$ one has $\mathbf{w}^j/\lambda \in \mathbf{V}^j(\mathbf{y})$, implying $F^0(\mathbf{y},\mathbf{v}) \geqq \lambda =$

$\text{Min}_{j=1,\ldots,J}F^j(\mathbf{y},\mathbf{w}^j)$. With the previously established inequality, this implies $F^0(\mathbf{y},\mathbf{v}) = \text{Sup}\{\text{Min}_{j=1,\ldots,J}F^j(\mathbf{y},\mathbf{v}^j)|\mathbf{v}^j \text{ positive}, \sum_{j=1}^J \mathbf{v}^j = \mathbf{v}\}$. By duality, P and S are then equivalent.

The factor price requirement set satisfies $\mathbf{R}^0(\mathbf{y}) = \text{Closure } \{\mathbf{r}|\mathbf{r} \text{ positive}, \sum_{j=1}^J C^j(\mathbf{y},\mathbf{r}) \geqq 1\}$. Then positive $\mathbf{r}$ is contained in $\mathbf{R}^0(\mathbf{y})$ if and only if there exist non-negative scalars z_j such that $\sum_{j=1}^J z_j = 1$ and $C^j(\mathbf{y},\mathbf{r}) \geqq z_j$, or $\mathbf{r} \in \bigcap_{j=1}^J(z_j\mathbf{R}^j(\mathbf{y}))$. Hence $\mathbf{R}^0(\mathbf{y}) = \text{Closure } \mathbf{A}$, where

$$\mathbf{A} = \bigcup_{\substack{z_j \geqq 0 \\ \sum_{j=1}^J z_j = 1}} \bigcap_{j=1}^J (z_j\mathbf{R}^j(\mathbf{y})).$$

Now suppose $\mathbf{r}^i \in \mathbf{A}$ and $\mathbf{r}^i \to \mathbf{r}^0$. Then there exist $z_{ji} \geqq 0$ such that $\sum_{j=1}^J z_{ji} = 1$ and $\mathbf{r}^i \in z_{ji}\mathbf{R}^j(\mathbf{y})$ for each j. Choose a subsequence of $(z_{1i},\ldots,z_{Ji})$ converging to $(z_{10},\ldots,z_{J0})$. Retain the index notation i for the subsequence. If $z_{j0} > 0$, then $\mathbf{r}^i/z_{ji} \to \mathbf{r}^0/z_{j0} \in \mathbf{R}^j(\mathbf{y})$, since $\mathbf{R}^j(\mathbf{y})$ is closed. If $z_{j0} = 0$, then $\mathbf{r}^0 \in z_{j0}\mathbf{R}^j(\mathbf{y}) = \mathbf{R}_+^N$. Hence, $\mathbf{r}^0 \in \bigcap_{j=1}^J (z_{j0}\mathbf{R}^j(\mathbf{y})) \subseteq \mathbf{A}$. Therefore, $\mathbf{A}$ is a closed set, and $\mathbf{R}^0(\mathbf{y}) = \mathbf{A}$. Duality then implies the equivalence of Q and T. This establishes Rule 5.

Rule 6 follows from Rule 5 by the formal duality of the distance and cost functions and of the input and factor price requirement sets.

Rules 7 and 8 – Consider Rule 7, and $F^0(\mathbf{y},\mathbf{v}) = F^*(\mathbf{y},\mathbf{v}),\ldots, F^J(\mathbf{y},\mathbf{v}))$. Since $F^*(\mathbf{y},\mathbf{z})$ is non-decreasing, linear homogeneous, and concave in positive $\mathbf{z}$, and the $F^j(\mathbf{y},\mathbf{v})$ have the same properties in positive $\mathbf{v}$, it is immediate that $F^0(\mathbf{y},\mathbf{v})$ is non-decreasing and linear homogeneous in positive $\mathbf{v}$. Consider positive $\mathbf{v},\mathbf{v}'$ and $0 < \theta < 1$. Then $F^i(\mathbf{y},\theta\mathbf{v} + (1-\theta)\mathbf{v}') \geqq \theta F^j(\mathbf{y},\mathbf{v}) + (1-\theta)F^j(\mathbf{y},\mathbf{v}')$, implying

$$\begin{aligned}
&F^0(\mathbf{y},\theta\mathbf{v} + (1-\theta)\mathbf{v}') \\
&= F^*(\mathbf{y},F^1(\mathbf{y},\theta\mathbf{v} + (1-\theta)\mathbf{v}'),\ldots,F^J(\mathbf{y},\theta\mathbf{v} + (1-\theta)\mathbf{v}')) \\
&\geqq F^*(\mathbf{y},\theta F^1(\mathbf{y},\mathbf{v}) + (1-\theta)F^1(\mathbf{y},\mathbf{v}'),\ldots,\theta F^J(\mathbf{y},\mathbf{v}) + (1-\theta)F^J(\mathbf{y},\mathbf{v}')) \\
&\geqq \theta F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}),\ldots,F^J(\mathbf{y},\mathbf{v})) + (1-\theta)F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}'),\ldots,F^J(\mathbf{y},\mathbf{v}')),
\end{aligned}$$

with the second inequality following from the concavity property of F^*. The value $F^0(\mathbf{y},\mathbf{v}) = +\infty$ can occur only if $F^*(\mathbf{y},\mathbf{z}) = +\infty$ for some positive finite $\mathbf{z}$, or $F^j(\mathbf{y},\mathbf{v}) = +\infty$ for some j. Since F^j and F^* are input-conventional, either case implies $\mathbf{y} = \mathbf{0}$. This establishes that $F^0(\mathbf{y},\mathbf{v})$ is input-conventional.

From equation (9),

$$\begin{aligned}\mathbf{V}^0(\mathbf{y}) &= \text{Closure } \{\mathbf{v} \gg \mathbf{0} | F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}),\ldots,F^J(\mathbf{y},\mathbf{v})) \geqq 1\}\\ &= \text{Closure } \{\mathbf{v} \gg \mathbf{0} | \text{ there exists } \mathbf{z} \gg \mathbf{0} \text{ such that } F^j(\mathbf{y},\mathbf{v})\\ &\quad \geqq z_j \text{ and } F^*(\mathbf{y},\mathbf{z}) \geqq 1\}\\ &= \text{Closure } \bigcup_{\substack{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})\\ \mathbf{z}\gg\mathbf{0}}} \bigcap_{j=1}^{J} \{\mathbf{v} \gg \mathbf{0} | F^j(\mathbf{y},\mathbf{v}) \geqq z_j\}.\end{aligned}$$

Define

$$\tilde{\mathbf{V}}^0(\mathbf{y}) = \bigcup_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} \bigcap_{j=1}^{J} (z_j\mathbf{V}^j(\mathbf{y})).$$

Clearly $\mathbf{V}^0(\mathbf{y}) \subseteq$ Closure $\tilde{\mathbf{V}}^0(\mathbf{y})$. If $\mathbf{v} \in \tilde{\mathbf{V}}^0(\mathbf{y})$, then $\mathbf{v} \in z_j\mathbf{V}^j(\mathbf{y})$ for some $\mathbf{z} \in \mathbf{V}^*(\mathbf{y})$. For a small positive scalar α, the vector $\mathbf{v} + \alpha\mathbf{e}_N$, where $\mathbf{e}_N$ is an N-vector of ones, is in the interior of $z_j\mathbf{V}^j(\mathbf{y})$. Hence, there exists a small positive scalar β such that $F^j(\mathbf{y},\mathbf{v} + \alpha\mathbf{e}_N) \geqq z_j + \beta$. Since $\mathbf{z} + \beta\mathbf{e}_J \in \mathbf{V}^*(\mathbf{y})$, this implies $\mathbf{v} + \alpha\mathbf{e}_N \in \mathbf{V}^0(\mathbf{y})$. Hence, closure $\tilde{\mathbf{V}}^0(\mathbf{y}) \subseteq \mathbf{V}^0(\mathbf{y})$. This establishes $\mathbf{V}^0(\mathbf{y}) =$ Closure $\tilde{\mathbf{V}}^0(\mathbf{y})$. Duality implies the equivalence of P and S.

Next consider the cost function defined by

$$\begin{aligned}C^0(\mathbf{y},\mathbf{r}) &= \text{Min}\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \mathbf{V}^0(\mathbf{y})\} = \inf\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \tilde{\mathbf{V}}^0(\mathbf{y})\}\\ &= \inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} \inf\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in z_j\mathbf{V}^j(\mathbf{y}) \text{ for all } j\}.\end{aligned}$$

For fixed $\mathbf{z} \in \mathbf{V}^*(\mathbf{y})$,

$$\inf\left\{\mathbf{r}\cdot\mathbf{v} | \mathbf{v} \in \bigcap_{j=1}^{J}(z_j\mathbf{V}^j(\mathbf{y}))\right\} = \sup\left\{\sum_{j=1}^{J} z_jC^j(\mathbf{y},\mathbf{r}^j) | \mathbf{r}^j \geqq \mathbf{0}, \sum_{j=1}^{J}\mathbf{r}^j = \mathbf{r}\right\},$$

by Rules 1 and 4. The function $f(\mathbf{z},(\mathbf{r}^j);\mathbf{y}) = \sum_{j=1}^{J} z_jC^j(\mathbf{y},\mathbf{r}^j)$, defined for $\mathbf{z} \in \mathbf{V}^*(\mathbf{y})$ and $(\mathbf{r}^j)$ in the set $\mathbf{A} = \{(\mathbf{r}^j) | \mathbf{r}^j \geqq \mathbf{0}, \sum_{j=1}^{J} \mathbf{r}^j = \mathbf{r}\}$, is continuous on $\mathbf{V}^*(\mathbf{y}) \times \mathbf{A}$, concave in $(\mathbf{r}^j)$ for each $\mathbf{z}$, and linear (and thus convex) in $\mathbf{z}$ for each $(\mathbf{r}^j)$. Since $\mathbf{A}$ is bounded, the general minimax theorem [Rockafellar (1970, Corollary 37.3.1)] implies

$$\inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} \sup_{(\mathbf{r}^j)\in\mathbf{A}} f(\mathbf{z},(\mathbf{r}^j);\mathbf{y}) = \sup_{(\mathbf{r}^j)\in\mathbf{A}} \inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} f(\mathbf{z},(\mathbf{r}^j);\mathbf{y}).$$

But

$$\inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} f(\mathbf{z},(\mathbf{r}^j);\mathbf{y}) = \inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} \sum_{j=1}^{J} z_jC^j(\mathbf{y},\mathbf{r}^j) = C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}^1),\ldots,C^J(\mathbf{y},\mathbf{r}^J)),$$

by the definition of C^*, implying

$$C^0(\mathbf{y},\mathbf{r}) = \inf_{\mathbf{z}\in\mathbf{V}^*(\mathbf{y})} \sup_{(\mathbf{r}^j)\in\mathbf{A}} f(\mathbf{z},(\mathbf{r}^j);\mathbf{y}) = \sup_{(\mathbf{r}^j)\in\mathbf{A}} C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}^1),\ldots,C^J(\mathbf{y},\mathbf{r}^J)).$$

Using duality, this establishes the equivalence of Q and S.

The factor price requirement set satisfies

$$\mathbf{R}^0(\mathbf{y}) = \{\mathbf{r}|C^0(\mathbf{y},\mathbf{r}) \geqq 1\} = \left\{\sum_{j=1}^{J} \mathbf{r}^j | C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}^1),\ldots,C^J(\mathbf{y},\mathbf{r}^J)) \geqq 1\right\}$$

$$= \left\{\sum_{j=1}^{J} \mathbf{r}^j | C^*(\mathbf{y},\mathbf{q}) \geqq 1 \text{ and } C^j(\mathbf{y},\mathbf{r}^j) \geqq q_j;\text{ for some } q_j \geqq 0\right\}$$

$$= \bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{y})} \sum_{j=1}^{J} \left\{\mathbf{r}^j | C^j(\mathbf{y},\mathbf{r}^j) \geqq q_j\right\} = \bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{y})} \sum_{j=1}^{J} q_j \mathbf{R}^j(\mathbf{y}).$$

With duality, this establishes the equivalence of Q and T. Hence, Rule 7 is established.

The formal duality of the distance and cost functions yields Rule 8 from Rule 7. Q.E.D.

A variety of implications for technological structure can be drawn from these composition rules. First, using Rule 7 and the Cobb–Douglas distance and cost functions given in equations (17) and (18), we obtain a *Cobb–Douglas composition of distance functions*: For $\alpha_1,\ldots,\alpha_J > 0$, $\sum_{j=1}^{J} \alpha_j = 1$,

$$\text{P:}\quad F^0(\mathbf{y},\mathbf{v}) = F^1(\mathbf{y},\mathbf{v})^{\alpha_1}\cdot\cdots\cdot F^J(\mathbf{y},\mathbf{v})^{\alpha_J},$$

$$\text{Q:}\quad C^0(\mathbf{y},\mathbf{r}) = \alpha_1^{-\alpha_1}\cdot\cdots\cdot\alpha_J^{-\alpha_J}$$

$$\cdot\sup\left\{C^1(\mathbf{y},\mathbf{r}^1)^{\alpha_1}\cdot\cdots\cdot C^J(\mathbf{y},\mathbf{r}^J)^{\alpha_J} | \mathbf{r}^j \text{ positive}, \sum_{j=1}^{J} \mathbf{r}^j = \mathbf{r}\right\},$$

$$\text{S:}\quad \mathbf{V}^0(\mathbf{y}) = \bigcup_{\sum_{j=1}^{J} z_j \geqq 0} \bigcap_{j=1}^{J} e^{z_j/\alpha_j}\mathbf{V}_j(\mathbf{y}),$$

$$\text{T:}\quad \mathbf{R}^0(\mathbf{y}) = \bigcup_{\sum_{j=1}^{J} z_j \geqq 0} \sum_{j=1}^{J} \alpha_j e^{z_j/\alpha_j}\mathbf{R}^j(\mathbf{y}).$$

Formal duality gives *Cobb–Douglas composition of cost functions*: For

$\alpha_1,\ldots,\alpha_J > 0$, $\sum_{j=1}^J \alpha_j = 1$,

$$\text{P:}\quad F^0(\mathbf{y},\mathbf{v}) = \alpha_1^{-\alpha_1}\cdot\cdots\cdot\alpha_J^{-\alpha_J} \sup\left\{F^1(\mathbf{y},\mathbf{v}^1)^{\alpha_1}\cdot\cdots\cdot F^J(\mathbf{y},\mathbf{v}^J)^{\alpha_J} \cdot \middle| \mathbf{v}^j \text{ positive}, \sum_{j=1}^J \mathbf{v}^j = \mathbf{v}\right\},$$

$$\text{Q:}\quad C^0(\mathbf{y},\mathbf{r}) = C^1(\mathbf{y},\mathbf{r})^{\alpha_1}.\ldots.C^J(\mathbf{y},\mathbf{r})^{\alpha_J},$$

$$\text{S:}\quad \mathbf{V}^0(\mathbf{y}) = \bigcup_{\sum_{j=1}^J z_j \geqq 0} \bigcap_{j=1}^J \alpha_j e^{z_j/\alpha_j}\mathbf{V}^j(\mathbf{y}),$$

$$\text{T:}\quad \mathbf{R}^0(\mathbf{y}) = \bigcup_{\sum_{j=1}^J z_j \geqq 0} \sum_{j=1}^J e^{z_j/\alpha_j}\mathbf{R}^j(\mathbf{y}).$$

Using Rule 7 and the C.E.S. distance and cost functions given in equations (19) and (20), we obtain a *C.E.S. composition of distance functions*:

$$\text{P:}\quad F^0(\mathbf{y},\mathbf{v}) = \left(\sum_{j=1}^J (F^j(\mathbf{y},\mathbf{v})/D_j(\mathbf{y}))^{1-1/\sigma}\right)^{1/(1-1/\sigma)},$$

$$\text{Q:}\quad C^0(\mathbf{y},\mathbf{r}) = \sup\left\{\left(\sum_{j=1}^J (C^j(\mathbf{y},\mathbf{r}^j)D_j(\mathbf{y}))^{1-\sigma}\right)^{1/(1-\sigma)} \middle| \mathbf{r}^j \geqq 0, \sum_{j=1}^J \mathbf{r}^j = \mathbf{r}\right\},$$

$$\text{S:}\quad \mathbf{V}^0(\mathbf{y}) = \bigcup_{\substack{z_j \geqq 0 \\ \sum_{j=1}^J z_j = 1}} \bigcap_{j=1}^J (z_j^{\sigma/(\sigma-1)} D_j(\mathbf{y})\mathbf{V}^j(\mathbf{y})),$$

$$\text{T:}\quad \mathbf{R}^0(\mathbf{y}) = \bigcup_{\substack{q_j \geqq 0 \\ \sum_{j=1}^J q_j = 1}} \sum_{j=1}^J (\mathbf{R}^j(\mathbf{y}) q_j^{1/(1-\sigma)}/D_j(\mathbf{y})).$$

Again, application of formal duality gives a *C.E.S. composition of cost functions*:

$$\text{P:}\quad F^0(\mathbf{y},\mathbf{v}) = \sup\left\{\left(\sum_{j=1}^J (F^j(\mathbf{y},\mathbf{v}^j)/D_j(\mathbf{y}))^{1-1/\sigma}\right)^{1/(1-1/\sigma)} \middle| \mathbf{v}^j \geqq 0, \sum_{j=1}^J \mathbf{v}^j = \mathbf{v}\right\},$$

$$\text{Q:}\quad C^0(\mathbf{y},\mathbf{r}) = \left(\sum_{j=1}^J (C^j(\mathbf{y},\mathbf{r})D_j(\mathbf{y}))^{1-\sigma}\right)^{1/(1-\sigma)},$$

$$\text{S:}\quad \mathbf{V}^0(\mathbf{y}) = \bigcup_{\substack{z_j \geqq 0 \\ \sum_{j=1}^J z_j = 1}} \sum_{j=1}^J (\mathbf{V}^j(\mathbf{y}) z_j^{1/(1-1/\sigma)} D_j(\mathbf{y})),$$

$$\text{T:}\quad \mathbf{R}^0(\mathbf{y}) = \bigcup_{\substack{q_j \geqq 0 \\ \sum_{j=1}^J q_j = 1}} \bigcap_{j=1}^J (q_j^{1/(1-\sigma)}\mathbf{R}^j(\mathbf{y})/D_j(\mathbf{y})).$$

A production possibility set is said to be input-homothetic[9] if there exists a positive function $\alpha(\lambda,\mathbf{y})$ of $\lambda \in \mathbf{E}_+$ and $\mathbf{y} \in \mathbf{E}_+^M$, increasing in λ, with $\alpha(0,\mathbf{y}) = 0$, such that for $\mathbf{y} \neq \mathbf{0}$, $\mathbf{V}(\mathbf{y}) = \alpha(|\mathbf{y}|, \mathbf{y}/|\mathbf{y}|)\ \mathbf{V}(\mathbf{y}/|\mathbf{y}|)$, where $|\mathbf{y}|$ is the norm of $\mathbf{y}$ and we assume $\mathbf{y}/|\mathbf{y}| \in \mathbf{Y}^*$. In the case of a single output, this reduces to the textbook definition $\mathbf{V}(\mathbf{y}) = \alpha(\mathbf{y},1)\mathbf{V}(1)$ of homotheticity. More generally, it satisfies the textbook definition for any fixed output proportions, and allows the shape of the scaling of inputs versus output to vary with the output proportions. A property of an input-homothetic technology is that for fixed output proportions, the cost minimizing input mix is determined solely by input prices, independent of the scale of output. Rule 1 in Lemma 9 yields the following conclusion, where $|\mathbf{y}|$ is the norm of $\mathbf{y}$.

For an input-conventional production possibility set, the following conditions are equivalent:

(a) *The production possibility set is input-homothetic.*

(b) *The distance function has the form*

$$F(\mathbf{y},\mathbf{v}) = F(\mathbf{y}/|\mathbf{y}|,\mathbf{v})/\alpha(|\mathbf{y}|,\mathbf{y}/|\mathbf{y}|) \quad \textit{for} \quad \mathbf{y} \neq \mathbf{0}. \tag{33}$$

(c) *The cost function has the form*

$$C(\mathbf{y},\mathbf{r}) = \alpha(|\mathbf{y}|,\mathbf{y}/|\mathbf{y}|)C(\mathbf{y}/|\mathbf{y}|,\mathbf{r}) \quad \textit{for} \quad \mathbf{y} \neq \mathbf{0}. \tag{34}$$

A technology is *input-output separable* if it can be defined by a condition of the form $\beta(\mathbf{v})\gamma(\mathbf{y}) \geqq 1$. The distance function for this technology satisfies $\beta(\mathbf{v}/F(\mathbf{y},\mathbf{v}))\gamma(\mathbf{y}) = 1$, and hence can be written in the form $F(\mathbf{y},\mathbf{v}) = f(\gamma(\mathbf{y}),\mathbf{v})$, with f linear homogeneous in $\mathbf{v}$. Then, the cost function can be written $C(\gamma(\mathbf{y}),\mathbf{r})$, and the input requirement set $\mathbf{V}(\gamma(\mathbf{y}))$, with $\gamma(\mathbf{y})$ interpretable as the level of a single intermediate output. From the preceding result, a technology is both input-homothetic and input–output-separable if and only if the distance and cost functions can be written in the separable forms $F(\mathbf{y},\mathbf{v}) = F^1(\mathbf{v})F^2(\mathbf{y})$ and $C(\mathbf{y},\mathbf{r}) = C^1(\mathbf{r})C^2(\mathbf{y})$. Note also that these forms are related directly by composition Rule 1 (with F^1 and C^1 independent of $\mathbf{y}$).

Composition Rule 2 can be used to deduce the implications of factor augmenting technical change, or output change, on the distance and cost functions. Composition Rules 3–6 allow the geometric or algebraic construction of cost functions and factor price requirement sets. For example, in (a) of Figure 14, suppose given a Cobb–Douglas input

[9]This definition is due to G. Hanoch.

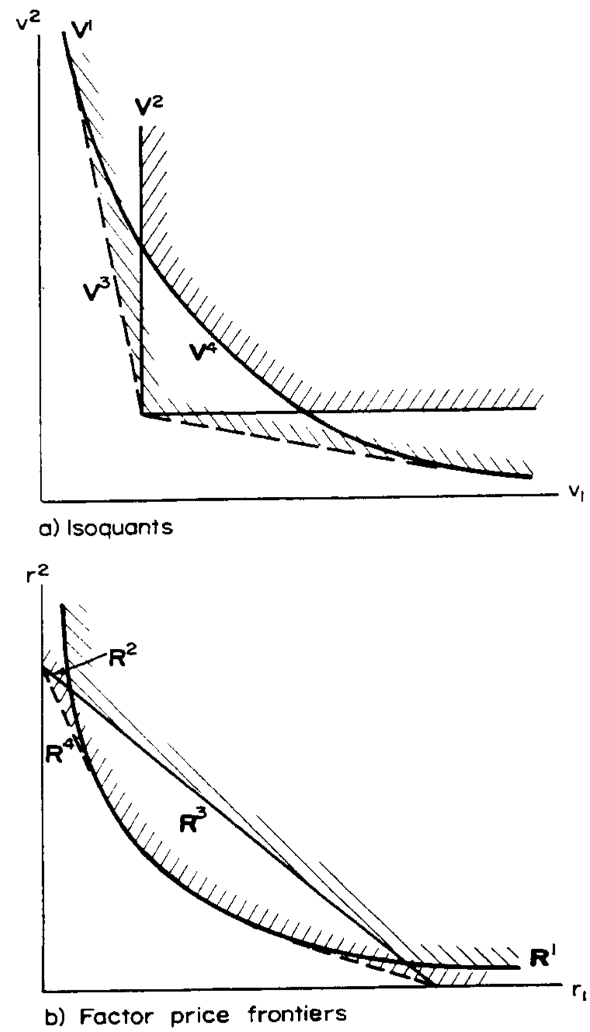


FIGURE 14

requirement set $\mathbf{V}^1$ and a Leontief input requirement set $\mathbf{V}^2$. The duals of these sets are the Cobb–Douglas factor price requirement set $\mathbf{R}^1$ and the linear factor price requirement set $\mathbf{R}^2$, respectively, illustrated in (b). The (convex hull of the) union of $\mathbf{V}^1$ and $\mathbf{V}^2$ is the set $\mathbf{V}^3$ in (a) of Figure 14. By Rule 3, the dual of $\mathbf{V}^3$ is the intersection $\mathbf{R}^3$ of $\mathbf{R}^1$ and $\mathbf{R}^2$. The intersection $\mathbf{V}^4$ of $\mathbf{V}^1$ and $\mathbf{V}^2$ has by Rule 4 the dual $\mathbf{R}^4$, given by the convex hull of the union of $\mathbf{R}^1$ and $\mathbf{R}^2$.

Composition Rules 7 and 8 yield a general result on separable distance and cost functions. Suppose the input vector $\mathbf{v}$ can be partitioned into sub-vectors, $\mathbf{v} = (\mathbf{v}_{(1)},\ldots,\mathbf{v}_{(J)})$, with a commensurate partition $\mathbf{r} = (\mathbf{r}_{(1)},\ldots,\mathbf{r}_{(J)})$ of the input price vector. Suppose a distance function F^j depends only on the sub-vector of inputs $\mathbf{v}_{(j)}$. Then, we can with a slight change of notation write the distance function $F^j(\mathbf{y},\mathbf{v}_{(j)})$. The dual cost function C^j then depends only on the sub-vector of prices $\mathbf{r}_{(j)}$, and can be written $C^j(\mathbf{y},\mathbf{r}_{(j)})$.

Lemma 10. Let $F^j(\mathbf{y},\mathbf{v}_{(j)})$ and $F^*(\mathbf{y},(z_1,\ldots,z_J))$ be input-conventional distance functions, and $C^j(\mathbf{y},\mathbf{r}_{(j)})$ and $C^*(\mathbf{y},(q_1,\ldots,q_J))$ their respective cost functions. Then $F^0(\mathbf{y},\mathbf{v}) = F^*(\mathbf{y},F^1(\mathbf{y},\mathbf{v}_{(1)}),\ldots,F^J(\mathbf{y},\mathbf{v}_{(J)}))$ is an input-conventional distance function with the dual cost function $C^0(\mathbf{y},\mathbf{r}) = C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}_{(1)}),\ldots,C^J(\mathbf{y},\mathbf{r}_{(J)}))$; i.e., the distance function is separable if and only if the cost function is separable.

Proof: The general concave composition Rule 7 implies that $F^0(\mathbf{y},\mathbf{v})$ is input-conventional, and that its cost function is

$$C^0(\mathbf{y},\mathbf{r}) = \sup\left\{C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}^1_{(1)}),\ldots,C^J(\mathbf{y},\mathbf{r}^J_{(J)}))\middle|\sum_{j=1}^{J}\mathbf{r}^j = \mathbf{r}\right\},$$

where $\mathbf{r}^j = (\mathbf{r}^j_{(1)},\ldots,\mathbf{r}^j_{(j)},\ldots,\mathbf{r}^j_{(J)})$. Since the cost functions are non-decreasing in prices, the supremum is achieved by $\mathbf{r}^j = (\mathbf{0},\ldots,\mathbf{0},\mathbf{r}_{(j)},\mathbf{0},\ldots,\mathbf{0})$ for $j = 1,\ldots,J$, or $C^0(\mathbf{y},\mathbf{r}) = C^*(\mathbf{y},C^1(\mathbf{y},\mathbf{r}_{(1)}),\ldots,C^J(\mathbf{y},\mathbf{r}_{(J)}))$. Q.E.D.

In the case of a single output and input-homotheticity, the separability property of the distance function implies a corresponding separability of the production function. However, in the absence of input-homotheticity, there is no simple relation between separability properties of the production and distance functions.

Composition Rules 7 and 8 can provide rules for computing distance or cost functions in cases of incomplete separability. For example, if the distance function is separable except for one input common to each F^j, then the cost function dual to the composite distance function will be given by a supremum involving the price of the one common input.

PART II. RESTRICTED PROFIT FUNCTIONS

13. The General Representation of Production Possibilities

In the previous sections of this chapter, the implications of input cost minimization with fixed outputs have been explored. More generally, optimization by the firm over any set of variable inputs and outputs can be analyzed. This approach leads to a general concept, the restricted profit function, which in special cases reduces to the cost function, a maximum revenue function, or an unrestricted maximum profit function.

Consider an environment for a firm in which N commodities, indexed $n = 1,2,...,N$, can be traded in competitive markets at a price vector $\mathbf{p} = (p_1,p_2,...,p_N)$. The firm treats these commodities as possible variable inputs or outputs. A *production plan* for the firm is an N-tuple of real numbers $\mathbf{x} = (x_1,x_2,...,x_N)$, with x_n interpreted as the quantity of net output (or, for compactness, *netput*) of commodity n, negative if the commodity is an input and positive if it is an output. The profit associated with a production plan $\mathbf{x}$ is given by $\pi = \mathbf{p}\cdot\mathbf{x}$, the inner product of $\mathbf{p}$ and $\mathbf{x}$.[10]

The technological limits on the actions of the firm can be described by a set $\mathbf{T}$ of possible production plans. Generally, the firm's possibilities are influenced by prior contracts to hire inputs or deliver outputs, and by the physical and economic environment. It is convenient to suppose that these effects can be summarized in an M-dimensional real vector $\mathbf{z} = (z_1,z_2,...,z_M)$ which can vary within some allowable set $\mathbf{Z}$. The production possibility set $\mathbf{T} = \mathbf{T}(\mathbf{z})$ then depends on the value of the vector $\mathbf{z}$.

Several examples will illustrate the generality of this formulation. If $\mathbf{z}$ is an output bundle, and all the commodities in the netput bundle are inputs, then $\mathbf{T}(\mathbf{z})$ is an input requirement set (with a negative sign) and $\pi = \mathbf{p}\cdot\mathbf{x}$ is the negative of cost. Under the appropriate interpretation of $\mathbf{T}(\mathbf{z})$, the problems of ex ante or ex post, or long or short run, cost minimization can be treated in this model.

If a firm is maximizing profit with a fixed input, then this input can be included in the parameter vector $\mathbf{z}$, and maximization can be carried out in terms of the variable commodities, yielding a maximum variable profit, net of the cost of the fixed input. Alternately, the fixed input can be included in the netput vector, with the production possibility set specifying its level. Maximization in this case yields a maximum total profit.

[10]The commodity price vector $\mathbf{p}$ is defined so that the prices of most commodities are non-negative. Then, output of a positively priced commodity contributes to revenue, and input of such a commodity contributes to cost. However, we do not rule out the possibility of negatively priced commodities. While this generalization is largely definitional, it proves useful in dealing with commodities for which there is no free disposal and for which net supply in an economy at zero price may be positive or negative. (Sawdust is an example.)

If the parameter vector $\mathbf{z}$ contains all inputs to the firm and all commodities in the netput bundle $\mathbf{x}$ are outputs, maximization leads to a maximum revenue for fixed inputs. If all inputs and outputs of the firm are in the netput bundle, then maximization leads to maximum unrestricted profits. Components of the vector $\mathbf{z}$ may be environmental or behavioral parameters other than commodity levels. The state of technical progress or the degree of learning by the firm may be included in $\mathbf{z}$. If the possibilities of the firm are influenced by ex ante decisions, then design parameters or anticipated prices and quantities can be included in $\mathbf{z}$. The parameter vector may include, in the case that the firm is subject to externalities, the production plans of other firms. Finally, $\mathbf{z}$ may include parameters introduced by the economic analyst to characterize the technology.

Three basic axioms on production possibilities, which involve little loss of economic generality, will be imposed in further analysis.

> *Axiom* 1. The set $\mathbf{Z}$ of possible production parameters is a non-empty subset of an M-dimensional Euclidean space $\mathbf{E}^M$. For each $\mathbf{z} \in \mathbf{Z}$, the set $\mathbf{T} = \mathbf{T}(\mathbf{z})$ of possible production plans is a closed non-empty subset of an N-dimensional Euclidean space $\mathbf{E}^N$.

The next axiom requires several definitions. The *normal cone* (barrier cone) of $\mathbf{T}(\mathbf{z})$, denoted by $\mathbf{P}(\mathbf{z})$, is the set of all price vectors $\mathbf{p} \in \mathbf{E}^N$ such that $\mathbf{p}\cdot\mathbf{x}$ is bounded above for $\mathbf{x} \in \mathbf{T}(\mathbf{z})$. Clearly, the normal cone will be the largest set of prices on which we can hope to define a maximum profit function. We denote the interior of the normal cone by $\mathbf{P}^0(\mathbf{z})$, and its closure by $\bar{\mathbf{P}}(\mathbf{z})$. The set $\mathbf{T}(\mathbf{z})$ is said to be *semi-bounded* if $\mathbf{P}^0(\mathbf{z})$ is non-empty. An example will illustrate the restriction placed on the structure of $\mathbf{T}(\mathbf{z})$ by a condition that it be semi-bounded: If $\mathbf{T}(\mathbf{z})$ contains both $\theta\mathbf{x}$ and $-\theta\mathbf{x}$ for some $\mathbf{x}$ and all large positive scalars θ, then the requirement that $\mathbf{p}\cdot(\theta\mathbf{x})$ and $\mathbf{p}\cdot(-\theta\mathbf{x})$ be bounded above for $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ implies that $\mathbf{p}$ must satisfy $\mathbf{p}\cdot\mathbf{x} = 0$. But this implies that $\mathbf{P}(\mathbf{z})$ is contained in a hyperplane, so that $\mathbf{P}^0(\mathbf{z})$ is empty and $\mathbf{T}(\mathbf{z})$ fails to be semi-bounded. Thus, the condition that $\mathbf{T}(\mathbf{z})$ be semi-bounded requires that at sufficiently large scale levels, production plans be irreversible in the sense that starting from a possible production plan $\mathbf{x}$, it is not feasible to reverse the role of inputs and outputs and produce the plan $-\mathbf{x}$. Most technologies can be expected to satisfy irreversibility, and hence be semi-bounded. This will be the case if labor cannot be produced, and all non-zero production plans require some labor input. Alternately, this will be the case if non-zero outputs in a production plan always require chronologically prior inputs.

An alternative definition of the semi-boundedness property of a set $\mathbf{T}(\mathbf{z})$, a condition that the asymptotic cone (recession cone) of $\mathbf{T}(\mathbf{z})$ be pointed,[11] is discussed in Appendix A.3. Result 11.3 in this appendix shows this definition to be equivalent to the requirement that $\mathbf{P}^0(\mathbf{z})$ be non-empty.

> *Axiom* 2. For each $\mathbf{z} \in \mathbf{Z}$, the production possibility set $\mathbf{T}(\mathbf{z})$ is semi-bounded.

In investigating the effects on profit levels of shifts in the parameter vector $\mathbf{z}$, it is useful to require that $\mathbf{T}(\mathbf{z})$ vary "regularly" with $\mathbf{z}$. We define $\mathbf{T}(\mathbf{z})$ to be *strongly continuous* on $\mathbf{Z}$ if for each $\mathbf{z}^0 \in \mathbf{Z}$ and sequence $\mathbf{z}^k \in \mathbf{Z}$ converging to $\mathbf{z}^0$, the following three conditions hold:

(i) If a sequence $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$ converges to $\mathbf{x}^0$, then $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$.
(ii) If $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$, then there exists a sequence $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$ which converges to $\mathbf{x}^0$.
(iii) If a sequence $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$ and a sequence of positive scalars θ_k have θ_k converging to zero and $\theta_k \mathbf{x}^k$ converging to $\mathbf{x}^0$, then there exists a sequence $\hat{\mathbf{x}}^k \in \mathbf{T}(\mathbf{z}^0)$ and a sequence of positive scalars $\hat{\theta}_k$ with $\hat{\theta}_k$ converging to zero and $\hat{\theta}_k \hat{\mathbf{x}}^k$ converging to $\mathbf{x}^0$.

In mathematical terminology, $\mathbf{T}(\mathbf{z})$ is a *correspondence*, conditions (i) and (ii) define *upper* and *lower hemicontinuous* correspondences, respectively, and (iii) states that the asymptotic cone of $\mathbf{T}(\mathbf{z})$ is an upper hemicontinuous correspondence.

Condition (i) requires that $\mathbf{T}(\mathbf{z})$ not "shrink" discontinuously as $\mathbf{z}$ varies, while (ii) requires that it not "expand" discontinuously. Condition (iii) requires that the set of directions in which $\mathbf{T}(\mathbf{z})$ is unbounded not "shrink" discontinuously as $\mathbf{z}$ varies. When $\mathbf{T}(\mathbf{z})$ satisfies (i) and (ii) alone, it is termed *continuous*. We shall show later that when the production possibility set is convex, the upper and lower hemicontinuity conditions (i) and (ii) imply condition (iii). Hence, a continuous convex production possibility correspondence is strongly continuous.

Define the set $\mathbf{Y} = \{(\mathbf{z},\mathbf{x}) \in \mathbf{E}^M \times \mathbf{E}^N | \mathbf{z} \in \mathbf{Z},\ \mathbf{x} \in \mathbf{T}(\mathbf{z})\}$. Note that $\mathbf{Y}$ is a

[11]The asymptotic cone (recession cone) of a set can be defined informally as the set of directions in which the set is unbounded. A cone is pointed if it contains no lines.

closed set if and only if the following two conditions hold: (i) $\mathbf{T}(\mathbf{z})$ is an upper hemicontinuous correspondence, and (iv) $(\mathbf{z}^k,\mathbf{x}^k) \in \mathbf{Y}$ and $(\mathbf{z}^k,\mathbf{x}^k)$ converging to $(\mathbf{z}^0,\mathbf{x}^0)$ with $\mathbf{x}^0$ finite implies $\mathbf{z}^0 \in \mathbf{Z}$. Then, strong continuity of $\mathbf{T}(\mathbf{z})$ is neither necessary nor sufficient for the set $\mathbf{Y}$ to be closed. However, (i) is common to both conditions.

Figure 15 illustrates the geometry of these continuity conditions. In each case except (c), $\mathbf{T}(\mathbf{z})$ is an upper hemicontinuous correspondence. In case (c), the point $\mathbf{x}^0$, obtained as a limit of points in $\mathbf{T}(\mathbf{z})$ for $\mathbf{z}$ approaching $\mathbf{z}^0$ from below, is not contained in $\mathbf{T}(\mathbf{z}^0) = \{\mathbf{x}|\mathbf{x} \leqq \mathbf{x}^1\}$. In cases (a), (b), (d), (e) the set $\mathbf{Y}$ is closed. Note that the set $\mathbf{Z}$ may or may not be closed [cases (a), (b), respectively] even though $\mathbf{Y}$ is closed. In case (c), $\mathbf{Y}$ fails to be closed because upper hemicontinuity of $\mathbf{T}(\mathbf{z})$ is absent. In case (f), $\mathbf{Y}$ fails to be closed because property (iv) fails to hold.

Lower hemicontinuity holds in Figure 15 in every case except (d), where it fails at the point $(\mathbf{z}^0,\mathbf{x}^0)$ for $\mathbf{z}$ approaching $\mathbf{z}^0$ from above. Finally, case (e) gives an example in which condition (iii) on the upper hemicontinuity of the asymptotic cone fails, since a sequence $\mathbf{z}^k$ converging to $\mathbf{z}^0$ from above and $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$ with $\mathbf{x}^k \to -\infty$ has $\theta_k \mathbf{x}^k = -1$ when $\theta_k = 1/|\mathbf{x}^k| \to 0$, whereas $\mathbf{T}(\mathbf{z}^0)$ is bounded, and any sequence $\hat{\theta}_k \hat{\mathbf{x}}^k$ with $\hat{\theta}_k$ converging to zero and $\hat{\mathbf{x}}^k \in \mathbf{T}(\mathbf{z}^0)$ must also converge to zero.

Insight into the economic restrictions imposed by strong continuity can be gained by two interpretations of the examples in Figure 15. First, suppose that $\mathbf{T}(\mathbf{z})$ determines the level of a single input (x) required to produce a specified output level (z). The normally imposed condition that the overall production possibility set $\mathbf{Y}$ be closed will imply upper hemicontinuity. Strong upper hemicontinuity seems to rule out only pathological cases such as (e). Lower hemicontinuity rules out cases such as (d) where there is a "plateau" at which additional input fails to yield more output, and in a multiple-input case rules out "thick" isoquants. Intuitively then, lower hemicontinuity implies that some small change in the input bundle must be productive.

For the second interpretation, suppose that $\mathbf{T}(\mathbf{z})$ specifies the level of a single input (x) required to produce a unit of output at different levels of technological knowledge (z) (with low z corresponding to advanced technology). Strong continuity then requires a steady progression of the state of the arts, without "breakthroughs" such as at z^0 in case (d).

Axiom 3. The production possibility set $\mathbf{T}(\mathbf{z})$ is strongly continuous on $\mathbf{Z}$.

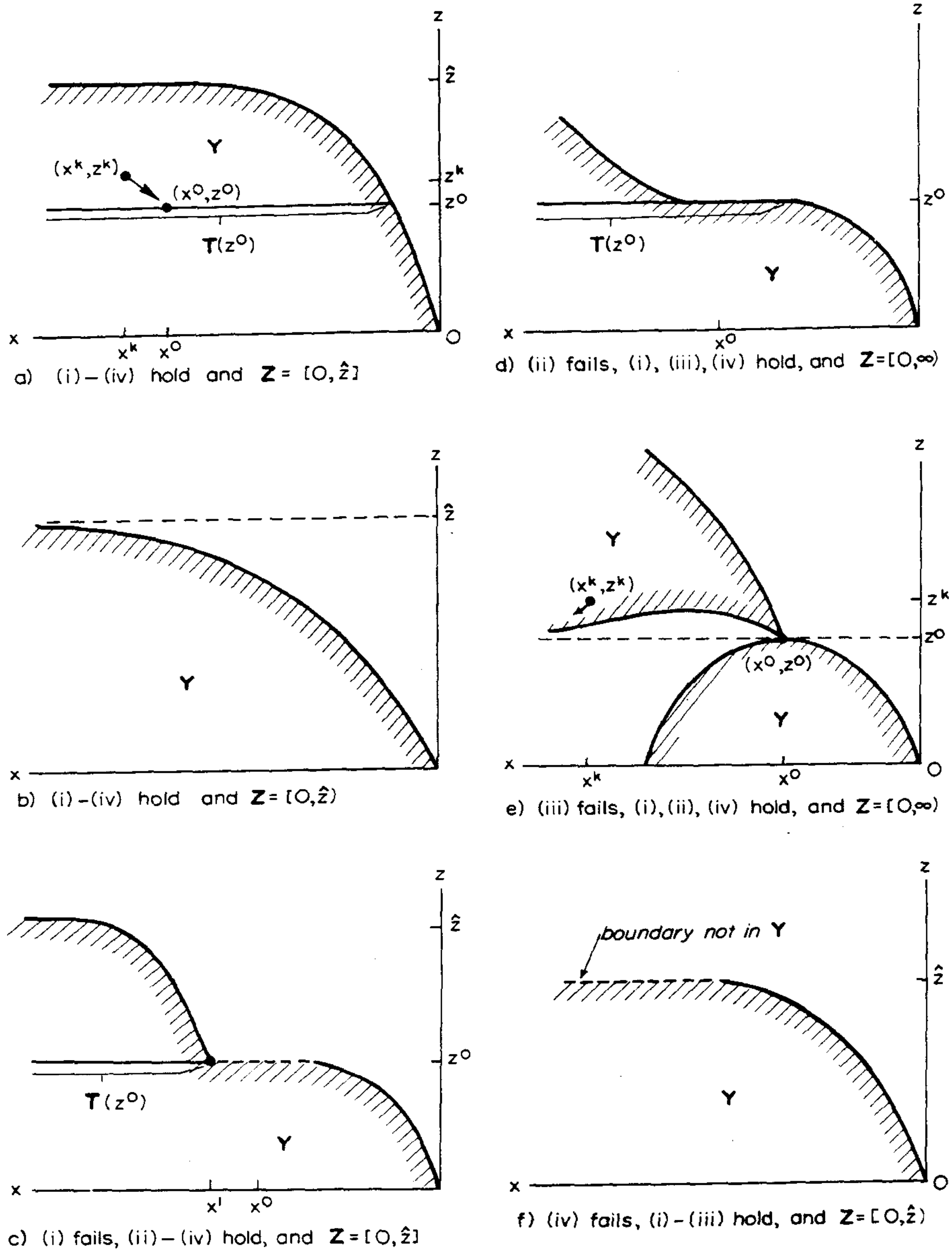

z
ẑ
Y
(xᵏ,zᵏ)
(x⁰,z⁰)
zᵏ
z⁰
T(z⁰)
x
xᵏ
x⁰
O
a) (i)–(iv) hold and Z = [O, ẑ]
z
z⁰
T(z⁰)
Y
x
x⁰
d) (ii) fails, (i), (iii), (iv) hold, and Z=[O,∞)
z
ẑ
Y
x
b) (i)-(iv) hold and Z = [O,ẑ)
z
Y
(xᵏ,zᵏ)
zᵏ
z⁰
(x⁰,z⁰)
Y
x
xᵏ
x⁰
O
e) (iii) fails, (i), (ii), (iv) hold, and Z = [O,∞)
z
ẑ
z⁰
T(z⁰)
Y
x
x'
x⁰
c) (i) fails, (ii)–(iv) hold, and Z = [O, ẑ]
z
boundary not in Y
ẑ
Y
x
O
f) (iv) fails, (i)-(iii) hold, and Z= [O,ẑ)

FIGURE 15

The basic Axioms 1–3 play the same role in the following analysis as did the assumption of input-regularity in the development of the cost function. The reader will recall that in the previous treatment, further assumptions of free disposal and convexity were often used with the argument that the economic behavior implied by cost minimization would always be consistent with these conditions. The same argument will be used to justify the following axiom.

Axiom 4. For each $\mathbf{z} \in \mathbf{Z}$, the technology $\mathbf{T}(\mathbf{z})$ is convex; i.e., for any netput bundles $\mathbf{x}^0$, $\mathbf{x}^1 \in \mathbf{T}(\mathbf{z})$ and weighted average $\mathbf{x}^* = \theta\mathbf{x}^0 + (1-\theta)\mathbf{x}^1$, $0 < \theta < 1$, it follows that $\mathbf{x}^* \in \mathbf{T}(\mathbf{z})$.

Lemma 13.3(2) in Appendix A.3 establishes that a technology satisfying Axioms 1, 2, and 4, plus the upper and lower hemicontinuity conditions (i) and (ii) in the definition of strong continuity, must also satisfy condition (iii), and hence Axiom 3.

The technology $\mathbf{T}(\mathbf{z})$ is said to exhibit *free disposal of inputs and outputs* if $\mathbf{x} \in \mathbf{T}(\mathbf{z})$ and $\mathbf{x}^1 \leqq \mathbf{x}$ imply $\mathbf{x}^1 \in \mathbf{T}(\mathbf{z})$. When $\mathbf{T}(\mathbf{z})$ satisfies this condition, all price vectors $\mathbf{p}$ in the normal cone $\mathbf{P}(\mathbf{z})$ of $\mathbf{T}(\mathbf{z})$ must be non-negative. Conversely, the existence of price vectors in $\mathbf{P}(\mathbf{z})$ with negative components indicates a lack of free disposability of the corresponding commodities. Free disposal and related assumptions will be discussed in the next section.

14. The General Restricted Profit Function

Consider a firm with a technology $\mathbf{T}(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, satisfying Axioms 1–3. Suppose that the firm faces a competitive price vector $\mathbf{p} \in \mathbf{E}^N$ for the commodities in its production plan, and desires to maximize its profit $\pi = \mathbf{p}\cdot\mathbf{x}$ over $\mathbf{x} \in \mathbf{T}(\mathbf{z})$. Recalling that $\mathbf{P}(\mathbf{z})$ is the set of price vectors $\mathbf{p}$ for which $\pi = \mathbf{p}\cdot\mathbf{x}$ is bounded above over $\mathbf{x} \in \mathbf{T}(\mathbf{z})$, define the *restricted profit function* of the firm by

$$\pi = \Pi(\mathbf{z},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x} \mid \mathbf{x} \in \mathbf{T}(\mathbf{z})\} \quad \text{for} \quad \mathbf{p} \in \mathbf{P}(\mathbf{z}). \tag{35}$$

The restricted profit function gives the least upper bound on the level of profits that can be attained with a parameter vector $\mathbf{z}$ and a price vector $\mathbf{p}$.

The restricted profit function Π is *convex* in $\mathbf{p}$ for fixed $\mathbf{z}$ if for any $\mathbf{p}$, $\mathbf{p}^0 \in \mathbf{P(z)}$ and weighted average $\mathbf{p}^1 = \theta\mathbf{p} + (1-\theta)\mathbf{p}^0$, with $0 < \theta < 1$, it follows that $\mathbf{p}^1 \in \mathbf{P(z)}$ and $\Pi(\mathbf{z},\mathbf{p}^1) \leqq \theta\Pi(\mathbf{z},\mathbf{p}) + (1-\theta)\Pi(\mathbf{z},\mathbf{p}^0)$. This function is *positively linear homogeneous* in $\mathbf{p}$ for fixed $\mathbf{z}$ if for any $\mathbf{p} \in \mathbf{P(z)}$ and $\lambda > 0$, it follows that $\lambda\mathbf{p} \in \mathbf{P(z)}$ and $\Pi(\mathbf{z},\lambda\mathbf{p}) = \lambda\Pi(\mathbf{z},\mathbf{p})$. This function is *closed* in $\mathbf{p}$ for fixed $\mathbf{z}$ if for any sequence $\mathbf{p}^k \in \mathbf{P(z)}$ converging to $\mathbf{p}^0$, either (a) $\mathbf{p}^0 \in \mathbf{P(z)}$ and $\Pi(\mathbf{z},\mathbf{p}^0) = \lim_{\mathbf{p}\in\mathbf{P(z)},\mathbf{p}\to\mathbf{p}^0} \inf \Pi(\mathbf{z},\mathbf{p})$, or (b) $\mathbf{p}^0 \notin \mathbf{P(z)}$ and $\lim_{\mathbf{p}\in\mathbf{P(z)},\mathbf{p}\to\mathbf{p}^0} \inf \Pi(\mathbf{z},\mathbf{p}) = +\infty$.[12] Restating the last condition less formally, the set of prices for which profit is bounded above contains a boundary point $\mathbf{p}^0$ if and only if profits are uniformly bounded above for some sequence of prices in $\mathbf{P(z)}$ approaching $\mathbf{p}^0$.

The next result establishes the basic properties of the restricted profit function.

Lemma 11. Suppose a technology $\mathbf{T(z)}$, $\mathbf{z} \in \mathbf{Z}$, satisfies Axioms 1 and 2. Then, the following conclusions hold:

(1) The set $\mathbf{P(z)}$ is a convex cone, and its interior $\mathbf{P}^0\mathbf{(z)}$ is non-empty.

(2) For each $\mathbf{z} \in \mathbf{Z}$, $\Pi(\mathbf{z},\mathbf{p})$ is a convex, positively linear homogeneous, closed function of $\mathbf{p} \in \mathbf{P(z)}$.

(3) For each $\mathbf{z} \in \mathbf{Z}$, $\Pi(\mathbf{z},\mathbf{p})$ is a continuous function of $\mathbf{p} \in \mathbf{P}^0\mathbf{(z)}$, and satisfies

$$\Pi(\mathbf{z},\mathbf{p}) = \text{Max}\{\mathbf{p}\cdot\mathbf{x} \mid \mathbf{x} \in \mathbf{T(z)}\}, \tag{36}$$

i.e., a profit maximizing netput bundle can be attained for each $\mathbf{p} \in \mathbf{P}^0\mathbf{(z)}$.

(4) The closed convex hull of $\mathbf{T(z)}$ is equal to the set

$$\tilde{\mathbf{T}}\mathbf{(z)} = \{\mathbf{x} \in \mathbf{E}^N \mid \mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p}) \quad \text{for all} \quad \mathbf{p} \in \mathbf{P(z)}\}. \tag{37}$$

If Axiom 4 holds, then $\mathbf{T(z)} = \tilde{\mathbf{T}}\mathbf{(z)}$.

Proof: In the terminology of Appendix A.3, Sections 8 and 9, $\mathbf{P(z)}$ is the normal cone of $\mathbf{T(z)}$ and $\Pi(\mathbf{z},\mathbf{p})$ is the support function of $\mathbf{T(z)}$. Axiom 2 implies $\mathbf{P}^0\mathbf{(z)}$ non-empty, and the convexity of the normal cone is a standard result (Appendix A.3, Section 10.18). Lemma 12.4 in the

[12]The notation $\lim_{\mathbf{p}\in\mathbf{P(z)},\mathbf{p}\to\mathbf{p}^0} \inf \Pi(\mathbf{z},\mathbf{p})$ means the greatest lower bound of the set of all limit points of $\Pi(\mathbf{z},\mathbf{p})$ for all sequences in $\mathbf{P(z)}$ converging to $\mathbf{p}^0$.

appendix establishes results (2) and (4).[13] Appendix A.3, Lemma 13.5(1) implies that a profit maximizing netput bundle can be attained for each $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, and Appendix A.3, Lemma 12.1(1) implies that Π is continuous in $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$. Q.E.D.

Figure 16 illustrates several simple technologies and the corresponding profit functions. (a) gives a case in which the normal cone of the technology fails to be closed. Note that the profit function for this case is not continuous at $\mathbf{p} = \mathbf{0}$. [E.g., $p_1 = p_2 + p_2^2$ and $p_2 \to 0$ implies $\Pi \to 1/4 \neq \Pi(0)$]. Hence, the conclusion of Lemma 11 that the profit function is continuous for prices in the interior of the normal cone and lower semicontinuous on all prices in the normal cone cannot be strengthened without further hypotheses.

A netput bundle $\mathbf{x} \in \mathbf{T}(\mathbf{z})$ is *exposed* if there exists a price vector $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ such that $\mathbf{p} \cdot \mathbf{x} > \mathbf{p} \cdot \mathbf{x}^1$ for all distinct $\mathbf{x}^1 \in \mathbf{T}(\mathbf{z})$. The next result gives conditions under which the profit function is continuous on $\mathbf{P}(\mathbf{z})$.

Lemma 12. If Axioms 1 and 2 hold, then any one of the following conditions is sufficient to imply that for each $\mathbf{z} \in \mathbf{Z}$, the restricted profit function is continuous in $\mathbf{p}$ on $\mathbf{P}(\mathbf{z})$:

(1) $\mathbf{P}(\mathbf{z})$ is closed and can be represented as the convex cone spanning a finite number of points.
(2) $\mathbf{T}(\mathbf{z})$ is bounded.
(3) The set of exposed points in $\mathbf{T}(\mathbf{z})$ is bounded.

Proof: Appendix A.3, Lemma 12.7 implies (1). If $\mathbf{T}(\mathbf{z})$ is bounded, then its asymptotic cone is empty and thus $\mathbf{P}(\mathbf{z}) = \mathbf{E}^N$ by result 10.16 in this appendix, and (2) is implied by (1). Finally, we prove (3).

[13]These proofs employ the fundamental mathematical theory of convex conjugate functions, from which many other implications can be easily derived. Alternately, Result (2) can be proved directly using the simple, pedagogically appealing arguments employed in deriving the properties of cost functions: If $\mathbf{p}, \mathbf{p}^0 \in \mathbf{P}(\mathbf{z})$ and $\mathbf{p}^1 = \theta\mathbf{p} + (1-\theta)\mathbf{p}^0$, $0 < \theta < 1$, and if $\mathbf{x}^k \in \mathbf{T}(\mathbf{z})$ is a sequence with $\mathbf{p}^1 \cdot \mathbf{x}^k \to \Pi(\mathbf{z},\mathbf{p}^1)$, then $\mathbf{p}^1 \cdot \mathbf{x}^k = \theta\mathbf{p}\cdot\mathbf{x}^k + (1-\theta)\mathbf{p}^0\cdot\mathbf{x}^k \leqq \theta\Pi(\mathbf{z},\mathbf{p}) + (1-\theta)\Pi(\mathbf{z},\mathbf{p}^0)$, implying in the limit $\mathbf{p}^1 \in \mathbf{P}(\mathbf{z})$ and $\Pi(\mathbf{z},\mathbf{p}^1) \leqq \theta\Pi(\mathbf{z},\mathbf{p}) + (1-\theta)\Pi(\mathbf{z},\mathbf{p}^0)$. Positive linear homogeneity is immediate from the definition of the profit function. Finally, a simple argument shows that Π is closed. Suppose $\mathbf{p}^0$ is in the boundary of $\mathbf{P}(\mathbf{x})$, and let $\mathbf{x}^j \in \mathbf{T}(\mathbf{z})$ be a sequence such that $\mathbf{p}^0\cdot\mathbf{x}^j \to \sup\{\mathbf{p}^0\cdot\mathbf{x} | \mathbf{x} \in \mathbf{T}(\mathbf{z})\}$. Then for any sequence $\mathbf{p}^k \in \mathbf{P}(\mathbf{z}), \mathbf{p}^k \to \mathbf{p}^0$, the inequality $\Pi(\mathbf{z},\mathbf{p}^k) \geqq \mathbf{p}^k\cdot\mathbf{x}^j$ implies in the limit $\lim_{k\to\infty} \inf \Pi(\mathbf{z},\mathbf{p}^k) \geqq \mathbf{p}^0\cdot\mathbf{x}^j$, and hence letting $j \to \infty$, $\lim_{k\to\infty} \inf \Pi(\mathbf{z},\mathbf{p}^k) \geqq \Pi(\mathbf{z},\mathbf{p}^0)$. Choosing the sequence $\mathbf{p}^k = k^{-1}\mathbf{p}^1 + (1-k^{-1})\mathbf{p}^0$ for some $\mathbf{p}^1 \in \mathbf{P}^0(\mathbf{z})$, one obtains from the convexity condition the opposite inequality $\lim_{k\to\infty} \Pi(\mathbf{z},\mathbf{p}^k) \leqq \Pi(\mathbf{z},\mathbf{p}^0)$. Hence, we conclude that either $\mathbf{p}^0 \notin \mathbf{P}(\mathbf{z})$ and $\lim_{\mathbf{p}\in\mathbf{P}(\mathbf{z}),\mathbf{p}\to\mathbf{p}^0} \inf \Pi(\mathbf{z},\mathbf{p}) = +\infty$, or $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z})$ and $\lim_{\mathbf{p}\in\mathbf{P}(\mathbf{z}),\mathbf{p}\to\mathbf{p}^0} \inf \Pi(\mathbf{z},\mathbf{p}) = \Pi(\mathbf{z},\mathbf{p}^0)$.

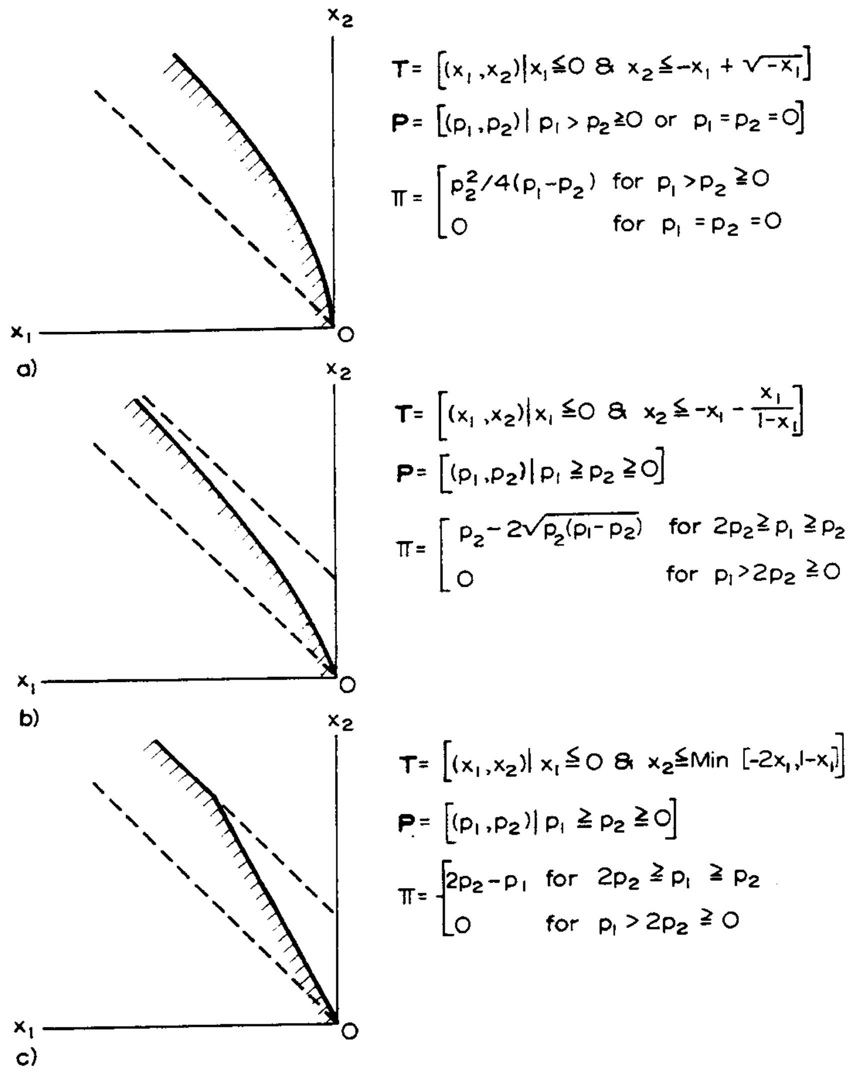


FIGURE 16

Consider a sequence $\hat{\mathbf{p}}^k \in \mathbf{P}^0(\mathbf{z})$ converging to $\mathbf{p}^0$. By results 12.1(2), 13.8(3), and 13.9 in Appendix A.3, there exists a vector $\mathbf{p}^k \in \mathbf{P}^0(\mathbf{z})$ arbitrarily close to $\hat{\mathbf{p}}^k$ and an exposed bundle $\mathbf{x}^k \in \mathbf{T}(\mathbf{z})$ such that $\Pi(\mathbf{z},\mathbf{p}^k) = \mathbf{p}^k \cdot \mathbf{x}^k$. By continuity of Π on $\mathbf{P}^0(\mathbf{z})$, $\mathbf{p}^k$ can be chosen so that $|\mathbf{p}^k - \hat{\mathbf{p}}^k| < k^{-1}$ and $|\Pi(\mathbf{z},\mathbf{p}^k) - \Pi(\mathbf{z},\hat{\mathbf{p}}^k)| < k^{-1}$. Since the exposed points $\mathbf{x}^k$ are bounded, we can extract a subsequence converging to a point $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z})$ such that $\Pi(\mathbf{z},\mathbf{p}^0) \geqq \mathbf{p}^0 \cdot \mathbf{x}^0 = \lim_{k\to\infty} \sup \Pi(\mathbf{z},\hat{\mathbf{p}}^k)$. Since Π is lower semicontinuous in $\mathbf{p}$ on $\mathbf{P}(\mathbf{z})$, we have established $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z})$ and $\Pi(\mathbf{z},\mathbf{p}^0) = \lim_{\mathbf{p}\in\mathbf{P}^0(\mathbf{z})\&\mathbf{p}\to\mathbf{p}^0} \Pi(\mathbf{z},\mathbf{p})$. Finally, note that for any sequence $\hat{\mathbf{p}}^k \in \mathbf{P}(\mathbf{z})$ converging to $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z})$, we have, by the result just proved, a sequence

$\mathbf{p}^k \in \mathbf{P}^0(\mathbf{z})$ with $|\mathbf{p}^k - \hat{\mathbf{p}}^k| < k^{-1}$ and $|\Pi(\mathbf{z},\mathbf{p}^k) - \Pi(\mathbf{z},\hat{\mathbf{p}}^k)| < k^{-1}$. Then, $\Pi(\mathbf{z},\mathbf{p}^0) = \lim_{\mathbf{p}\in\mathbf{P}(\mathbf{z})\ \&\ \mathbf{p}\to\mathbf{p}^0} \Pi(\mathbf{z},\mathbf{p})$. Q.E.D.

We note from the proof of Lemma 12 that condition (3) implies $\mathbf{P}(\mathbf{z})$ closed. This lemma has one immediate corollary: *If $N = 2$ and $\mathbf{P}(\mathbf{z})$ is closed, then* (1) *is satisfied and Π is continuous in* $\mathbf{p}$ *on* $\mathbf{P}(\mathbf{z})$. Figure 16, (b) and (c), illustrates cases in which $\mathbf{P}(\mathbf{z})$ is closed, and the resulting profit functions are continuous in $\mathbf{p}$ on $\mathbf{P}(\mathbf{z})$ in accord with this corollary.

When the technology exhibits free disposal of inputs and outputs, it is convenient to distinguish bundles $\mathbf{x} \in \mathbf{T}(\mathbf{z})$ which are *efficient* in that no distinct bundle $\mathbf{x}^1 \in \mathbf{T}(\mathbf{z})$ has $\mathbf{x}^1 \geqq \mathbf{x}$. Since under this assumption any exposed $\mathbf{x}$ must be efficient, (3) in Lemma 12 implies the following corollary: *If Axioms* 1 *and* 2 *hold, the technology exhibits free disposal of inputs and outputs, and the set of efficient points is bounded, then the profit function is continuous in* $\mathbf{p}$ *on* $\mathbf{P}(\mathbf{z})$.

A technology $\mathbf{T}(\mathbf{z})$ is *bounded above* if there exists $\mathbf{x}^0$ such that $\mathbf{x} \leqq \mathbf{x}^0$ for all $\mathbf{x} \in \mathbf{T}(\mathbf{z})$. In the case of cost minimization, in which the netput bundles in $\mathbf{T}(\mathbf{z})$ are the negative of input bundles, $\mathbf{T}(\mathbf{z})$ is bounded above by the origin. In the case of profit maximization with all inputs fixed, or with some input fixed which is essential to production, the technology is generally bounded above by some positive $\mathbf{x}^0$. For these cases, the following corollary is useful: *If Axioms* 1 *and* 2 *hold, and the technology is bounded above and exhibits free disposal of inputs and outputs, then the profit function is continuous in* $\mathbf{p}$ *on* $\mathbf{P}(\mathbf{z})$. Under the hypotheses of this corollary, the normal cone of the technology is the non-negative orthant of $\mathbf{E}^N$, and (1) in Lemma 12 implies the conclusion.

For $N > 2$, $\mathbf{P}(\mathbf{z})$ closed is not in general sufficient to imply continuity of the profit function over $\mathbf{P}(\mathbf{z})$. A counter-example of Gale, Klee, and Rockafellar (1968, Lemma and proof of Theorem 2) gives a convex function which is lower semicontinuous (closed), but not upper semicontinuous. The domain of this function can be taken to be the set of $\mathbf{p} \in \mathbf{E}^N$ satisfying $\mathbf{p}\cdot\mathbf{p} \leqq 1$ and $\sum_{n=1}^{N} p_n = 1/2$. Form the closed convex semi-bounded cone spanned by this domain, and define a linear homogeneous extension of this function on the cone. Then, Lemma 12.5 in Appendix A.3 implies that the resulting function is the profit function for a technology satisfying Axioms 1 and 2.

Lemma 11(3) establishes that a profit maximizing netput bundle can be attained for $\mathbf{p}$ in the interior of the normal cone of the technology. (b) in Figure 16 illustrates a case in which a maximum cannot be achieved at the boundary price vector $\mathbf{p} = (1,1)$, and (c) illustrates a case in which a maximum can be achieved at this boundary price vector. Let $\mathbf{P}^1(\mathbf{z})$ denote the set of $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ for which a profit maximizing netput bundle can be attained. These examples show that $\mathbf{P}^1(\mathbf{z})$ may be neither open nor closed in general. A further example given in Appendix A.3, 13.6, shows that $\mathbf{P}^1(\mathbf{z})$ need not be convex, although its interior, equal to $\mathbf{P}^0(\mathbf{z})$, is convex. The following result gives one condition under which the normal cone of the technology is closed and a profit maximizing bundle can be achieved for each price vector in this cone.

Lemma 13. If Axioms 1 and 2 hold, and the set of exposed points in $\mathbf{T}(\mathbf{z})$ is bounded, then $\mathbf{P}(\mathbf{z})$ is closed and $\mathbf{P}^1(\mathbf{z}) = \mathbf{P}(\mathbf{z})$.

Proof: Let $\mathbf{T}^1$ denote the closed convex hull of $\mathbf{T}(\mathbf{z})$, $\mathbf{T}^2$ denote the asymptotic cone of $\mathbf{T}^1$, and $\mathbf{T}^3$ denote the closed convex hull of the set of exposed points in $\mathbf{T}(\mathbf{z})$. By hypothesis, $\mathbf{T}^3$ is closed and bounded. By Appendix A.3, Lemma 14.3, $\mathbf{T}^1 = \mathbf{T}^2 + \mathbf{T}^3$. For any $\mathbf{p}^0$ in the closure of $\mathbf{P}(\mathbf{z})$, $\mathbf{p}^0 \cdot \mathbf{x} \leqq 0$ for any $\mathbf{x} \in \mathbf{T}^2$. Hence, the supremum of $\mathbf{p}^0 \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbf{T}^1$ is approached by $\mathbf{x} \in \mathbf{T}^3$, and is therefore achieved at some $\mathbf{x}^0 \in \mathbf{T}^3 \subseteq \mathbf{T}^1$. But if a linear function achieves a maximum on the convex hull of a closed set, then it achieves a maximum on the set. Hence, a profit maximizing bundle for $\mathbf{p}^0$ can be found in $\mathbf{T}(\mathbf{z})$, implying $\mathbf{p}^0 \in \mathbf{P}^1(\mathbf{z})$ and $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z})$. Q.E.D.

Further conditions for the convexity and closedness of $\mathbf{P}^1(\mathbf{z})$ have been given by Winter (forthcoming). We next establish several additional properties of profit maximizing netput bundles. For $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, let $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ denote the set of netput bundles in $\mathbf{T}(\mathbf{z})$ which maximize profit.

Lemma 14. Suppose Axioms 1 and 2 hold. Then, $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ has the following properties:

(1) For $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is closed, and bounded.

(2) For any closed, bounded subset $\mathbf{R}$ of $\mathbf{P}^0(\mathbf{z})$, the set $\bigcup_{\mathbf{p} \in \mathbf{R}} \mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is bounded.

(3) $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is an upper hemicontinuous correspondence in $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$ for each $\mathbf{z} \in \mathbf{Z}$; i.e., if $\mathbf{p}^k \in \mathbf{P}^0(\mathbf{z})$, $\mathbf{p}^k \to \mathbf{p}^0 \in \mathbf{P}^0(\mathbf{z})$, $\mathbf{x}^k \in \mathbf{\Phi}(\mathbf{z},\mathbf{p}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$, then $\mathbf{x}^0 \in \mathbf{\Phi}(\mathbf{z},\mathbf{p}^0)$.

(4) If Axiom 4 holds, then $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is convex set.

(5) $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is positively homogeneous of degree zero in $\mathbf{p}$; i.e., for $\lambda > 0$, $\mathbf{\Phi}(\mathbf{z},\lambda\mathbf{p}) = \mathbf{\Phi}(\mathbf{z},\mathbf{p})$.

Proof: Appendix A.3, Lemma 13.5 establishes results (1)–(4); the proof of result (5) is trivial.

Now consider the behavior of the restricted profit function under joint variation of the parameter vector $\mathbf{z}$ and the price vector $\mathbf{p}$. The first result establishes the behavior of the normal cone of the technology under variations in $\mathbf{z}$.

Lemma 15. Suppose Axioms 1–3 hold. Then, $\mathbf{P}(\mathbf{z})$ has the following properties:

(1) $\mathbf{P}(\mathbf{z})$ is a lower hemicontinuous correspondence on $\mathbf{Z}$; i.e., $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z}^0)$ implies the existence of $\mathbf{p}^k \in \mathbf{P}(\mathbf{z}^k)$ such that $\mathbf{p}^k \to \mathbf{p}^0$.

(2) If $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, and $\mathbf{R}$ is a non-empty, closed, bounded subset of $\mathbf{P}^0(\mathbf{z}^0)$, then there exists a k_0 such that $\mathbf{R} \subseteq \mathbf{P}^0(\mathbf{z}^k)$ for $k \geqq k_0$ and the set $\bigcup_{k \geqq k_0} \bigcup_{\mathbf{p} \in \mathbf{R}} \mathbf{\Phi}(\mathbf{z}^k, \mathbf{p})$ is bounded.

Proof: Appendix A.3, Lemma 15.2.

Figure 17 illustrates the results of Lemmas 14 and 15 for a simple case in which $\mathbf{Z}$ is the non-negative real line and

$$\mathbf{T}(z) = \{(x_1, x_2) \in \mathbf{E}^2 | x_1 \leqq -\theta,\ x_2 \leqq \theta - z\theta^2 \text{ for some } \theta \geqq 0\}. \quad (38)$$

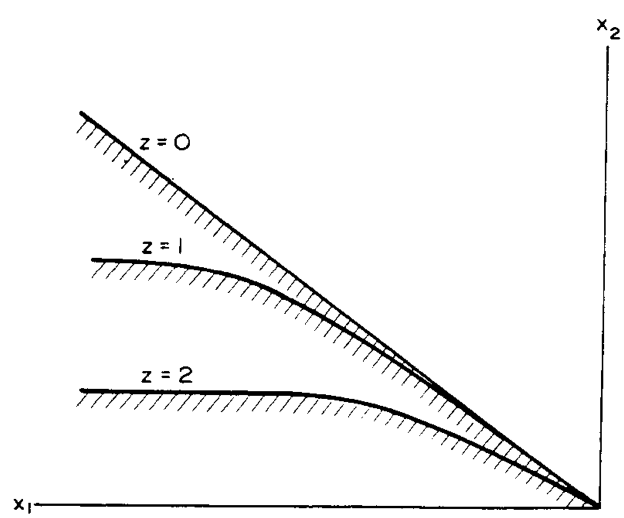


FIGURE 17

The normal cone of this technology is the non-negative orthant of $\mathbf{E}^2$ when z is positive, and is the set of (p_1,p_2) satisfying $0 \leqq p_2 \leqq p_1$ when $z = 0$. Note that this cone is a continuous correspondence at any positive z, but is only a lower hemicontinuous correspondence at $z = 0$. The restricted profit function for this example is

$$\begin{aligned} \Pi(z,\mathbf{p}) &= (p_2 - p_1)^2/4zp_2 \quad \text{if} \quad z > 0 \quad \text{and} \quad p_2 > p_1 \geqq 0, \\ &= 0 \quad \text{if} \quad 0 \leqq p_2 \leqq p_1 \quad \text{and} \quad z \geqq 0. \end{aligned} \tag{39}$$

The set of profit maximizing netput bundles for $z > 0$ is

$$\begin{aligned} \mathbf{\Phi}(z,\mathbf{p}) &= \{((p_1 - p_2)/2zp_2, (p_2^2 - p_1^2)/4zp_2^2)\} \quad \text{for} \quad p_2 > p_1 \geqq 0, \\ &= \{(0,0)\} \quad \text{for} \quad 0 < p_2 \leqq p_1 \\ & \qquad \text{or} \quad 0 = p_2 < p_1, \\ &= \mathbf{T}(\mathbf{z}) \quad \text{for} \quad p_1 = p_2 = 0. \end{aligned} \tag{40a}$$

The set of profit maximizing netput bundles for $z = 0$ is

$$\begin{aligned} \mathbf{\Phi}(0,\mathbf{p}) &= \{(0,0)\} \quad \text{for} \quad 0 \leqq p_2 < p_1, \\ &= \{(-\theta,\theta) | 0 \leqq \theta < +\infty\} \quad \text{for} \quad 0 < p_1 = p_2, \\ &= \mathbf{T}(0) \quad \text{for} \quad 0 = p_1 = p_2. \end{aligned} \tag{40b}$$

For each $z \in \mathbf{Z}$, the set of maximands is seen to be bounded for $\mathbf{p} \in \mathbf{P}^0(z)$, and to be upper semicontinuous on $\mathbf{P}(z)$. Note in this example that for a point such as $\mathbf{p} = (1,2) \in \mathbf{P}(z)$ for $z > 0$ with $\mathbf{p} \notin \mathbf{P}(0)$, one has $\lim_{z\to 0} \Pi(z,\mathbf{p}) = +\infty$, while for a point such as $\mathbf{p} = (2,1) \in \mathbf{P}^0(z)$ for $z \geqq 0$, one has $\lim_{z\to 0} \Pi(z,\mathbf{p}) = \Pi(0,\mathbf{p})$. This property that the profit function is continuous in z and $\mathbf{p}$ jointly at a point with $\mathbf{p} \in \mathbf{P}^0(z)$, and that $\Pi(z,\mathbf{p})$ approaches infinity as $(z,\mathbf{p})$ approach a point at which the price vector is not contained in the corresponding normal cone, is a general one, as the following result shows.

Lemma 16. Suppose Axioms 1–3 hold. Then, the restricted profit function $\Pi(\mathbf{z},\mathbf{p})$ is continuous jointly in $\mathbf{z}$ and $\mathbf{p}$ at each $\mathbf{z}^0 \in \mathbf{Z}$ and $\mathbf{p}^0 \in \mathbf{P}^0(\mathbf{z}^0)$. Further, at any $\mathbf{z}^0 \in \mathbf{Z}$ and $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z}^0)$, $\Pi(\mathbf{z},\mathbf{p})$ is lower semicontinuous in $(\mathbf{z},\mathbf{p})$; i.e., if $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z}^0)$, then $\Pi(\mathbf{z}^0,\mathbf{p}^0) = \lim_{\mathbf{p}^k \in \mathbf{P}(\mathbf{z}^k), (\mathbf{z}^k,\mathbf{p}^k) \to (\mathbf{z}^0,\mathbf{p}^0)} \inf \Pi(\mathbf{z}^k,\mathbf{p}^k)$. Finally, if a sequence $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{p}^k \in \mathbf{P}(\mathbf{z}^k)$ converges to $\mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{p}^0 \notin \mathbf{P}(\mathbf{z}^0)$, then $\lim_k \Pi(\mathbf{z}^k,\mathbf{p}^k) = +\infty$.

Proof: Appendix A.3, Lemma 15.3.

15. The Derivative Property of the Restricted Profit Function

In Section 5, the cost function was shown to have the useful property that its partial derivatives with respect to input prices were equal, when they existed, to the corresponding cost minimizing input demands. We now establish a similar property for the restricted profit function: the vector of partial derivatives of this function with respect to commodity prices, when it exists, equals a unique profit maximizing netput bundle. Further, the vector of partial derivatives is found to exist for almost all commodity vectors. Finally, employing a generalization of the ordinary concept of a derivative, the identification of the "derivative" with the set of profit maximizing netput bundles can be shown to hold for all commodity prices. The first result concerns the differentiability of the restricted profit function.

> *Lemma* 17. Suppose Axioms 1 and 2 hold. For fixed $\mathbf{z} \in \mathbf{Z}$, the profit function $\Pi(\mathbf{z},\mathbf{p})$, considered as a function of $\mathbf{p}$, possesses a first and second differential on a set $\mathbf{P}^2(\mathbf{z}) \subseteq \mathbf{P}^0(\mathbf{z})$, where the set of points in $\mathbf{P}^0(\mathbf{z})$, but not in $\mathbf{P}^2(\mathbf{z})$, has Lebesgue measure zero. The vector of first order partial derivatives of Π with respect to $\mathbf{p}$, denoted by $\Pi_p(\mathbf{z},\mathbf{p})$ and termed the *gradient*, is continuous in $\mathbf{P}^2(\mathbf{z})$. At each $\mathbf{p} \in \mathbf{P}^2(\mathbf{z})$, the matrix of second-order partial derivatives of Π with respect to $\mathbf{p}$, denoted by $\Pi_{pp}(\mathbf{z},\mathbf{p})$ and termed the *Hessian*, is symmetric and non-negative definite.

Proof: Appendix A.3, Lemma 12.1.

From the derivative property of the cost function and its relation to the curvature of the boundary of an input requirement set, as illustrated in Figures 3 and 11; it is clear that the *set* of minimizing bundles coincides with the *set* of normals to "tangent planes", or supporting planes, to the cost function, appropriately scaled. To generalize this concept, we define the *sub-differential* of Π with respect to $\mathbf{p}$ at a point $\mathbf{z} \in \mathbf{Z}$ and $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$ as the set of points $\mathbf{x} \in \mathbf{E}^N$ such that for all $\mathbf{q} \in \mathbf{E}^N$, it follows that

$$\mathbf{q}\cdot\mathbf{x} \leqq \lim_{\theta\to 0^+} \inf(\Pi(\mathbf{z},\mathbf{p}+\theta\mathbf{q}) - \Pi(\mathbf{z},\mathbf{p}))/\theta. \tag{41}$$

When Π is differentiable in $\mathbf{p}$ at $(\mathbf{z},\mathbf{p})$, then the limit of the right-hand side of (41) exists and equals $\mathbf{q}\cdot\Pi_p(\mathbf{z},\mathbf{p})$. Hence, the sub-differential equals $\{\Pi_p(\mathbf{z,p})\}$ when the gradient $\Pi_p(\mathbf{z,p})$ exists. In Figure 11, the sub-differential of a function with the illustrated contour at the "kink" $\mathbf{v}^0$ is the closed line segment joining $\mathbf{r}^0$ and $\mathbf{r}^1$. The next result establishes the properties of the sub-differential of Π with respect to $\mathbf{p}$, which will be denoted hereafter by $\Gamma(\mathbf{z,p})$.

Lemma 18. Suppose Axioms 1 and 2 hold. Then, the sub-differential $\Gamma(\mathbf{z,p})$ exists for all $\mathbf{z} \in \mathbf{Z}$, $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, with the following properties:

(1) $\Gamma(\mathbf{z,p})$ is a non-empty, convex, closed, and bounded set, with $\mathbf{x} \in \Gamma(\mathbf{z,p})$ if and only if $\Pi(\mathbf{z,p}) = \mathbf{p \cdot x}$ and $\Pi(\mathbf{z,q}) \geqq \mathbf{q \cdot x}$ for all $\mathbf{q} \in \mathbf{P}(\mathbf{z})$.

(2) $\Gamma(\mathbf{z,p})$ is an upper hemicontinuous correspondence in $\mathbf{p}$; i.e., if $\mathbf{p}^k \in \mathbf{P}^0(\mathbf{z})$, $\mathbf{x}^k \in \Gamma(\mathbf{z,p}^k)$, $(\mathbf{p}^k,\mathbf{x}^k) \rightarrow (\mathbf{p}^0,\mathbf{x}^0)$ with $\mathbf{p}^0 \in \mathbf{P}^0(\mathbf{z})$, then $\mathbf{x}^0 \in \Gamma(\mathbf{z,p}^0)$.

Proof: Appendix A.3, Lemma 13.8.

We can now state the basic derivative property of the restricted profit function. Recall that $\Phi(\mathbf{z,p})$ denotes the set of profit maximizing netput bundles for $\mathbf{z} \in \mathbf{Z}$ and $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$.

Lemma 19. Suppose Axioms 1 and 2 hold. Then, for $\mathbf{z} \in \mathbf{Z}$ and $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, the sub-differential $\Gamma(\mathbf{z,p})$ equals the convex hull of the set of profit maximizing netput bundles $\Phi(\mathbf{z,p})$. If, for any given $(\mathbf{z,p})$, the sub-gradient contains the unique vector $\Pi_p(\mathbf{z,p})$ [i.e., Π is differentiable at $(\mathbf{z,p})$], then there is a unique profit maximizing netput vector equal to $\Pi_p(\mathbf{z,p})$. If Axiom 4 holds, then $\Phi(\mathbf{z,p}) = \Gamma(\mathbf{z,p})$.

Proof: Appendix A.3, Corollary 13.9, except the last statement, which follows from Lemma 14(4) above.

Figure 18 illustrates the relation established in this result. For the price vector $\mathbf{p}$, the set of profit maximizing netput bundles for this technology is $\Phi(\mathbf{z,p}) = \{\mathbf{x}^1,\mathbf{x}^2\}$, whereas the sub-differential is the set $\Gamma(\mathbf{z,p}) = \{\mathbf{x}|\mathbf{x} = \theta\mathbf{x}^1 + (1-\theta)\mathbf{x}^2,\ 0 \leqq \theta \leqq 1\}$. Hence, all extreme points in $\Gamma(\mathbf{z,p})$ are also in $\Phi(\mathbf{z,p})$, but $\Gamma(\mathbf{z,p})$ may contain additional non-extreme points which are not possible to produce. However, if $\mathbf{T}(\mathbf{z})$ is convex by Axiom 4, these sets coincide exactly.

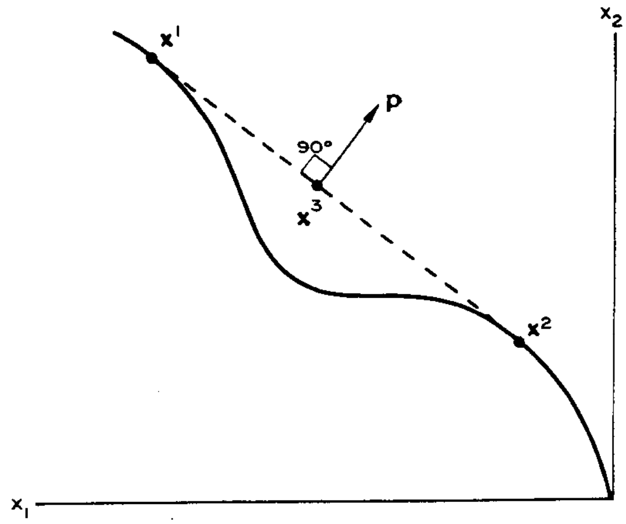


FIGURE 18

16. The Gauge Function for Production Possibilities

The introduction of a distance function to define input requirement sets yielded a convenient symmetry in the treatment of cost functions. A similar concept, the *gauge function* of the production possibility set, plays the same role in analysis of the restricted profit function.

Consider a family of production possibility sets $\mathbf{T}'(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, satisfying Axioms 1–4. By Result 15.6 in Appendix A.3, there exists a continuous function $\mathbf{x}^*(\mathbf{z})$ from $\mathbf{Z}$ into $\mathbf{E}^N$ with $\mathbf{x}^*(\mathbf{z}) \in \mathbf{T}'(\mathbf{z})$. [Further, $\mathbf{x}^*(\mathbf{z})$ can be chosen so that for each $\mathbf{x}' \in \mathbf{T}'(\mathbf{z})$, it follows that $\theta\mathbf{x}' + (1-\theta)\mathbf{x}^*(\mathbf{z}) \in \mathbf{T}'(\mathbf{z})$ for any sufficiently small positive or negative scalar θ. Then $\mathbf{x}^*(\mathbf{z})$ is said to be in the *relative interior* of $\mathbf{T}'(\mathbf{z})$.] Suppose one now redefines quantities of commodities by measuring them from $\mathbf{x}^*(\mathbf{z})$, so that $\mathbf{x} = \mathbf{x}' - \mathbf{x}^*(\mathbf{z})$ becomes a "translated" commodity bundle and $\mathbf{T}(\mathbf{z}) = \{\mathbf{x}' - \mathbf{x}^*(\mathbf{z}) | \mathbf{x}' \in \mathbf{T}'(\mathbf{z})\}$ becomes the "translated" technology. The translated technology continues to satisfy Axioms 1–4, and has the property that $\mathbf{0} \in \mathbf{T}(\mathbf{z})$ for all $\mathbf{z} \in \mathbf{Z}$. If, further, $\mathbf{x}^*(\mathbf{z})$ is in the relative interior of $\mathbf{T}'(\mathbf{z})$, then for any $\mathbf{x} \in \mathbf{T}(\mathbf{z})$, one has $\theta\mathbf{x} \in \mathbf{T}(\mathbf{z})$ for any sufficiently small positive or negative scalar θ, and $\mathbf{0}$ is in the *relative interior* of $\mathbf{T}(\mathbf{z})$. If $\Pi'(\mathbf{z},\mathbf{p})$, $\mathbf{p} \in \mathbf{P}'(\mathbf{z})$, is the restricted profit function of the original technology and $\Pi(\mathbf{z},\mathbf{p})$, $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, is the restricted profit function of the translated technology, for *any* translation $\mathbf{x}^*(\mathbf{z})$, then these functions are related by $\mathbf{P}'(\mathbf{z}) = \mathbf{P}(\mathbf{z})$ and $\Pi'(\mathbf{z},\mathbf{p}) = \Pi(\mathbf{z},\mathbf{p}) + \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$.

Consider a translated technology $\mathbf{T}(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, satisfying Axioms 1–4, and containing the origin. Define the set

$$\mathbf{W}(\mathbf{z}) = \left\{ \mathbf{x} \in \mathbf{E}^N \,\middle|\, \frac{1}{\lambda}\mathbf{x} \in \mathbf{T}(\mathbf{z}) \text{ for some } \lambda > 0 \right\}. \tag{42}$$

Define the *gauge function* of the translated technology by

$$\begin{aligned} H(\mathbf{z},\mathbf{x}) &= +\infty && \text{if} \quad \mathbf{x} \notin \mathbf{W}(\mathbf{z}), \\ &= \inf \left\{ \lambda > 0 \,\middle|\, \frac{1}{\lambda}\mathbf{x} \in \mathbf{T}(\mathbf{z}) \right\} && \text{if} \quad \mathbf{x} \in \mathbf{W}(\mathbf{z}). \end{aligned} \tag{43}$$

Figure 19 illustrates the definition of this function. The following lemma gives its basic properties.

Lemma 20. Suppose a translated technology $\mathbf{T}(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, satisfies Axioms 1, 2, and 4 and contains the origin. Then, the following properties hold:

(1) $\mathbf{W}(\mathbf{z})$ is a convex cone in $\mathbf{E}^N$ for each $\mathbf{z} \in \mathbf{Z}$, and if Axiom 3 holds is a lower hemicontinuous correspondence [i.e., $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$ implies the existence of $\mathbf{x}^k \in \mathbf{W}(\mathbf{z}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$]. If the origin is in the relative interior of $\mathbf{T}(\mathbf{z})$, then $\mathbf{W}(\mathbf{z})$ is a linear subspace.

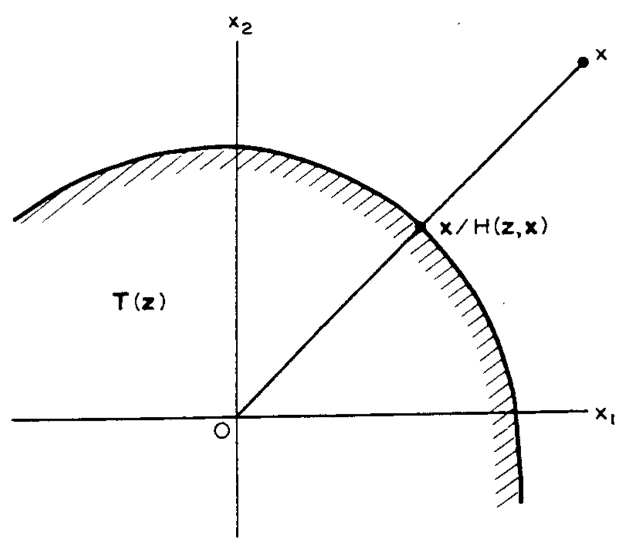


FIGURE 19

(2) For each $\mathbf{z} \in \mathbf{Z}$, $H(\mathbf{z},\mathbf{x})$ is a non-negative convex, closed,[14] positively linear homogeneous function of $\mathbf{x} \in \mathbf{W}(\mathbf{z})$.
(3) For each $\mathbf{z} \in \mathbf{Z}$, $\mathbf{T}(\mathbf{z}) = \mathbf{T}'(\mathbf{z})$, where

$$\mathbf{T}'(\mathbf{z}) = \{\mathbf{x} \in \mathbf{W}(\mathbf{z}) | H(\mathbf{z},\mathbf{x}) \leqq 1\}, \tag{44}$$

and $\mathbf{T}'(\mathbf{z})$ is semi-bounded; i.e., its asymptotic cone, given by the set of $\mathbf{x}$ with $H(\mathbf{z},\mathbf{x}) = 0$, is semi-bounded.
(4) For each $\mathbf{z} \in \mathbf{Z}$, $H(\mathbf{z},\mathbf{x})$ is continuous in $\mathbf{x}$ in the relative interior of $\mathbf{W}(\mathbf{z})$.
(5) If Axiom 3 holds, then $H(\mathbf{z},\mathbf{x})$ is lower semicontinuous jointly in $\mathbf{z}$ and $\mathbf{x}$ at each $\mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$; i.e., for each sequence $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \rightarrow \mathbf{z}^0$, one has $H(\mathbf{z}^0,\mathbf{x}^0) = \lim_{\mathbf{x}^k \in \mathbf{W}(\mathbf{z}^k), \mathbf{x}^k \rightarrow \mathbf{x}^0} \inf H(\mathbf{z}^k,\mathbf{x}^k)$.
(6) If Axiom 3 holds, and if $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \rightarrow \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^k \in \mathbf{W}(\mathbf{z}^k)$, $\mathbf{x}^k \rightarrow \mathbf{x}^0 \notin \mathbf{W}(\mathbf{z}^0)$, then $\lim_k H(\mathbf{z}^k,\mathbf{x}^k) = +\infty$.

Proof: By Lemma 16.2 in Appendix A.3, $\mathbf{W}(\mathbf{z})$ is a cone, and if $\mathbf{0}$ is in the relative interior of $\mathbf{T}(\mathbf{z})$, $\mathbf{W}(\mathbf{z})$ is a linear subspace. The convexity of $\mathbf{W}(\mathbf{z})$ is trivial. Hence (2), (3) hold. To show that $\mathbf{W}(\mathbf{z})$ is lower hemicontinuous, note that $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$ implies $\mathbf{x}^0/\lambda \in \mathbf{T}(\mathbf{z}^0)$ for some $\lambda > 0$. By Axiom 3, there exist $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$ for $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \rightarrow \mathbf{z}^0$ such that $\mathbf{x}^k \rightarrow \mathbf{x}^0/\lambda$. Since $\mathbf{T}(\mathbf{z}^k) \subseteq \mathbf{W}(\mathbf{z}^k)$, this establishes lower hemicontinuity. Result (4) is an implication of Appendix A.3, Lemma 12.1. To show (5), consider a sequence of positive scalars λ_k satisfying $\lambda_k > H(\mathbf{z}^k,\mathbf{x}^k) > \lambda_k - k^{-1}$, where $(\mathbf{z}^k,\mathbf{x}^k)$ is a sequence with $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{x}^k \in \mathbf{W}(\mathbf{z}^k)$ converging to $\mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$. Then, $\mathbf{x}^k/\lambda_k \in \mathbf{T}(\mathbf{z}^k)$. By strong continuity, if $\lim_k \lambda_k = 0$, then $\mathbf{x}^0$ is in the asymptotic cone of $\mathbf{T}(\mathbf{z}^0)$, implying $H(\mathbf{z}^0,\mathbf{x}^0) = 0$ by the definition of H. If $\lim_k \lambda_k = +\infty$, then $H(\mathbf{z}^0,\mathbf{x}^0) \leqq \lim_k H(\mathbf{z}^k,\mathbf{x}^k)$ trivially. Finally, if $\lim_k \lambda_k = \lambda_0 > 0$, then $\mathbf{x}^0/\lambda_0 \in \mathbf{T}(\mathbf{z}^0)$ by strong continuity, implying $H(\mathbf{z}^0,\mathbf{x}^0) \leqq \lim_k H(\mathbf{z}^k,\mathbf{x}^k)$. We next show that there exist $\{\mathbf{x}^k\}$ for which equality is achieved in this limit; this will prove lower hemicontinuity. If $H(\mathbf{z}^0,\mathbf{x}^0) = \lambda > 0$, then $\mathbf{x}^0/\lambda \in \mathbf{T}(\mathbf{z}^0)$, and Axiom 3 implies the existence of $\mathbf{x}^k/\lambda \in \mathbf{T}(\mathbf{z}^k)$, $\mathbf{x}^k \rightarrow \mathbf{x}^0$. Then, $\lim_k H(\mathbf{z}^k,\mathbf{x}^k) \leqq H(\mathbf{z}^0,\mathbf{x}^0)$, implying with the previously established inequality that equality is achieved. If $H(\mathbf{z}^0,\mathbf{x}^0) = 0$, then $j\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$ for each integer j. By Axiom 3, there exists $\mathbf{x}^{jk} \in \mathbf{T}(\mathbf{z}^k)$ with $\mathbf{x}^{jk} \rightarrow j\mathbf{x}^0$. Then, $\mathbf{x}^k = \mathbf{x}^{kk}/k$ has $\mathbf{x}^k \rightarrow \mathbf{x}^0$ and $H(\mathbf{z}^k,\mathbf{x}^k) \leqq k^{-1}$, and $H(\mathbf{z}^0,\mathbf{x}^0) = \lim_k H(\mathbf{z}^k,\mathbf{x}^k)$.

To show (6), note that $\mathbf{x}^k \in \mathbf{W}(\mathbf{z}^k)$ implies the existence of $\lambda_k > 0$ such

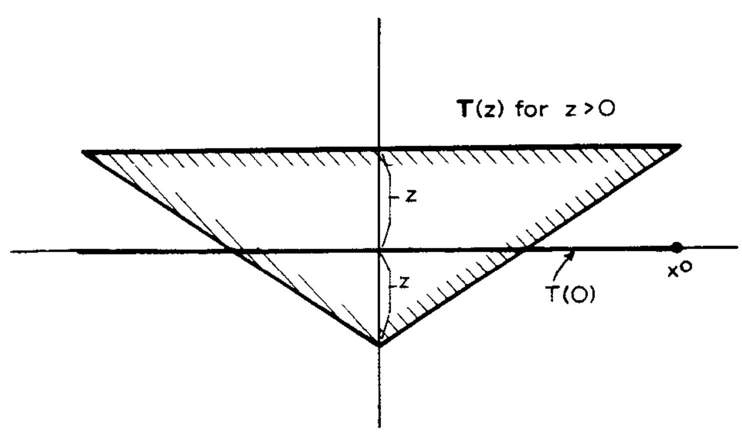
[14]The function $H(\mathbf{z},\mathbf{x})$ is *closed* in $\mathbf{x}$ (for fixed $\mathbf{z}$) if the set $\{(\mathbf{x},h) \in \mathbf{E}^N \times \mathbf{E} | h \geqq H(\mathbf{z},\mathbf{x})\}$ is closed.

that $\mathbf{x}^k/\lambda_k \in \mathbf{T}(\mathbf{z}^k)$ and $\lambda_k < H(\mathbf{z}^k,\mathbf{x}^k) + k^{-1}$. If λ_k is bounded, then there exists a subsequence (retain notation) converging to a scalar λ. If $\lambda > 0$, then $\mathbf{x}^0/\lambda \in \mathbf{T}(\mathbf{z}^0)$ by continuity of $\mathbf{T}$, implying $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$. If $\lambda = 0$, then $\mathbf{x}^0$ is contained in the asymptotic cone of $\mathbf{T}(\mathbf{z}^0)$ by the strong continuity of $\mathbf{T}$, implying $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$. Hence, $\mathbf{x}^0 \notin \mathbf{W}(\mathbf{z}^0)$ implies λ_k unbounded, and $\lim_k H(\mathbf{z}^k,\mathbf{x}^k) = +\infty$. Q.E.D.

Figure 20 shows that conclusion (5) cannot in general be strengthened to imply continuity. However, the following corollary can be established.

Corollary 21. Suppose a translated technology $\mathbf{T}(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, satisfies Axioms 1–4 and contains the origin in its interior. Then $\mathbf{W}(\mathbf{z}) = \mathbf{E}^N$ for all $\mathbf{z} \in \mathbf{Z}$, and $H(\mathbf{z},\mathbf{x})$ is continuous jointly in $\mathbf{z}$ and $\mathbf{x}$ at each $\mathbf{z} \in \mathbf{Z}$, $\mathbf{x} \in \mathbf{E}^N$.

Proof: Suppose $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, and $\mathbf{x}^k \to \mathbf{x}^0$. For any $\lambda > H(\mathbf{z}^0,\mathbf{x}^0)$, it follows from the continuity of H in $\mathbf{x}$ that $\lambda > H(\mathbf{z}^0,\mathbf{x}^k)$ for $k \geqq k_1$, with k_1 some large integer. The set $\mathbf{R} = \{\mathbf{x}^0/\lambda, \mathbf{x}^{k_1}/\lambda, \mathbf{x}^{k_1+1}/\lambda, \ldots\}$ is then closed and bounded, and is contained in the interior of $\mathbf{T}(\mathbf{z}^0)$. By Lemma 13.3(3) in Appendix A.3, there then exists $k_0 \geqq k_1$ such that $\mathbf{R}$ is contained in the interior of $\mathbf{T}(\mathbf{z}^k)$ for $k \geqq k_0$. But this implies $H(\mathbf{z}^k,\mathbf{x}^k) \leqq \lambda$, and hence, taking $\lambda \to H(\mathbf{z}^0,\mathbf{x}^0)$, $\lim_k H(\mathbf{z}^k,\mathbf{x}^k) \leqq H(\mathbf{z}^0,\mathbf{x}^0)$. Combined with lower semi-



$\mathbf{Z} = [0,1]$. Define $\mathbf{x}^1(z) = (1,z)$, $\mathbf{x}^2(z) = (-1,z)$, $\mathbf{x}^3(z) = (0,-z)$, and define $\mathbf{T}(z)$ to be the convex hull of these three points. Then, for $\mathbf{x}^0 = (1,0)$, one has $H(z,\mathbf{x}^0) = 2$ for $z > 0$ and $H(0,\mathbf{x}^0) = 1$.

FIGURE 20

continuity established in Lemma 20(5), this establishes the result. Q.E.D.

It should be noted that the gauge function has many of the same mathematical properties as the restricted profit function (compare Lemmas 11 and 15 versus 20). In particular, under the hypotheses of Corollary 21, the gauge function has all the properties that were demonstrated for the restricted profit function. The next result establishes that all the implications that can be drawn for the gauge function from Axioms 1–4 can also be drawn from the properties given by Lemma 20.

Lemma 22. Let $\mathbf{Z}$ be a non-empty subset of $\mathbf{E}^M$, $\mathbf{W}(\mathbf{z})$ be a linear subspace in $\mathbf{E}^N$ which is a lower hemicontinuous correspondence, and $H(\mathbf{z},\mathbf{x})$ be a non-negative convex, closed, positively linear homogeneous function of $\mathbf{x} \in \mathbf{W}(\mathbf{z})$ for each $\mathbf{z} \in \mathbf{Z}$. Suppose that $H(\mathbf{z},\mathbf{x})$ satisfies properties (5) and (6) of Lemma 20, and that the set of $\mathbf{x}$ satisfying $H(\mathbf{z},\mathbf{x}) = 0$ is semi-bounded for each $\mathbf{z} \in \mathbf{Z}$. Then, the correspondence $\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{W}(\mathbf{z}) | H(\mathbf{z},\mathbf{x}) \leqq 1\}$ satisfies Axioms 1–4.

Proof: It is immediate that $\mathbf{T}(\mathbf{z})$ satisfies Axioms 1, 2, and 4. To verify that Axiom 3 holds, consider $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$. Suppose $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$. Then, $H(\mathbf{z}^k,\mathbf{x}^k) \leqq 1$ implies by property (6) in Lemma 20 that $\mathbf{x}^0 \in \mathbf{W}(\mathbf{z}^0)$, and implies by property (5) in Lemma 20 that $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$. Alternately, suppose $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$. By property (5) of Lemma 20, there exist $\mathbf{y}^k \in \mathbf{W}(\mathbf{z}^k)$ such that $\mathbf{y}^k \to \mathbf{x}^0$ and $\lim_k H(\mathbf{z}^k,\mathbf{y}^k) = H(\mathbf{z}^0,\mathbf{x}^0)$. Choose $\lambda_k > 0$ such that $\lambda_k - k^{-1} < H(\mathbf{z}^k,\mathbf{x}^k) < \lambda_k$. Then, $\lim \lambda_k \leqq 1$. Define $\mathbf{x}^k = \mathbf{y}^k/\text{Max}(\lambda_k,1)$. Then, $\lim_k \mathbf{y}^k = \lim_k \mathbf{x}^k = \mathbf{x}^0$ and $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$. Thus, $\mathbf{T}(\mathbf{z})$ is continuous. Lemma 13.3(2) in Appendix A.3 then implies that $\mathbf{T}(\mathbf{z})$ is strongly continuous. Q.E.D.

Using Lemmas 20 and 22, we can take the production possibility set and the gauge function as interchangeable descriptions of the technology when Axioms 1–4 hold. We conclude our analysis of the gauge function by noting its relation to the "gauge function" of the original technology $\mathbf{T}'(\mathbf{z})$. Recalling that $\mathbf{T}(\mathbf{z}) = \{\mathbf{x}' - \mathbf{x}^*(\mathbf{z}) \in \mathbf{E}^N | \mathbf{x}' \in \mathbf{T}'(\mathbf{z})\}$, define the *gauge function* relative to $\mathbf{x}^*(\mathbf{z})$ of the original technology $\mathbf{T}'(\mathbf{z})$ by

$$H'(\mathbf{z},\mathbf{x}') = H(\mathbf{z},\mathbf{x}' - \mathbf{x}^*(\mathbf{z})), \tag{45}$$

where $H(\mathbf{z},\mathbf{x})$ is the gauge function of the translated technology $\mathbf{T}(\mathbf{z})$. The reader can verify that H' is finite on an affine subspace of $\mathbf{x} \in \mathbf{E}^N$

for each $\mathbf{z} \in \mathbf{Z}$, that $\mathbf{T}'(\mathbf{z}) = \{\mathbf{x}' \in \mathbf{E}^N | H'(\mathbf{z},\mathbf{x}') \leqq 1\}$, and that H' has the same mathematical properties as H, with the exception of linear homogeneity.

17. Duality for the Restricted Profit Function

It has been established that a technology satisfying Axioms 1 and 2 yields a restricted profit function with the properties stated in Lemma 11, and that the technology can be completely recovered from a knowledge of the restricted profit function, provided in addition Axiom 4 holds, by use of the mapping (37). We will now show, conversely, that a function with the properties of a restricted profit function will yield, via the mapping (37), a technology satisfying Axioms 1, 2, and 4, and that this technology returns the original function via the mapping (35). From these results, we will have obtained a generalization of the Shephard–Uzawa duality theorem to the case of the restricted profit function. This generalization will allow the use of the duality principle in applications for a broad range of environments of the competitive firm.

> *Lemma* 23. Suppose $\mathbf{Z}$ is a non-empty subset of $\mathbf{E}^M$, and that for each $\mathbf{z} \in \mathbf{Z}$, $\mathbf{P}(\mathbf{z})$ is a convex cone with a non-empty interior, and $\Pi(\mathbf{z},\mathbf{p})$ is a convex, positively linear homogeneous, closed function of $\mathbf{p} \in \mathbf{P}(\mathbf{z})$. Then, $\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{E}^N | \mathbf{p} \cdot \mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p})$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})\}$ satisfies Axioms 1, 2, and 4, and $\Pi(\mathbf{z},\mathbf{p})$ is the restricted profit function of $\mathbf{T}(\mathbf{z})$ as defined by equation (35).

Proof: Appendix A.3, Lemma 12.5.

Note that the basic duality theorem expressed by Lemmas 11 and 23 is an immediate consequence of the basic mathematical theory of convex conjugate functions. As in the case of cost functions, the restricted profit function will fail to distinguish between distinct technologies which have the same convex hull, reflecting the fact that two competitive firms with these respective technologies may exhibit identical behavior.

Utilizing the concept of a gauge function introduced in the previous section, we can summarize these duality relations in the following formal duality theorem.

Theorem 24. Suppose $\mathbf{Z}$ is a non-empty subset of $\mathbf{E}^M$ and $\mathbf{x}^*(\mathbf{z})$ is a function from $\mathbf{Z}$ into $\mathbf{E}^N$.

(a) Let $\mathscr{T}$ denote the class of sets $\mathbf{T}(\mathbf{z})$, $\mathbf{z} \in \mathbf{Z}$, which satisfy Axioms 1, 2, and 4 and have $\mathbf{x}^*(\mathbf{z})$ in the relative interior of $\mathbf{T}(\mathbf{z})$; i.e., for each $\mathbf{x} \in \mathbf{T}(\mathbf{z})$, it follows that $\theta\mathbf{x} + (1-\theta)\mathbf{x}^*(\mathbf{z}) \in \mathbf{T}(\mathbf{z})$ for small positive or negative scalars θ.

(b) Let $\mathscr{H}$ denote the class of pairs $\langle H,\mathbf{X}\rangle$, where for each $\mathbf{z} \in \mathbf{Z}$, the set $\{\mathbf{x}-\mathbf{x}^*(\mathbf{z})|\mathbf{x} \in \mathbf{X}(\mathbf{z})\}$ is a linear subspace of $\mathbf{E}^N$ and $H(\mathbf{z},\mathbf{x})$ is a non-negative convex, closed function of $\mathbf{x} \in \mathbf{X}(\mathbf{z})$ with $H(\mathbf{z},\mathbf{x}^*(\mathbf{z}) + \theta(\mathbf{x}-\mathbf{x}^*(\mathbf{z}))) = \theta H(\mathbf{z},\mathbf{x})$ for $\mathbf{x} \in \mathbf{X}(\mathbf{z})$, $\theta > 0$, and with the set of $\mathbf{x} \in \mathbf{X}(\mathbf{z})$ for which $H(\mathbf{z},\mathbf{x}) = 0$ a semi-bounded set.

(c) Let $\mathscr{P}$ denote the class of pairs $\langle \Pi,\mathbf{P}\rangle$, where for each $\mathbf{z} \in \mathbf{Z}$, the set $\mathbf{P}(\mathbf{z})$ is a convex cone in $\mathbf{E}^N$ with a non-empty interior and $\Pi(\mathbf{z},\mathbf{p})$ is a convex, closed, positively linear homogeneous function of $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ such that if $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ and either $-\mathbf{p} \notin \mathbf{P}(\mathbf{z})$, or $-\mathbf{p} \in \mathbf{P}(\mathbf{z})$ and $\Pi(\mathbf{z},\mathbf{p}) \neq -\Pi(\mathbf{z},-\mathbf{p})$, then $\Pi(\mathbf{z},\mathbf{p}) > \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$.

(Note that $\mathscr{T}$ is a class of production possibility sets, $\mathscr{H}$ is a class of non-translated gauge functions, and $\mathscr{P}$ is a class of restricted profit functions.)

(d) Define a *profit mapping* from $\mathbf{T} \in \mathscr{T}$ to pairs $\langle \Pi,\mathbf{P}\rangle$ satisfying $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ if and only if $\sup\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x} \in \mathbf{T}(\mathbf{z})\} < +\infty$, and $\Pi(\mathbf{z},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x} \in \mathbf{T}(\mathbf{z})\}$ for $\mathbf{p} \in \mathbf{P}(\mathbf{z})$. Then, the image of the profit mapping is a unique element in $\mathscr{P}$.

(e) Define an *implicit technology mapping* from $\langle \Pi,\mathbf{P}\rangle \in \mathscr{P}$ to sets $\mathbf{T}(\mathbf{z}) \subseteq \mathbf{E}^N$ satisfying $\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{E}^N|\mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p})$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})\}$. Then, the image of the implicit technology mapping is a unique element in $\mathscr{T}$. Further, the profit and implicit technology mappings are mutual inverses.

(f) Define a *gauge mapping* from $\mathbf{T} \in \mathscr{T}$ to pairs $\langle H,\mathbf{X}\rangle$ satisfying $\mathbf{x} \in \mathbf{X}(\mathbf{z})$ if and only if $\theta\mathbf{x} + (1-\theta)\mathbf{x}^*(\mathbf{z}) \in \mathbf{T}(\mathbf{z})$ for small positive θ, and $H(\mathbf{z},\mathbf{x}) = \inf\{\lambda > 0|\mathbf{x}^*(\mathbf{z}) + (\mathbf{x}-\mathbf{x}^*(\mathbf{z}))/\lambda \in \mathbf{T}(\mathbf{z})\}$ for $\mathbf{x} \in \mathbf{X}(\mathbf{z})$. Then, the image of the gauge mapping is a unique element in $\mathscr{H}$.

(g) Define an *inverse gauge mapping* from $\langle H,\mathbf{X}\rangle \in \mathscr{H}$ to sets $\mathbf{T}(\mathbf{z}) \subseteq \mathbf{E}^N$ satisfying $\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{X}(\mathbf{z})|H(\mathbf{x},\mathbf{z}) \leqq 1\}$. Then, the image of the inverse gauge mapping is a unique element in $\mathscr{T}$. Further, the gauge and inverse gauge mappings are mutual inverses.

(h) Define an *implicit profit mapping* from $\langle H,\mathbf{X}\rangle \in \mathscr{H}$ to pairs $\langle \Pi,\mathbf{P}\rangle$ satisfying $\Pi(\mathbf{z},\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \inf\{\lambda > 0|\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}^*(\mathbf{z})) \leqq \lambda H(\mathbf{z},\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}(\mathbf{z})\}$ for $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, with $\mathbf{P}(\mathbf{z})$ defined as the set of $\mathbf{p} \in \mathbf{E}^N$

for which the set of λ in the right-hand side of this expression is non-empty. Then, the implicit profit mapping is the composition of the inverse gauge mapping and the profit mapping, and its image is a unique element in $\mathcal{P}$.

(i) Define an *implicit gauge mapping* from $\langle \Pi, \mathbf{P} \rangle \in \mathcal{P}$ to pairs $\langle H, \mathbf{X} \rangle$ satisfying $H(\mathbf{z},\mathbf{x}) = \inf\{\lambda > 0 | \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda(\Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}))$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})\}$ for $\mathbf{x} \in \mathbf{X}(\mathbf{z})$, with $\mathbf{X}(\mathbf{z})$ defined as the set of $\mathbf{x} \in \mathbf{R}^N$ for which the set of λ in the right-hand side of this expression is non-empty. Then, the implicit gauge mapping is the composition of the implicit technology mapping and the gauge mapping, and its image is a unique element in $\mathcal{H}$. Further, the implicit profit and implicit gauge mappings are mutual inverses.

Corollary. Suppose in Theorem 24, condition (a) is modified to require only that $\mathbf{x}^*(\mathbf{z})$ be contained in $\mathbf{T}(\mathbf{z})$ rather than in its relative interior, condition (b) is modified to require that the set $\{\mathbf{x} - \mathbf{x}^*(\mathbf{z}) | \mathbf{x} \in \mathbf{X}(\mathbf{z})\}$ be a convex cone rather than a linear subspace, and condition (c) is modified to weaken the implication $\Pi(\mathbf{z},\mathbf{p}) > \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$ to $\Pi(\mathbf{z},\mathbf{p}) \geqq \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$. Then the conclusions of the theorem continue to hold.

Proof: Consider statement (d). Lemma 11 establishes $\langle \Pi, \mathbf{P} \rangle$ to be a unique element in $\mathcal{P}$, provided that Π satisfies the last property in (c). But (a) implies that if the conclusion of this property is false, so that $\Pi(\mathbf{z},\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$ for some $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, then $\mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) = \mathbf{p}\cdot\mathbf{x}$ for all $\mathbf{x} \in \mathbf{T}(\mathbf{z})$. Then, $\Pi(\mathbf{z},\mathbf{p}) = \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) = -\Pi(\mathbf{z},-\mathbf{p})$, and the hypothesis for the last property in (c) is also false.

Consider statement (e). Lemma 23 and Lemma 11(4) imply all the results, provided we show that $\mathbf{x}^*(\mathbf{z})$ is in the relative interior of $\mathbf{T}(\mathbf{z})$. If this conclusion failed to hold, then by Result 10.12 in Appendix A.3, there would exist $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ with $\mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) \geqq \Pi(\mathbf{z},\mathbf{p}) > \mathbf{p}\cdot\mathbf{x}$ for some $\mathbf{x} \in \mathbf{T}(\mathbf{z})$, and the last condition defining the class $\mathcal{P}$ would be contradicted. Hence, $\mathbf{x}^*(\mathbf{z})$ is in the relative interior of $\mathbf{T}(\mathbf{z})$.

Consider statements (f) and (g). Except for two propositions, Lemmas 20 and 22 imply the results directly. First, we need to show that the composition of the mappings from $\mathcal{H}$ to $\mathcal{T}$ and from $\mathcal{T}$ to $\mathcal{H}$ is the identity mapping. If not, there exist distinct gauge functions H^1 and H^2 such that H^1 maps into T^1 in $\mathcal{T}$ and T^1 maps into H^2. But Lemma 20(3) implies that H^2 maps into T^1. Hence $H^1(\mathbf{z},\mathbf{x}) = 1$ implies $H^2(\mathbf{z},\mathbf{x}) = 1$, and positive linear homogeneity in $\mathbf{x}$ implies $H^1 = H^2$. Second, we need to show that $\mathbf{x}^*(\mathbf{z})$ is in the image of the inverse gauge mapping, and in its relative interior when $\mathbf{X}(\mathbf{z})$ is an affine linear subspace. The definition of H implies $H(\mathbf{z},\mathbf{x}^*(\mathbf{z})) = 0$, implying $\mathbf{x}^*(\mathbf{z})$ is in $\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{X}(\mathbf{z}) | H(\mathbf{z},\mathbf{x}) \leqq 1\}$. Since H is convex and closed, it is continuous on $\mathbf{X}(\mathbf{z})$ when $\mathbf{X}(\mathbf{z})$ is an affine linear subspace, implying $\mathbf{x}^*(\mathbf{z})$ is in the relative interior of $\mathbf{T}(\mathbf{z})$.

Finally, consider statements (h) and (i). In (h), $\mathbf{x} \in \mathbf{X}(\mathbf{z})$ implies either $H(\mathbf{z},\mathbf{x}) = 0$ and $\mathbf{p}\cdot\mathbf{x} \leqq \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, or $H(\mathbf{z},\mathbf{x}) > 0$ and $\mathbf{x}^*(\mathbf{z}) + (\mathbf{x} - \mathbf{x}^*(\mathbf{z}))/H(\mathbf{z},\mathbf{x}) \in \mathbf{T}(\mathbf{z})$, with $\mathbf{T}$ defined by the inverse gauge mapping. Then,

$$\begin{aligned}\Pi(\mathbf{z},\mathbf{p}) &= \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \sup\{\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))/H(\mathbf{z},\mathbf{x}) | \mathbf{x} \in \mathbf{X}(\mathbf{z}),\ H(\mathbf{z},\mathbf{x}) > 0\} \\ &= \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \inf\{\lambda > 0 | \lambda > \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))/H(\mathbf{z},\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{X}(\mathbf{z}) \\ &\quad \text{such that } H(\mathbf{z},\mathbf{x}) > 0\} \\ &= \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \inf\{\lambda > 0 | \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda H(\mathbf{z},\mathbf{x}) \text{for all } \mathbf{x} \in \mathbf{X}(\mathbf{z})\}.\end{aligned}$$

Hence, the implicit profit mapping is a composition of the inverse gauge and profit mappings. In (i),

$$\begin{aligned}H(\mathbf{z},\mathbf{x}) &= \inf\{\lambda > 0 | \mathbf{x}^*(\mathbf{z}) + (\mathbf{x} - \mathbf{x}^*(\mathbf{z}))/\lambda \in \mathbf{T}(\mathbf{z})\} \\ &= \inf\{\lambda > 0 | \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))/\lambda \leqq \Pi(\mathbf{z},\mathbf{p}) \text{for all } \mathbf{p} \in \mathbf{P}(\mathbf{z})\} \\ &= \inf\{\lambda > 0 | \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda(\Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})) \text{for all } \mathbf{p} \in \mathbf{P}(\mathbf{z})\}.\end{aligned}$$

Hence, the implicit gauge mapping is a composition of the implicit technology mapping and the gauge mapping. Q.E.D.

Note that the implicit profit mapping and the implicit gauge mapping have similar formal structures, making the "duality" essentially complete. If one takes $\mathbf{x}^*(\mathbf{z}) = \mathbf{0}$ and considers a subclass of $\mathscr{H}$ for which $\mathbf{X}(\mathbf{z}) = \mathbf{E}^N$ and a subclass of $\mathscr{P}$ for which $\mathbf{P}(\mathbf{z})$ is semi-bounded, then the duality is complete in the sense that the members of H and Π have symmetric properties, and the implicit profit and gauge mappings are identical, except for the change of variables. For dual profit and gauge functions $\Pi(\mathbf{z},\mathbf{p})$ and $H(\mathbf{z},\mathbf{x})$, the inequality

$$\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq H(\mathbf{z},\mathbf{x})(\Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})) \tag{46}$$

holds for all $\mathbf{p}$ and $\mathbf{x}$ in $\mathbf{E}^N$, with equality when $\mathbf{x} \in \Phi(\mathbf{z},\mathbf{p})$.

For the classes of gauge and profit functions defined in Theorem 24, we can establish further structural relations of the form "the gauge function has property 'P' if and only if the restricted profit function has property 'Q'," in the way that Lemma 7 established structural relationships for cost and distance functions. Hereafter, we shall assume the translation of production possibility sets described in Section 16 has been carried out, so that $\mathbf{x}^*(\mathbf{z}) \equiv \mathbf{0}$ in the hypothesis of Theorem 24, the origin is contained in the relative interior of sets $\mathbf{T} \in \mathscr{T}$, $\mathbf{X}(\mathbf{z})$ is a linear subspace of $\mathbf{E}^N$ for each $\langle H,\mathbf{X}\rangle \in \mathscr{H}$, and each $\langle \Pi,\mathbf{P}\rangle \in \mathscr{P}$ satisfies $\Pi(\mathbf{z},\mathbf{p}) > 0$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ such that either $\Pi(\mathbf{z},\mathbf{p}) \neq -\Pi(\mathbf{z},-\mathbf{p})$ or $-\mathbf{p} \notin \mathbf{P}(\mathbf{z})$. Let $\mathscr{T}^0$, $\mathscr{H}^0$, and $\mathscr{P}^0$ denote the classes $\mathscr{T}$, $\mathscr{H}$, $\mathscr{P}$, respectively, for this case of $\mathbf{x}^*(\mathbf{z}) \equiv \mathbf{0}$. The qualitative structural relationships derived under this assumption continue to hold in the case of a general non-zero $\mathbf{x}^*(\mathbf{z})$, with an appropriate modification of the definition of the sub-differential for the gauge function and of an exposed value of $\mathbf{x}$ for this function (defined below). This last stage of generalization, which considerably complicates notation without adding new results, will be left to the reader.

Recalling that $\mathbf{z} \in \mathbf{Z}$ is a vector in $\mathbf{E}^M$, let $\mathbf{z} = (\mathbf{z}_{(1)},\mathbf{z}_{(2)})$ denote a partition of $\mathbf{z}$ into sub-vectors. The gauge function $H(\mathbf{z},\mathbf{x})$ is defined to be *non-increasing* in $\mathbf{z}_{(1)}$ if for any $\mathbf{z}',\mathbf{z}'' \in \mathbf{Z}$ with $\mathbf{z}'_{(1)} \geqq \mathbf{z}''_{(1)}$ and $\mathbf{z}'_{(2)} = \mathbf{z}''_{(2)}$, it follows that $H(\mathbf{z}',\mathbf{x}) \leqq H(\mathbf{z}'',\mathbf{x})$, where $H(\mathbf{z},\mathbf{x})$ is allowed to assume the value $+\infty$ if $\mathbf{x} \notin \mathbf{X}(\mathbf{z})$. The gauge function is defined to be *uniformly decreasing* in $\mathbf{z}_{(1)}$ if for any distinct $\mathbf{z}'$, $\mathbf{z}'' \in \mathbf{Z}$ with $\mathbf{z}'_{(1)} \geqq \mathbf{z}''_{(1)}$ and $\mathbf{z}'_{(2)} = \mathbf{z}''_{(2)}$, it follows that there exists a positive scalar λ such that $(1+\lambda)H(\mathbf{z}',\mathbf{x}) \leqq H(\mathbf{z}'',\mathbf{x})$, where again the value $+\infty$ is allowed for $H(\mathbf{z},\mathbf{x})$ when $\mathbf{x}$ fails to lie in $\mathbf{X}(\mathbf{z})$. Clearly, H is non-increasing in $\mathbf{z}_{(1)}$ if and only if the production possibility set $\mathbf{T}(\mathbf{z})$ satisfies $\mathbf{T}(\mathbf{z}'') \subseteq \mathbf{T}(\mathbf{z}')$. This could be expected to be the case, for example, if the components of $\mathbf{z}_{(1)}$ are fixed inputs to the production process (measured with a positive sign) or are indices of the level of technical progress. Analogous definitions can be made for the gauge function non-decreasing or uniformly increasing in a sub-vector $\mathbf{z}_{(1)}$, or for the restricted profit function weakly or uniformly monotone in $\mathbf{z}_{(1)}$.

Theorem 25. Suppose $\mathbf{Z}$ is a non-empty subset of $\mathbf{E}^M$, and consider the class of gauge functions $\mathscr{H}^0$ and the class of restricted profit functions $\mathscr{P}^0$. Then, for gauge and restricted profit functions in these classes which are dual under the implicit profit and implicit gauge mappings of Theorem 24, the structural relationships given in Table 3 hold; i.e., the gauge function has the property "P" if and only if the restricted profit function has the property "Q".

Proof: *Result* 1 – If $\mathbf{X}(\mathbf{z}) = \mathbf{E}^N$, then for any $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, $\mathbf{p} \neq \mathbf{0}$, one has $H(\mathbf{z},\mathbf{p}) < +\infty$, implying $H(\mathbf{z},\lambda\mathbf{p}) \leqq 1$ for some $\lambda > 0$, and hence $\Pi(\mathbf{z},\mathbf{p}) \geqq$

TABLE 3
Property "P" holds for a gauge function $\langle H,\mathbf{X}\rangle \in \mathscr{H}^0$ if and only if property "Q" holds for its restricted profit function $\langle \Pi,\mathbf{P}\rangle \in \mathscr{P}^0$.[a]

	"P" on the gauge function $H(\mathbf{z},\mathbf{x})$, $\mathbf{x} \in \mathbf{X}(\mathbf{z})$	"Q" on the restricted profit function $\Pi(\mathbf{z},\mathbf{p})$, $\mathbf{p} \in \mathbf{P}(\mathbf{z})$
1.[b]	$\mathbf{X}(\mathbf{z}) = \mathbf{E}^N$	$\Pi(\mathbf{z},\mathbf{p}) > 0$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, $\mathbf{p} \neq 0$
2.[c]	$H(\mathbf{z},\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{X}(\mathbf{z}), \mathbf{x} \neq \mathbf{0}$	$\mathbf{P}(\mathbf{z}) = \mathbf{E}^N$
3.	Non-increasing (non-decreasing) in a subset of variables $\mathbf{z}_{(1)}$ of $\mathbf{z}$	Non-decreasing (non-increasing) in a subset of variables $\mathbf{z}_{(1)}$ of $\mathbf{z}$
4.	Uniformly decreasing (uniformly increasing) in a subset of variables $\mathbf{z}_{(1)}$ of $\mathbf{z}$	Uniformly increasing (uniformly decreasing) in a subset of variables $\mathbf{z}_{(1)}$ of $\mathbf{z}$
5.	$\langle H,\mathbf{X}\rangle$ has the joint continuity property in $(\mathbf{z},\mathbf{x})$ defined by statements (1), (4), (5), and (6) in Lemma 20	$\langle \Pi,\mathbf{P}\rangle$ has the joint continuity property in $(\mathbf{z},\mathbf{p})$ defined by statement (1) in Lemma 15 and the conclusions of Lemma 16

[a]By formal duality, the implications of this table continue to hold when properties "P" and "Q" are reversed.
[b]This condition is equivalent to requiring that the origin be an interior point of the production possibility set $\mathbf{T} \in \mathscr{T}^0$.
[c]This condition is equivalent to requiring that the production possibility set $\mathbf{T} \in \mathscr{T}^0$ be bounded.

$\lambda\mathbf{p}\cdot\mathbf{p} > 0$. Alternately, if $\mathbf{X}(\mathbf{z}) \neq \mathbf{E}^N$, then taking $\mathbf{p} \neq \mathbf{0}$ in the subspace orthogonal to $\mathbf{X}(\mathbf{z})$ implies $\mathbf{p} \in \mathbf{P}(\mathbf{z})$ and $\Pi(\mathbf{z},\mathbf{p}) = 0$.

Result 2 – This result follows from Result 1 by formal duality.

Result 3 – Consider $\mathbf{z}'$, $\mathbf{z}'' \in \mathbf{Z}$ with $\mathbf{z}'_{(1)} \geqq \mathbf{z}''_{(1)}$ and $\mathbf{z}'_{(2)} = \mathbf{z}''_{(2)}$. Since either H non-increasing or Π non-decreasing are equivalent to $\mathbf{T}(\mathbf{z}'') \subseteq \mathbf{T}(\mathbf{z}')$ under the inverse gauge or implicit technology mappings, respectively, the result follows.

Result 4 – Consider distinct $\mathbf{z}'$, $\mathbf{z}'' \in \mathbf{Z}$ with $\mathbf{z}'_{(1)} \geqq \mathbf{z}''_{(1)}$ and $\mathbf{z}'_{(2)} = \mathbf{z}''_{(2)}$. As in the previous result, each condition is equivalent to $(1+\lambda)\mathbf{T}(\mathbf{z}'') \subseteq \mathbf{T}(\mathbf{z}')$ for some $\lambda > 0$, and the result follows.

Result 5 – We first establish that property "Q" on restricted profit functions in $\mathscr{P}^0$ implies that dual $\mathbf{T} \in \mathscr{T}^0$ satisfy Axiom 3; the remaining properties of $\mathbf{T}$, Axioms 1, 2, and 4, are immediate from Lemmas 15 and 16 and Theorem 24. Suppose $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$. Consider any $\mathbf{p}^0 \in \mathbf{P}^0(\mathbf{z}^0)$. The lower hemicontinuity of $\mathbf{P}(\mathbf{z})$ implies there exist $\mathbf{p}^k \in \mathbf{P}(\mathbf{z}^k)$ such that $\mathbf{p}^k \to \mathbf{p}^0$. By definition of $\mathbf{T}(\mathbf{z})$, $\mathbf{p}^k\cdot\mathbf{x}^k \leqq \Pi(\mathbf{z}^k,\mathbf{p}^k)$. Then, $\lim_k \mathbf{p}^k\cdot\mathbf{x}^k = \mathbf{p}^0\cdot\mathbf{x}^0 \leqq \lim_k \Pi(\mathbf{z}^k,\mathbf{p}^k)$ by the continuity of Π. The condition $\mathbf{p}\cdot\mathbf{x}^0 \leqq \Pi(\mathbf{z}^0,\mathbf{p})$ for $\mathbf{p} \in \mathbf{P}^0(\mathbf{z}^0)$ and the lower semicontinuity of Π on $\mathbf{P}(\mathbf{z})$ implies $\mathbf{p}\cdot\mathbf{x}^0 \leqq \Pi(\mathbf{z}^0,\mathbf{p})$ for all $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, and hence $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$.

Next suppose $\mathbf{z}^k \in \mathbf{Z}$, $\mathbf{z}^k \to \mathbf{z}^0 \in \mathbf{Z}$, $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$. Suppose there exists $\epsilon > 0$ such that $(1-\epsilon)\mathbf{x}^0 \notin \mathbf{T}(\mathbf{z}^k)$ for k large. Then there exists $\mathbf{p}^k \in \mathbf{P}(\mathbf{z}^k)$, $|\mathbf{p}^k| = 1$ such that $(1-\epsilon)\mathbf{p}^k \cdot \mathbf{x}^0 > \Pi(\mathbf{z}^k,\mathbf{p}^k)$. Then $\mathbf{p}^k$ has a subsequence (retain notation) converging to $\mathbf{p}^0$. Then $\Pi(\mathbf{z}^k,\mathbf{p}^k) < (1-\epsilon)\mathbf{p}^k \cdot \mathbf{x}^0 \to (1-\epsilon)\mathbf{p}^0 \cdot \mathbf{x}^0$ implies $\mathbf{p}^0 \in \mathbf{P}(\mathbf{z}^0)$ by the last continuity property of $\Pi(\mathbf{z},\mathbf{p})$ given in the statement of Lemma 16. Hence, $\mathbf{p}^0 \cdot \mathbf{x}^0 \leqq \Pi(\mathbf{z}^0,\mathbf{p}^0)$, implying $\Pi(\mathbf{z}^k,\mathbf{p}^k) < (1-\epsilon)\Pi(\mathbf{z}^0,\mathbf{p}^0)$. But this contradicts the lower semicontinuity property of Π. Hence there exist $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$. Therefore, $\mathbf{T}$ is continuous and convex-valued, and hence (Appendix A.3, Lemma 13.3) strongly continuous. Hence, $\mathbf{T}$ satisfies Axiom 3. Similarly, Lemmas 20 and 22 along with Theorem 24 imply that property "P" on gauge functions in $\mathscr{H}^0$ is equivalent to $\mathbf{T} \in \mathscr{T}^0$ satisfying Axioms 1–4. Hence, the result follows. Q.E.D.

The next result relates differentiability properties of the gauge function to curvature properties of the restricted profit function, and vice versa. We shall now add to the conditions determining the class of gauge functions $\mathscr{H}^0$ the assumption that $\mathbf{X}(\mathbf{z}) = \mathbf{E}^N$ for each $\mathbf{z} \in \mathbf{Z}$. From Result 1 in Table 3, this is equivalent to assuming that the origin of corresponding production possibility sets is an interior point of these sets, or that the restricted profit function is positive for all non-zero $\mathbf{p} \in \mathbf{P}(\mathbf{z})$. Let $\mathscr{T}^1$, $\mathscr{H}^1$, $\mathscr{P}^1$ denote the subclasses of $\mathscr{T}^0$, $\mathscr{H}^0$, $\mathscr{P}^0$, respectively, on which this added restriction is satisfied.

By Lemma 19, the sub-differential $\mathbf{\Gamma}(\mathbf{z},\mathbf{p})$ of $\Pi(\mathbf{z},\mathbf{p})$ with $\langle \Pi,\mathbf{P}\rangle \in \mathscr{P}^1$ exists for $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$, and satisfies $\mathbf{\Gamma}(\mathbf{z},\mathbf{p}) = \mathbf{\Phi}(\mathbf{z},\mathbf{p})$, where $\mathbf{\Phi}(\mathbf{z},\mathbf{p})$ is the set of maximands of $\mathbf{p}\cdot\mathbf{x}$ for $\mathbf{x} \in \mathbf{T}(\mathbf{z})$. Further, $\mathbf{x} \in \mathbf{\Phi}(\mathbf{z},\mathbf{p})$ implies $H(\mathbf{z},\mathbf{x}) = 1$. Define $\mathbf{X}^*(\mathbf{z}) = \{\mathbf{x} \in \mathbf{E}^N | \mathbf{x} \in \mathbf{\Phi}(\mathbf{z},\mathbf{p}) \text{ for some } \mathbf{p} \in \mathbf{P}^0(\mathbf{z})\}$.

The gauge function $H(\mathbf{z},\mathbf{x})$, $\langle H,\mathbf{E}^N\rangle \in \mathscr{H}^1$, is *exposed* at $\mathbf{x} \in \mathbf{E}^N$ if $H(\mathbf{z},\mathbf{x}') - H(\mathbf{z},\mathbf{x}) > \mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})$ for all $\mathbf{x}' \in \mathbf{E}^N$, $\mathbf{x}'$ not proportional to $\mathbf{x}$, and all $\mathbf{p}$ in the relative interior of $\mathbf{\Lambda}(\mathbf{z},\mathbf{x})$, where $\mathbf{\Lambda}(\mathbf{z},\mathbf{x})$ is the sub-differential of $H(\mathbf{z},\mathbf{x})$. (a) in Figure 21 illustrates a case in which H is not exposed at $\mathbf{x}^0$, since the point $\mathbf{x}^1$ has $H(\mathbf{z},\mathbf{x}^1) = H(\mathbf{z},\mathbf{x}^0)$, $\mathbf{p}^0\cdot\mathbf{x}^1 = \mathbf{p}^0\cdot\mathbf{x}^0$, where $\mathbf{p}^0$ is the price vector satisfying $\{\mathbf{p}^0\} = \mathbf{\Lambda}(\mathbf{z},\mathbf{x}^0)$. In (b), (c), (d), the point $\mathbf{x}^0$ is exposed. Note that in case (c), the set $\mathbf{\Lambda}(\mathbf{z},\mathbf{x}^0)$ is the closed line segment connecting $\mathbf{p}^1$ and $\mathbf{p}^2$, and the relative interior of $\mathbf{\Lambda}(\mathbf{z},\mathbf{x}^0)$ is the open line segment connecting, but excluding, $\mathbf{p}^1$ and $\mathbf{p}^2$. For any $\mathbf{p}^0$ in this open line segment, note that the strict inequality $H(\mathbf{z},\mathbf{x}') - H(\mathbf{z},\mathbf{x}^0) > \mathbf{p}^0\cdot(\mathbf{x}'-\mathbf{x}^0)$ holds for $\mathbf{x}'$ not proportional to $\mathbf{x}^0$, but that for the end points $\mathbf{p}^1$ and $\mathbf{p}^2$, points such as $\mathbf{x}^1$ and $\mathbf{x}^2$ will result in equality holding. In case (d), the strict inequality $H(\mathbf{z},\mathbf{x}') - H(\mathbf{z},\mathbf{x}^0) > \mathbf{p}^0\cdot(\mathbf{x}'-\mathbf{x}^0)$ holds for $\mathbf{x}'$ not propor-

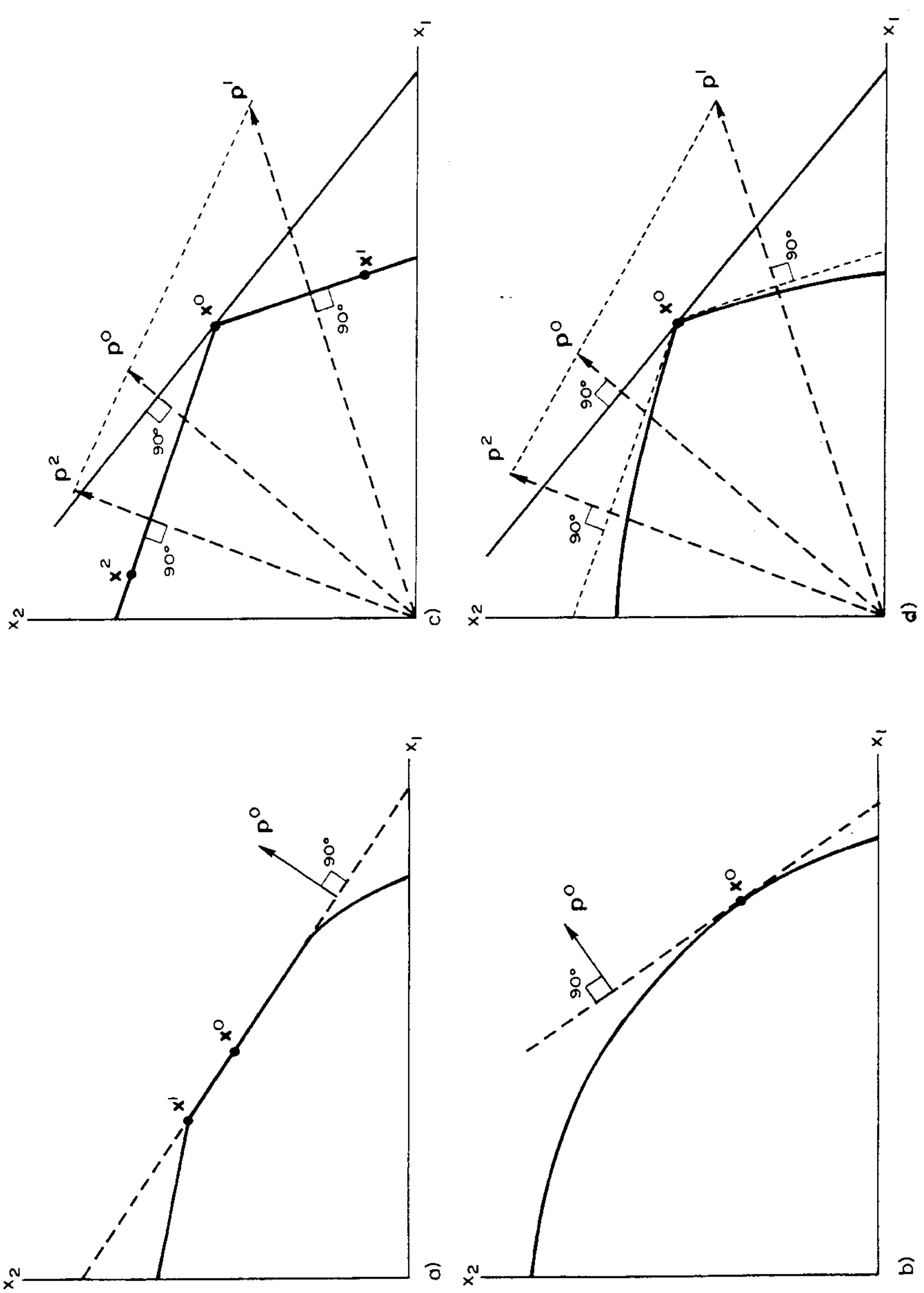


FIGURE 21

tional to $\mathbf{x}$ and all points $\mathbf{p}^0$ in $\Lambda(\mathbf{z},\mathbf{x}^0)$, including the end points $\mathbf{p}^1$ and $\mathbf{p}^2$ which are not in the relative interior of $\Lambda(\mathbf{z},\mathbf{x}^0)$.

The gauge function $H(\mathbf{z},\mathbf{x})$, $\langle H,\mathbf{E}^N\rangle \in \mathscr{H}^1$, is *strictly quasiconvex at* $\mathbf{x} \in \mathbf{E}^N$ if $H(\mathbf{z},\theta\mathbf{x}+(1-\theta)\mathbf{x}') < \theta H(\mathbf{z},\mathbf{x}) + (1-\theta)H(\mathbf{z},\mathbf{x}')$ for all $\mathbf{x}'$ not proportional to $\mathbf{x}$, $0 < \theta < 1$.[15] (a) and (c) of Figure 21 illustrate cases in which the gauge function fails to be strictly quasi-convex at $\mathbf{x}^0$, since in each case an average of $\mathbf{x}^0$ and $\mathbf{x}^1$ has $H(\mathbf{z},\theta\mathbf{x}^0+(1-\theta)\mathbf{x}^1) = \theta H(\mathbf{z},\mathbf{x}^0) + (1-\theta)H(\mathbf{z},\mathbf{x}^1)$. Cases (b) and (d) in this figure have H strictly quasi-convex at $\mathbf{x}^0$. Lemma 16.5 in Appendix A.3 establishes that H strictly quasi-convex at $\mathbf{x}$ implies H exposed at $\mathbf{x}$, and that the converse implication holds provided the strict inequality required for H to be exposed at $\mathbf{x}$ holds for all points in the sub-differential of H, and not just those points in its relative interior.

The gauge function $H(\mathbf{z},\mathbf{x})$, $\langle H,\mathbf{E}^N\rangle \in \mathscr{H}^1$, is *strictly differentially quasi-convex* at $\mathbf{x}^0 \in \mathbf{E}^N$ if it has a first and second differential in $\mathbf{x}$ in a neighborhood of $\mathbf{x}^0$, and its Hessian matrix of second partial derivatives in $\mathbf{x}$ is non-negative definite and is of rank $N-1$. Lemma 16.5 in Appendix A.3 establishes that H strictly differentiably quasi-convex implies H strictly quasi-convex, and also gives a partial converse. The following result relates the structural properties of curvature and differentiability.

> *Theorem* 26. Suppose $\mathbf{Z}$ is a non-empty subset of $\mathbf{E}^M$, and consider the class of gauge functions $\mathscr{H}^1$ and the class of restricted profit functions $\mathscr{P}^1$. Then, for the gauge and restricted profit functions in these classes which are dual under the implicit profit and implicit gauge mappings of Theorem 24, the structural relationships given in Table 4 hold; i.e., the gauge function has the property "P" if and only if the restricted profit function has the property "Q".

Proof: Most of the results are established in Lemma 16.7 in Appendix A.3, as follows: Result 1, "P" implies "Q", is established by 16.7(3), and the converse is trivial. Result 2, "Q" implies "P", is established by 16.7(4), and the converse is trivial. Result 3, "Q" implies "P", is established by 16.7(6). The converse is established by the proof of 16.7(9), using Appendix A.3, Lemma 16.5(1) rather than 16.5(2). Result 4 follows from an argument dual to that for Result 3. Result 5, "P" implies "Q", is

[15]Since H is linear homogeneous in $\mathbf{x}$, this definition is equivalent to the requirement that an open line segment between $\mathbf{x}$ and any point $\mathbf{x}'$ in the lower contour set $\{\mathbf{x}' | H(\mathbf{z},\mathbf{x}') \leqq H(\mathbf{z},\mathbf{x})\}$ be contained in the interior of this set. The condition that H be strictly quasi-convex for all $\mathbf{z} \in \mathbf{E}^N$ is then equivalent to the usual definition of strict quasi-convexity.

TABLE 4
Property "P" holds for the gauge function $\langle H,\mathbf{X}\rangle \in \mathscr{H}^1$ if and only if property "Q" holds for its restricted profit function $\langle \Pi,\mathbf{P}\rangle \in \mathscr{P}^1$.

	"P" on the gauge function $H(\mathbf{z},\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^N$	"Q" on the restricted profit function $\Pi(\mathbf{z},\mathbf{p})$, $\mathbf{p} \in \mathbf{P}(\mathbf{z})$
1.	$H(\mathbf{z},\mathbf{x})$ differentiable at $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ exposed at $\mathbf{p}$, where $\{\mathbf{p}\} = \Lambda(\mathbf{z},\mathbf{x})$, $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$
2.	$H(\mathbf{z},\mathbf{x})$ exposed at $\mathbf{x}$, where $\{\mathbf{x}\} = \Gamma(\mathbf{z},\mathbf{p})$, $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ differentiable at $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$
3.	$H(\mathbf{z},\mathbf{x})$ differentiable at all $\mathbf{x}$ in the relative interior of $\Gamma(\mathbf{z},\mathbf{p})$ where $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ exposed at $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$
4.	$H(\mathbf{z},\mathbf{x})$ exposed at $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ differentiable at all $\mathbf{p}$ in the relative interior of $\Lambda(\mathbf{z},\mathbf{x})$, where $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$
5.	$H(\mathbf{z},\mathbf{x})$ strictly quasi-convex at $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ differentiable at all $\mathbf{p} \in \Lambda(\mathbf{z},\mathbf{x})$, where $\mathbf{x} \in \mathbf{X}^*(\mathbf{z})$
6.	$H(\mathbf{z},\mathbf{x})$ differentiable at all $\mathbf{x} \in \Gamma(\mathbf{z},\mathbf{p})$, where $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$	$\Pi(\mathbf{z},\mathbf{p})$ strictly quasi-convex at $\mathbf{p} \in \mathbf{P}^0(\mathbf{z})$
7.	$H(\mathbf{z},\mathbf{x})$ possesses a continuous first and second differential in $\mathbf{x}$ in a neighborhood of $\mathbf{x}^0 \in \mathbf{X}^*(\mathbf{z})$, and is strictly differentially quasi-convex on this neighborhood	$\Pi(\mathbf{z},\mathbf{p})$ possesses a continuous first and second differential in $\mathbf{p}$ in a neighborhood of $\mathbf{p}^0$, where $\{\mathbf{p}^0\} = \Lambda(\mathbf{z},\mathbf{x}^0)$, and is strictly differentially quasi-convex on this neighborhood
8.	$H(\mathbf{z},\mathbf{x})$ possesses a continuous first and second differential in $\mathbf{x}$ in a neighborhood of $\mathbf{x}^0$, where $\{\mathbf{x}^0\} = \Gamma(\mathbf{z},\mathbf{p}^0)$, and is strictly differentially quasi-convex on this neighborhood	$\Pi(\mathbf{z},\mathbf{p})$ possesses a continuous first and second differential in $\mathbf{p}$ in a neighborhood of $\mathbf{p}^0 \in \mathbf{P}^0(\mathbf{z})$, and is strictly differentially quasi-convex on this neighborhood

established by 16.7(7), and the converse is established by 16.7(10). Result 6 follows from an argument dual to that for Result 5. Results 7 and 8 follow from 16.7(11) and 16.7(12), plus the observation that $\{\mathbf{p}^0\} = \Lambda(\mathbf{z},\mathbf{x}^0)$ and $\{\mathbf{x}^1\} = \Gamma(\mathbf{z},\mathbf{p}^0)$ imply $\mathbf{x}^1 = \mathbf{x}^0$. Q.E.D.

The geometry of the structural relationships in Theorem 26 is summarized in Figure 22. As in the geometry of the two-input cost function, "kinks" are mapped into "flats", and vice versa. The linear segment $\mathbf{x}^1\mathbf{x}^2$ maps into the point $\mathbf{p}^1$. Note that the "price frontier" is not horizontal at $\mathbf{p}^1$, allowing this price frontier to be supported at $\mathbf{p}^1$ by planes with normals in the line segment through $\mathbf{x}^1\mathbf{x}^2$ (and extending indefinitely to

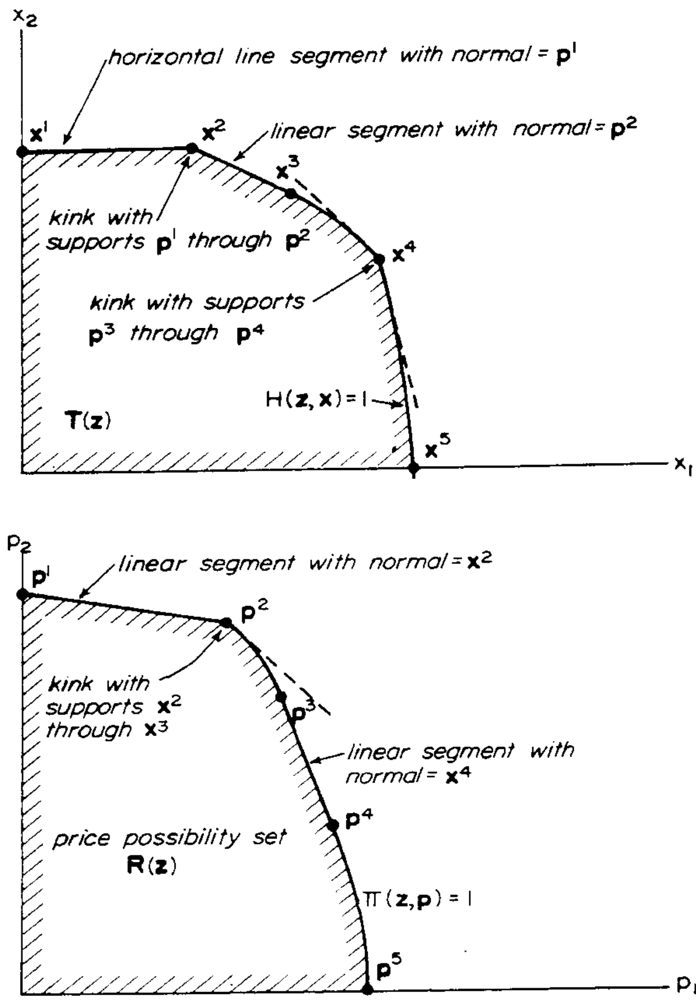


FIGURE 22

the left). The kink $\mathbf{x}^2$ maps into the linear segment $\mathbf{p}^1\mathbf{p}^2$, and the normal to this linear segment is $\mathbf{x}^2$. The linear segment $\mathbf{x}^2\mathbf{x}^3$ maps into the point $\mathbf{p}^2$. Note that the price frontier is linear to the left of $\mathbf{p}^2$ and curved to the right of $\mathbf{p}^2$, reflecting the fact that a kink occurred at the $\mathbf{x}^2$ end of the line segment $\mathbf{x}^2\mathbf{x}^3$, but the boundary of $\mathbf{T}$ is smooth at the $\mathbf{x}^3$ end of the line segment. The curve $\mathbf{x}^3\mathbf{x}^4$ maps into the curve $\mathbf{p}^2\mathbf{p}^3$, and the kink $\mathbf{x}^4$ maps into the line segment $\mathbf{p}^3\mathbf{p}^4$. Finally, the curve $\mathbf{x}^4\mathbf{x}^5$ maps into the curve $\mathbf{p}^4\mathbf{p}^5$, with the vertical tangent at $\mathbf{p}^5$ corresponding to the absence of a "flat" above $\mathbf{x}^5$. Note that the boundary of $\mathbf{T}$ is differentiable at $\mathbf{x}^1$, $\mathbf{x}^3$, and $\mathbf{x}^5$, exposed at $\mathbf{x}^2$, and strictly quasi-convex at $\mathbf{x}^4$. Correspondingly, the price frontier is exposed at $\mathbf{p}^1$ and $\mathbf{p}^2$, and differentiable at $\mathbf{p}^3$, $\mathbf{p}^4$, and $\mathbf{p}^5$.

The *price possibility set* used in Figure 22 can be introduced formally as

$$\begin{aligned}\mathbf{R}(\mathbf{z}) &= \{\mathbf{p} \in \mathbf{P}(\mathbf{z}) | \Pi(\mathbf{z},\mathbf{p}) \leqq 1\} \\ &= \{\mathbf{p} \in \mathbf{E}^N | \mathbf{p}\cdot\mathbf{x} \leqq 1 \text{ for all } \mathbf{x} \in \mathbf{T}(\mathbf{z})\}. \end{aligned} \tag{47}$$

This set plays the same role as the factor price requirement set in the analysis of cost functions, and can be viewed as the formal dual of the set $\mathbf{T}(\mathbf{z})$. Figure 23 gives two further illustrations of this relationship. In case (a), with commodity 1 an input to production and commodity 2 an output, a translation of the production frontier has been made so that the

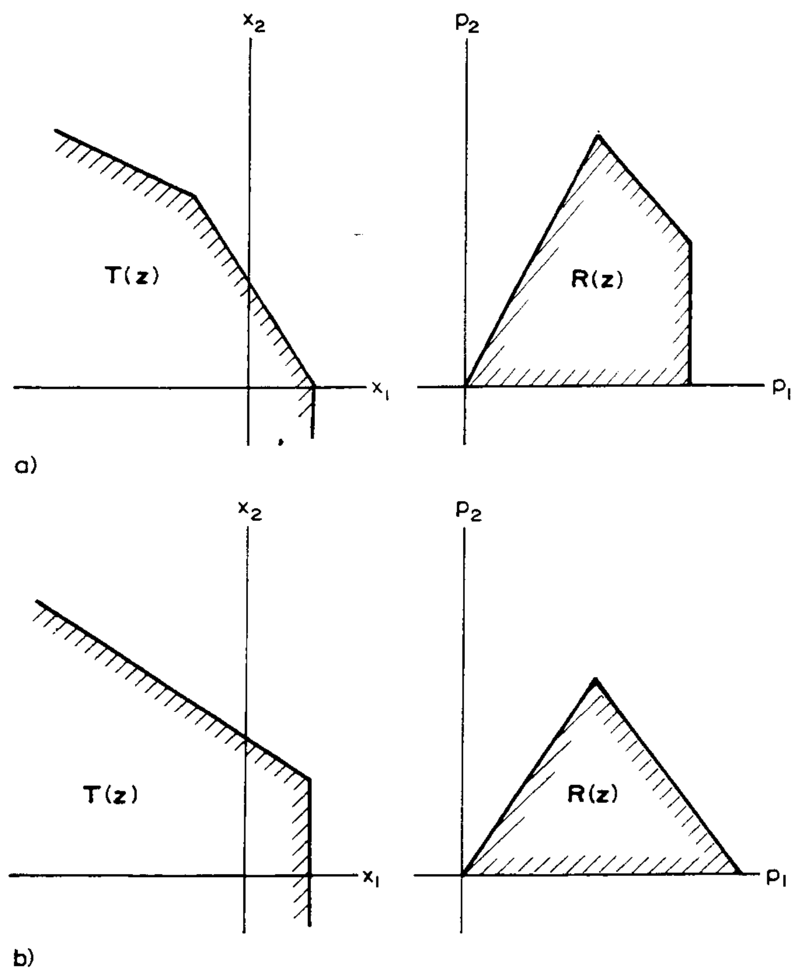


FIGURE 23

origin is interior to the possibility set. Correspondingly, the restricted profit function is positive for all non-zero $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, and the price frontier is bounded. Case (b) illustrates a constant returns technology [i.e., the set $\mathbf{T}(\mathbf{z})$ is a convex cone] translated again so that the origin is an interior point.

18. Examples of Restricted Profit Functions

Examples of restricted profit functions can be given for the Cobb–Douglas and C.E.S. production functions introduced in Section 9, and for a C.E.S. production frontier. Consider a homogeneous version of the Cobb–Douglas transformation function (17), defined for some $\mathbf{z}^0 \in \mathbf{Z}$,

$$x_{N+1}^{1/\mu} \leqq D(-x_1)^{\theta_1}(-x_2)^{\theta_2} \cdots (-x_N)^{\theta_N}, \tag{48}$$

where $x_1,...,x_N \leqq 0$, $x_{N+1} \geqq 0$, $\theta_i > 0$, and $\sum_{i=1}^N \theta_i = 1$, and the scale parameter μ is less than one. The profit function when all inputs and output are variable is

$$\Pi(\mathbf{z}^0,p_1,...,p_{N+1}) = D''p_1^{-\theta_1\eta}p_2^{-\theta_2\eta} \cdots p_N^{-\theta_N\eta}p_{N+1}^{1+\eta}, \tag{49}$$

where $\eta = \mu/(1-\mu)$ and $D'' = (1-\mu)D^\eta\mu^\eta\theta_1^{\eta\theta_1} \cdots \theta_N^{\eta\theta_N}$.

When inputs $S+1,...,N$ are held fixed, and the remaining inputs and output are variable, the restricted profit function is

$$\Pi(\mathbf{z}^0,x_{S+1},...,x_N,p_1,...,p_S,p_{N+1}) = D^*p_1^{-\theta_1\nu} \cdots p_S^{-\theta_S\nu}p_{N+1}^{1+\nu\nu'}, \tag{50}$$

where $\nu = \mu/(1-\mu\nu')$, $\nu' = \sum_{i=1}^S \theta_i$, and $D^* = (1-\mu\nu')D^\nu\mu^{\nu\nu'}\theta_1^{\nu\theta_1} \cdots \theta_S^{\nu\theta_S}(-x_{S+1})^{\nu\theta_{S+1}} \cdots (-x_N)^{\nu\theta_N}$. Note that this function is defined for all positive p_{N+1} provided $\mu\nu' < 1$ ($\mu < 1$ is not required).

Consider a homogeneous version of the C.E.S. transformation function (19), defined for some $\mathbf{z}^0 \in \mathbf{Z}$,

$$x_{N+1}^{1/\mu} \leqq [(-x_1/D_1)^{1-1/\sigma} + \cdots + (-x_N/D_N)^{1-1/\sigma}]^{1/(1-1/\sigma)}, \tag{51}$$

where $x_1,...,x_N \leqq 0$, $x_{N+1} \geqq 0$, $D_1,...,D_N > 0$, $\sigma > 0$, $\sigma \neq 1$, and $\mu < 1$. The profit function when all inputs and output are variable is

$$\Pi(\mathbf{z}^0,p_1,...,p_{N+1}) = (1-\mu)\mu^\eta p_{N+1}^{1+\eta}[(p_1D_1)^{1-\sigma} + \cdots + (p_ND_N)^{1-\sigma}]^{-\eta/(1-\sigma)}. \tag{52}$$

When inputs $S+1,...,N$ and output $N+1$ are held fixed and the remaining inputs are variable, then the restricted profit function, equal to the negative of the cost function for the variable inputs, has been shown by Knut Mork (1976) to have the form

$$\pi^* = \Pi^*(\mathbf{z}^0, x_{S+1}, \ldots, x_{N+1}, p_1, \ldots, p_S) \quad \text{if} \quad x_{N+1}^{1/\mu(1-1/\sigma)} > \sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma},$$

$$= 0 \quad \text{if} \quad x_{N+1}^{(1-1/\sigma)/\mu} \leqq \sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma} \quad \text{and} \quad \sigma > 1, \tag{52a}$$

$$= -\infty \quad \text{if} \quad x_{N+1}^{(1-1/\sigma)/\mu} \leqq \sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma} \quad \text{and} \quad 0 < \sigma < 1,$$

where the last alternative corresponds to a non-producible value of x_{N+1}, and where

$$\Pi^* = -\left[\sum_{i=1}^{S} (p_i D_i)^{1-\sigma}\right]^{1/(1-\sigma)} \left\{x_{N+1}^{(1-1/\sigma)/\mu} - \sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma}\right\}^{-\sigma/(1-\sigma)}. \tag{52b}$$

Suppose now that x_{N+1} is made variable, so that the restricted profit function becomes

$$\pi' = \Pi'(z^0, x_{S+1}, \ldots, x_N, p_1, \ldots, p_S, p_{N+1}) = \operatorname*{Max}_{x_{N+1}} [p_{N+1} x_{N+1} + \pi^*], \tag{52c}$$

with π^* given in (52a). For $\mu = 1$, Mork shows this restricted profit function to have the form

$$\pi' = \Pi'(\mathbf{z}^0, x_{S+1}, \ldots, x_N, p_1, \ldots, p, p_{N+1}) \quad \text{if} \quad p_{N+1}^{1-\sigma} > \sum_{i=1}^{S} (D_i p_i)^{1-\sigma},$$

$$= 0 \quad \text{if} \quad p_{N+1}^{1-\sigma} \leqq \sum_{i=1}^{S} (D_i p_i)^{1-\sigma} \quad \text{and} \quad 0 < \sigma < 1, \tag{52d}$$

$$= +\infty \quad \text{if} \quad p_{N+1}^{1-\sigma} \leqq \sum_{i=1}^{S} (D_i p_i)^{1-\sigma} \quad \text{and} \quad \sigma > 1,$$

where

$$\Pi' = \left[p_{N+1}^{1-\sigma} - \sum_{i=1}^{S} (D_i p_i)^{1-\sigma}\right]^{1/(1-\sigma)} \left[\sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma}\right]^{1/(1-1/\sigma)}. \tag{52e}$$

For $\mu \neq 1$, the maximand of (52c) satisfies

$$\mu p_{N+1} \left\{x_{N+1}^{(1-1/\sigma)/\mu} - \sum_{i=S+1}^{N} (-x_i/D_i)^{1-1/\sigma}\right\}^{1/(1-\sigma)}$$

$$= x_{N+1}^{-1+(1-1/\sigma)/\mu} \left[\sum_{i=1}^{S} (p_i D_i)^{1-\sigma}\right]^{1/(1-\sigma)}. \tag{52f}$$

In general, (52f) does not have a closed form solution for x_{N+1}, and consequently (52c) fails to have a closed form.

Consider a C.E.S. production frontier, defined for some $\mathbf{z}^0 \in \mathbf{Z}$ by

$$(-x_{N+1})^{\mu} \geqq [(x_1/D_1)^{1+1/\sigma} + \cdots + (x_N/D_N)^{1+1/\sigma}]^{1/(1+1/\sigma)}, \tag{53}$$

where $x_1,\ldots,x_N \geqq 0$, $x_{N+1} \leqq 0$, $D_1,\ldots,D_N > 0$, $\sigma > 0$, and $\mu < 1$. The revenue function when all outputs are variable and the input is fixed is

$$\Pi(\mathbf{z}^0,x_{N+1},p_1,\ldots,p_N) = (-x_{N+1})^{\mu}[(p_1D_1)^{1+\sigma} + \cdots + (p_ND_N)^{1+\sigma}]^{1/(1+\sigma)}. \tag{54}$$

The profit function when all outputs and input are variable is

$$\Pi(\mathbf{z}^0,p_1,\ldots,p_{N+1}) = (1-\mu)\mu^{\eta}[(p_1D_1)^{1+\sigma} + \cdots + (p_ND_N)^{1+\sigma}]^{(1+\eta)/(1+\sigma)}p_{N+1}^{-\eta}, \tag{55}$$

where $\eta = \mu/(1-\mu)$.

19. Composition Rules for Profit Functions

Composition rules of the sort established for cost functions can also be derived for restricted profit functions. These rules allow the construction of complex functions from simple known forms.

Throughout the remainder of this section, we consider a set of parameters $\mathbf{Z}$ and a family of production possibility sets $\mathbf{T}^j(\mathbf{z})$, $j = 1,\ldots,J$, each satisfying Axioms 1–4, containing the origin in its interior, and having the free disposal property. Let $\mathbf{P}^j(\mathbf{z})$ denote the normal cone, $\Pi^j(\mathbf{z},\mathbf{p})$ the profit function, $H^j(\mathbf{z},\mathbf{x})$ the gauge function, and $\mathbf{R}^j(\mathbf{z})$ the price possibility set of $\mathbf{T}^j(\mathbf{z})$. The first result is due to Fenchel (1949, Result 41), and is also proved by Karlin (1959, 7.6.1).

Lemma 27. If $\mathbf{T}'(\mathbf{z}) = \bigcap_{j=1}^J \mathbf{T}^j(\mathbf{z})$, then $\mathbf{T}'(\mathbf{z})$ satisfies Axioms 1–4, contains the origin in its interior, has the free disposal property, and satisfies

(i) $H'(\mathbf{z},\mathbf{x}) = \text{Max}_j\, H^j(\mathbf{z},\mathbf{x})$.

(ii) $\sum_{j=1}^J \mathbf{P}^j(\mathbf{z}) \subseteq \mathbf{P}'(\mathbf{z}) \subseteq \text{Closure}\ (\sum_{j=1}^J \mathbf{P}^j(\mathbf{z}))$.

(iii) For $p \in \sum_{j=1}^J \mathbf{P}^j(\mathbf{z})$, $\Pi'(\mathbf{z},\mathbf{p}) = \inf\{\sum_{j=1}^J \Pi^j(\mathbf{z},\mathbf{p}^j) | \mathbf{p}^j \in \mathbf{P}^j(\mathbf{z}), \sum_{j=1}^J \mathbf{p}^j = \mathbf{p}\}$.

(iv) $\mathbf{R}'(\mathbf{z}) = \text{Convex Hull of } \bigcup_{j=1}^J \mathbf{R}^j(\mathbf{z})$.

The next result gives one general-purpose composition rule. Let $\mathbf{W}^+(\mathbf{z})$

denote a convex closed bounded subset of the non-negative orthant of $\mathbf{E}^J$ which contains a strictly positive vector, and let $\mathbf{W}(\mathbf{z})$ denote the set obtained from $\mathbf{W}^+(\mathbf{z})$ by free disposal; i.e., $\mathbf{W}(\mathbf{z}) = \{\mathbf{w} \in \mathbf{E}^J | \mathbf{w} \leqq \mathbf{w}' \in \mathbf{W}^+(\mathbf{z})\}$. Then, the origin is in the interior of $\mathbf{W}(\mathbf{z})$. For $\mathbf{q} \in \mathbf{R}^J$, $\mathbf{q}\cdot\mathbf{w}$ is bounded above on $\mathbf{W}(\mathbf{z})$ if and only if $\mathbf{q}$ is non-negative. Define $\Pi^*(\mathbf{z},\mathbf{q})$ to be the restricted profit function of $\mathbf{W}(\mathbf{z})$ for $\mathbf{z} \in \mathbf{Z}$, $\mathbf{q} \geqq \mathbf{0}$. Define $H^*(\mathbf{z},\mathbf{w})$ to be the gauge function of $\mathbf{W}(\mathbf{z})$ (centered at the origin of $\mathbf{E}^J$). Define $\mathbf{R}^*(\mathbf{z}) = \{\mathbf{q} \in \mathbf{E}^J | \Pi^*(\mathbf{z},\mathbf{q}) \leqq 1\}$ to be the price possibility set of $\mathbf{W}(\mathbf{z})$.

Theorem 28. If $H'(\mathbf{z},\mathbf{x}) = H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}),\ldots,H^J(\mathbf{z},\mathbf{x}))$, then

(i) $H'(\mathbf{z},\mathbf{x})$ is the gauge function of a convex set in $\mathbf{E}^N$,

(ii) $\mathbf{T}'(\mathbf{z}) = \text{Closure } \bigcup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})} \bigcap_{j=1}^J (w_j\mathbf{T}^j(\mathbf{z}))$.

(iii) $\sum_{j=1}^J \mathbf{P}^j(\mathbf{z}) \subseteq \mathbf{P}'(\mathbf{z}) \subseteq \text{Closure } (\sum_{j=1}^J \mathbf{P}^j(\mathbf{z}))$.

(iv) For $\mathbf{p} \in \sum_{j=1}^J \mathbf{P}^j(\mathbf{z})$, $\Pi'(\mathbf{z},\mathbf{p}) = \inf\{\Pi^*(\mathbf{z},\Pi^1(\mathbf{z},\mathbf{p}^1),\ldots,\Pi^J(\mathbf{z},\mathbf{p}^J)) | \mathbf{p}^j \in \mathbf{P}^j(\mathbf{z}), \sum_{j=1}^J \mathbf{p}^j = \mathbf{p}\}$.

(v) $\mathbf{R}'(\mathbf{z}) = \text{Closure } \bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{z})} \sum_{j=1}^J (q_j\mathbf{R}^j(\mathbf{z}))$.

Proof: It is immediate that $H'(\mathbf{z},\mathbf{x})$ is positively linear homogeneous in $\mathbf{x}$, with $H'(\mathbf{z},\mathbf{0}) = 0$. By convexity, $H^*(\mathbf{z},\mathbf{w}) \leqq H^*(\mathbf{z},\mathbf{w}-\mathbf{w}') + H^*(\mathbf{z},\mathbf{w}')$. If $\mathbf{w}' \geqq \mathbf{w}$, then $H^*(\mathbf{z},\mathbf{w}-\mathbf{w}') = 0$ by the free disposal property of $\mathbf{W}(\mathbf{z})$. Hence, $H^*(\mathbf{z},\mathbf{w})$ is non-decreasing in $\mathbf{w}$. Now consider $\mathbf{x}', \mathbf{x}'' \in \mathbf{R}^N$, $\mathbf{x}^* = \theta\mathbf{x}' + (1-\theta)\mathbf{x}''$ for some $0 < \theta < 1$. Then, $H^j(\mathbf{z},\mathbf{x}^*) \leqq \theta H^j(\mathbf{z},\mathbf{x}') + (1-\theta)H^j(\mathbf{z},\mathbf{x}'')$. Hence,

$$\begin{aligned} &H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}^*),\ldots,H^J(\mathbf{z},\mathbf{x}^*)) \\ &\leqq H^*(\mathbf{z},\theta H^1(\mathbf{z},\mathbf{x}') + (1-\theta)H^1(\mathbf{z},\mathbf{x}''),\ldots,\theta H^J(\mathbf{z},\mathbf{x}') + (1-\theta)H^J(\mathbf{z},\mathbf{x}'')) \\ &\leqq \theta H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}'),\ldots,H^J(\mathbf{z},\mathbf{x}')) + (1-\theta)H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}''),\ldots,H^J(\mathbf{z},\mathbf{x}'')), \end{aligned}$$

and $H'(\mathbf{z},\mathbf{x})$ is convex in $\mathbf{x}$. This establishes (i).

By definition, $\mathbf{T}'(\mathbf{z}) = \{\mathbf{x} | H'(\mathbf{z},\mathbf{x}) \leqq 1\}$. But $H'(\mathbf{z},\mathbf{x}) \leqq 1$ if and only if there exists $\mathbf{w} \in \mathbf{E}^J$, $w \geqq 0$, such that $H^*(\mathbf{z},\mathbf{w}) \leqq 1$ and $H^j(\mathbf{z},\mathbf{x}) \leqq w_j$. Then, we may without loss of generality choose $\mathbf{w} \in \mathbf{W}^+(\mathbf{z})$. If $w_j = 0$, then $\{\mathbf{x} | H^j(\mathbf{z},\mathbf{x}) = 0\} = \mathbf{AT}^j(\mathbf{z})$, the asymptotic cone of $\mathbf{T}^j(\mathbf{z})$. If $w_j > 0$, then $\{\mathbf{x} | H^j(\mathbf{z},\mathbf{x}) \leqq w_j\} = w_j\mathbf{T}^j(\mathbf{z})$, and $w_j\mathbf{T}^j(\mathbf{z})$ contains $\mathbf{AT}^j(\mathbf{z})$. Since $\mathbf{W}^+(\mathbf{z})$ contains a positive vector, every vector in $\mathbf{W}^+(\mathbf{z})$ is a limit of strictly positive vectors in $\mathbf{W}^+(\mathbf{z})$. Hence,

$$\begin{aligned}\mathbf{T}'(\mathbf{z}) &= \bigcup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})}\bigcap_{j=1}^{J}\{\mathbf{x}|H^j(\mathbf{z},\mathbf{x})\leqq w_j\}\\ &= \bigcup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})}\bigcap_{j=1}^{J}(w_j\mathbf{T}^j(\mathbf{z})+\mathbf{AT}^j(\mathbf{z}))\\ &= \text{Closure}\bigcup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})}\bigcap_{j=1}^{J}(w_j\mathbf{T}^j(\mathbf{z})).\end{aligned}$$

This proves (ii).

To verify (iii), note first that the normal cone of $w_j\mathbf{T}^j(\mathbf{z})+\mathbf{AT}^j(\mathbf{z})$ is $\mathbf{P}^j(\mathbf{z})$ for every $w_j\geqq 0$. Hence by Lemma 27, the normal cone of $\bigcap_{j=1}^{J}(w_j\mathbf{T}^j(\mathbf{z})+\mathbf{AT}^j(\mathbf{z}))$ contains $\sum_{j=1}^{J}\mathbf{P}^j(\mathbf{z})$ and is contained in the closure of this set. Now, the normal cone of the union of an arbitrary collection of sets is contained in the intersection of the normal cones of its members. Hence, the normal cone of $\mathbf{T}'(\mathbf{z})$ is contained in the closure of $\sum_{j=1}^{J}\mathbf{P}^j(\mathbf{z})$. Finally, since $\mathbf{W}^+(\mathbf{z})$ is bounded, there exists a vector $\bar{\mathbf{w}}\geqq\mathbf{w}$ for all $\mathbf{w}\in\mathbf{W}^+(\mathbf{z})$, implying $\mathbf{T}'(\mathbf{z})\subseteq\bigcap_{j=1}^{J}(\bar{w}_j\mathbf{T}^j(\mathbf{z}))$. The normal cone of this last set contains $\sum_{j=1}^{J}\mathbf{P}^j(\mathbf{z})$, and is contained in the normal cone of $\mathbf{T}'(\mathbf{z})$. Hence, (iii) holds.

Consider $\mathbf{p}\in\sum_{j=1}^{J}\mathbf{P}^j(\mathbf{z})$. By Lemma 27,

$$\sup\left\{\mathbf{p}\cdot\mathbf{x}|\mathbf{x}\in\bigcap_{j=1}^{J}(w_j\mathbf{T}^j(\mathbf{z}))\right\}=\inf\left\{w_1\Pi^1(\mathbf{z},\mathbf{p}^1)+\cdots+w_J\Pi^J(\mathbf{z},\mathbf{p}^J)|\mathbf{p}^j\in\mathbf{P}^j(\mathbf{z}),\sum_{j=1}^{J}\mathbf{p}^j=\mathbf{p}\right\}.$$

Hence,

$$\begin{aligned}\Pi'(\mathbf{z},\mathbf{p}) &= \sup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})}\inf\left\{w_1\Pi^1(\mathbf{z},\mathbf{p}^1)+\cdots+w_J\Pi^J(\mathbf{z},\mathbf{p}^J)|\mathbf{p}^j\in\mathbf{P}^j(\mathbf{z}),\sum_{j=1}^{J}\mathbf{p}^j=\mathbf{p}\right\}\\ &= \inf\left\{\sup_{\mathbf{w}\in\mathbf{W}^+(\mathbf{z})}(w_1\Pi^1(\mathbf{z},\mathbf{p}^1)+\cdots+w_J\Pi^J(\mathbf{z},\mathbf{p}^J))|\mathbf{p}^j\in\mathbf{P}^j(\mathbf{z}),\sum_{j=1}^{J}\mathbf{p}^j=\mathbf{p}\right\}\\ &= \inf\left\{\Pi^*(\mathbf{z},\Pi^1(\mathbf{z},\mathbf{p}^1),\ldots,\Pi^J(\mathbf{z},\mathbf{p}^J))|\mathbf{p}^j\in\mathbf{P}^j(\mathbf{z}),\sum_{j=1}^{J}\mathbf{p}^j=\mathbf{p}\right\},\end{aligned}$$

with the first equality holding by the minimax theorem [Rockafellar

(1970, Corollary 37.3.2)] since $\mathbf{W}^+(\mathbf{z})$ is bounded, and the second equality holding by the definition of Π^*. This proves (iv).

If $\mathbf{p} \in \bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{z})} \sum_{j=1}^{J} q_j\mathbf{R}^j(\mathbf{z})$, then there exist $\mathbf{q} \in \mathbf{R}^*(\mathbf{z})$ and $\mathbf{p}^j \in \mathbf{R}^j(\mathbf{z})$ such that $\mathbf{p} = \sum_{j=1}^{J} q_j\mathbf{p}^j$, implying $\Pi'(\mathbf{z},\mathbf{p}) \leqq \Pi^*(\mathbf{z},q_1\Pi^1(\mathbf{z},\mathbf{p}^1),\ldots,q_J\Pi^J(\mathbf{z},\mathbf{p}^J)) \leqq \Pi^*(\mathbf{z},\mathbf{q}) \leqq 1$, and hence $\mathbf{p} \in \mathbf{R}'(\mathbf{z}) = \{\mathbf{p} \in \mathbf{R}^N | \Pi'(\mathbf{z},\mathbf{p}) \leqq 1\}$. Alternately, suppose $\mathbf{p} \in \mathbf{R}'(\mathbf{z})$. Then for any $\epsilon > 0$, there exist $\mathbf{p}^j \in \mathbf{P}^j(\mathbf{z})$ such that $\Pi^*(\mathbf{z},\Pi^1(\mathbf{z},\mathbf{p}^1),\ldots,\Pi^J(\mathbf{z},\mathbf{p}^J)) \leqq 1+\epsilon$ and $\mathbf{p} = \sum_{j=1}^{J} \mathbf{p}^j$. Let $q_j = \Pi^j(\mathbf{z},\mathbf{p}^j)$. Then $\Pi^*(\mathbf{z},\mathbf{q}) \leqq 1+\epsilon$ and $\Pi^j(\mathbf{z},\mathbf{p}^j) \leqq q_j$, implying $\mathbf{p}^j \in q_j\mathbf{R}^j(\mathbf{z})$. Hence, $\mathbf{p} \in \sum_{j=1}^{J} (q_j\mathbf{R}^j(\mathbf{z}))$, and therefore $\mathbf{p} \in \bigcup_{\mathbf{q}\in(1+\epsilon)\mathbf{R}^*(\mathbf{z})} \sum_{j=1}^{J} (q_j\mathbf{R}^j(\mathbf{z}))$. Letting $\epsilon \to 0$ establishes

$$\mathbf{R}'(\mathbf{z}) \subseteq \text{Closure} \bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{z})} \sum_{j=1}^{J} (q_j\mathbf{R}^j(\mathbf{z})).$$

With the preceding inclusion, this proves (v). Q.E.D.

Using the formal duality of the gauge and profit functions, one obtains the following corollary to Lemma 27 and Theorem 28.

> *Corollary* 28a. Suppose the production possibility sets $\mathbf{T}^j(\mathbf{z})$, price possibility sets $\mathbf{R}^j(\mathbf{z})$, gauge functions $H^j(\mathbf{z},\mathbf{x})$, and profit functions $\Pi^j(\mathbf{z},\mathbf{p})$ satisfy the assumptions preceding Lemma 27 for $j = 1,\ldots,J$. Suppose the production possibility set $\mathbf{W}(\mathbf{z})$, price possibility set $\mathbf{R}^*(\mathbf{z})$, gauge function $H^*(\mathbf{z},\mathbf{w})$, and profit function $\Pi^*(\mathbf{z},\mathbf{q})$ satisfy the assumptions preceding Theorem 28. Then composition Rules 1–3 in Table 5 hold. Under the additional assumption that the interior of $\bigcap_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$ is non-empty, Rules 4–6 in this table hold. (In Rules 2 and 4, $\mathbf{\Delta}$ is the unit simplex.)

Proof: Rules 1 and 3 are restatements of Lemma 27 and Theorem 28, while Rule 2 is a special case of Rule 3 when $\mathbf{W}^+(\mathbf{z}) = \mathbf{\Delta}$. Rule 4 is a formal dual of Rule 2, Rule 5 is a formal dual of Rule 1, and Rule 6 is a formal dual of Rule 3, with the expression for $\mathbf{P}'(\mathbf{z})$ following in these rules from application of Rockafellar (1970, Corollary 8.3.3 and Corollary 16.4.2) to the dual asymptotic and normal cones of the $\mathbf{T}^j(\mathbf{z})$. The assumption that $\mathbf{P}'(\mathbf{z})$ has a non-empty interior in Rules 4, 5, 6 implies that the technologies $\mathbf{T}'(\mathbf{z})$ are semi-bounded. Hence, they satisfy Axiom 1, and the set $\mathbf{T}'(\mathbf{z})$ defined in Rules 4 and 5 is closed. Q.E.D.

An example illustrates the use of these composition rules to generate new functional forms. Let $\mathbf{A} = (a_{ij})$ denote a symmetric, positive definite,

TABLE 5
Composition rules for gauge and profit functions.

	Rule 1	Rule 2	Rule 3
$\mathbf{T}'(\mathbf{z})$	$\bigcap_{j=1}^{J} \mathbf{T}^j(\mathbf{z})$	Closure $\bigcup_{\mathbf{w}\in\Delta} \bigcap_{j=1}^{J} \mathbf{T}^j(\mathbf{z})w_j$	Closure $\bigcup_{\mathbf{w}\in\mathbf{W}^*(\mathbf{z})} \bigcap_{j=1}^{J} \mathbf{T}^j(\mathbf{z})w_j$
$H'(\mathbf{z},\mathbf{x})$	$\mathrm{Max}_j\, H^j(\mathbf{z},\mathbf{x})$	$\sum_{j=1}^{J} H^j(\mathbf{z},\mathbf{x})$	$H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}),\ldots,H^J(\mathbf{z},\mathbf{x}))$
The interior of $\mathbf{P}'(\mathbf{z})$ equals the interior of the following set	$\sum_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$	$\sum_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$	$\sum_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$
$\Pi'(\mathbf{z},\mathbf{p})$	$\inf\left\{\sum_{j=1}^{J} \Pi^j(\mathbf{z},\mathbf{p}^j) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{p}^j = \mathbf{p}\right\}$	$\inf\left\{\mathrm{Max}_j\, \Pi^j(\mathbf{z},\mathbf{p}^j) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{p}^j = \mathbf{p}\right\}$	$\inf\left\{\Pi^*(\mathbf{z},\Pi^1(\mathbf{z},\mathbf{p}^1),\ldots,\Pi^J(\mathbf{z},\mathbf{p}^J)) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{p}^j = \mathbf{p}\right\}$
$\mathbf{R}'(\mathbf{z})$	Convex Hull of $\bigcup_{j=1}^{J} \mathbf{R}^j(\mathbf{z})$	$\sum_{j=1}^{J} \mathbf{R}^j(\mathbf{z})$	Closure $\bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{z})} \sum_{j=1}^{J} (q_j\mathbf{R}^j(\mathbf{z}))$

	Rule 4	Rule 5	Rule 6
$\mathbf{T}'(\mathbf{z})$	$\sum_{j=1}^{J} \mathbf{T}^j(\mathbf{z})$	Convex Hull of $\bigcup_{j=1}^{J} \mathbf{T}^j(\mathbf{z})$	Closure $\bigcup_{\mathbf{w}\in\mathbf{W}^*(\mathbf{z})} \sum_{j=1}^{J} (w_j\mathbf{T}^j(\mathbf{z}))$
$H'(\mathbf{z},\mathbf{x})$	$\inf\left\{\mathrm{Max}_j\, H^j(\mathbf{z},\mathbf{x}^j) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{x}^j = \mathbf{x}\right\}$	$\inf\left\{\sum_{j=1}^{J} H^j(\mathbf{z},\mathbf{x}^j) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{x}^j = \mathbf{x}\right\}$	$\inf\left\{H^*(\mathbf{z},H^1(\mathbf{z},\mathbf{x}^1),\ldots,H^J(\mathbf{z},\mathbf{x}^J)) \,\middle\vert\, \sum_{j=1}^{J} \mathbf{x}^j = \mathbf{x}\right\}$
The interior of $\mathbf{P}'(\mathbf{z})$ equals the interior of the following set	$\bigcap_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$	$\bigcap_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$	$\bigcap_{j=1}^{J} \mathbf{P}^j(\mathbf{z})$
$\Pi'(\mathbf{z},\mathbf{p})$	$\sum_{j=1}^{J} \Pi^j(\mathbf{z},\mathbf{p})$	$\mathrm{Max}_j\, \Pi^j(\mathbf{z},\mathbf{p})$	$\Pi^*(\mathbf{z},\Pi^1(\mathbf{z},\mathbf{p}),\ldots,\Pi^j(\mathbf{z},\mathbf{p}))$
$\mathbf{R}'(\mathbf{z})$	Closure $\bigcup_{\mathbf{q}\in\Delta} \bigcap_{j=1}^{J} (q_i\mathbf{R}^j(\mathbf{z}))$	$\bigcap_{j=1}^{J} \mathbf{R}^j(\mathbf{z})$	Closure $\bigcup_{\mathbf{q}\in\mathbf{R}^*(\mathbf{z})} \bigcap_{j=1}^{J} \mathbf{R}^j(\mathbf{z})q_i$

non-negative matrix of order J. Then,

$$\begin{aligned} \Pi^*(\mathbf{q}) &= (\mathbf{q}'\mathbf{A}\mathbf{q})^{1/2} \quad \text{if} \quad \mathbf{q} \geqq \mathbf{0}, \\ &= +\infty \qquad \text{otherwise}, \end{aligned} \tag{56}$$

is a profit function which is non-decreasing in $\mathbf{q} \geqq \mathbf{0}$. The dual gauge function is

$$H^*(\mathbf{w}) = \inf\{(\hat{\mathbf{w}}'A^{-1}\hat{\mathbf{w}})^{1/2} | \hat{\mathbf{w}} \geqq \mathbf{w}\}. \tag{57}$$

[The duality of the functions $(\mathbf{q}'\mathbf{A}\mathbf{q})^{1/2}$ and $(\mathbf{w}'\mathbf{A}^{-1}\mathbf{w})^{1/2}$ is established from equation (46) using Schwartz's inequality; see Rockafellar (1970, p. 136). The modification in equations (56) and (57) can be obtained from composition Rule 5.] For a sequence of non-negative profit functions $\Pi^i(\mathbf{z},\mathbf{p})$ and dual gauge functions $H^i(\mathbf{z},\mathbf{w})$, the composites

$$\Pi^0(\mathbf{z},\mathbf{p}) = \left[\sum_{i,j=1}^{J} a_{ij}\Pi^i(\mathbf{z},\mathbf{p})\Pi^j(\mathbf{z},\mathbf{p})\right]^{1/2}, \tag{58}$$

and

$$H^0(\mathbf{z},\mathbf{x}) = \left[\sum_{i,j=1}^{J} b_{ij}H^i(\mathbf{z},\mathbf{x})H^j(\mathbf{z},\mathbf{x})\right]^{1/2}, \tag{59}$$

where the b_{ij} are elements of $\mathbf{A}^{-1}$, define new profit and gauge functions. Taking the coefficients a_{ij} to be non-negative parameters and the Π^j to be concrete functions (such as Cobb–Douglas profit functions in all prices or subsets of prices), one obtains a broad parametric class of profit functions with netput supply functions

$$\mathbf{x}_k = \frac{1}{\Pi^0} \sum_{i,j=1}^{J} a_{ij}\Pi^i \frac{\partial \Pi^j}{\partial p^k},$$

which are linear in the coefficients a_{ij}. Such linear-in-parameters forms have obvious econometric applications; this topic is discussed further in Chapter II.2.

Composition rules can also be used to deduce results on the structure of technology and separability properties of netputs. In Chapter I.3, Lau has several applications, including conditions for non-joint production (Theorem III.6) and for separability of inputs (Theorem II.6). In Chapter V.1, Denny establishes a condition for separability of inputs and outputs.

20. Profit Saddle-Functions

Suppose a production possibility set **Y** contains production plans (**z**,**x**), where $\mathbf{z} = (z_1,\ldots,z_M)$ and $\mathbf{x} = (x_1,\ldots,x_N)$ are netput vectors which are distinguished from each other in an application. For example, **z** may denote netputs which are fixed in the short run and variable in the long run, while **x** denotes short run variable netputs. Alternately, **z** may represent netputs of primary goods (capital and labor), with **x** representing all other goods; or **z** may represent outputs and **x** inputs (expressed as netputs). It is possible to define gauge and profit functions with

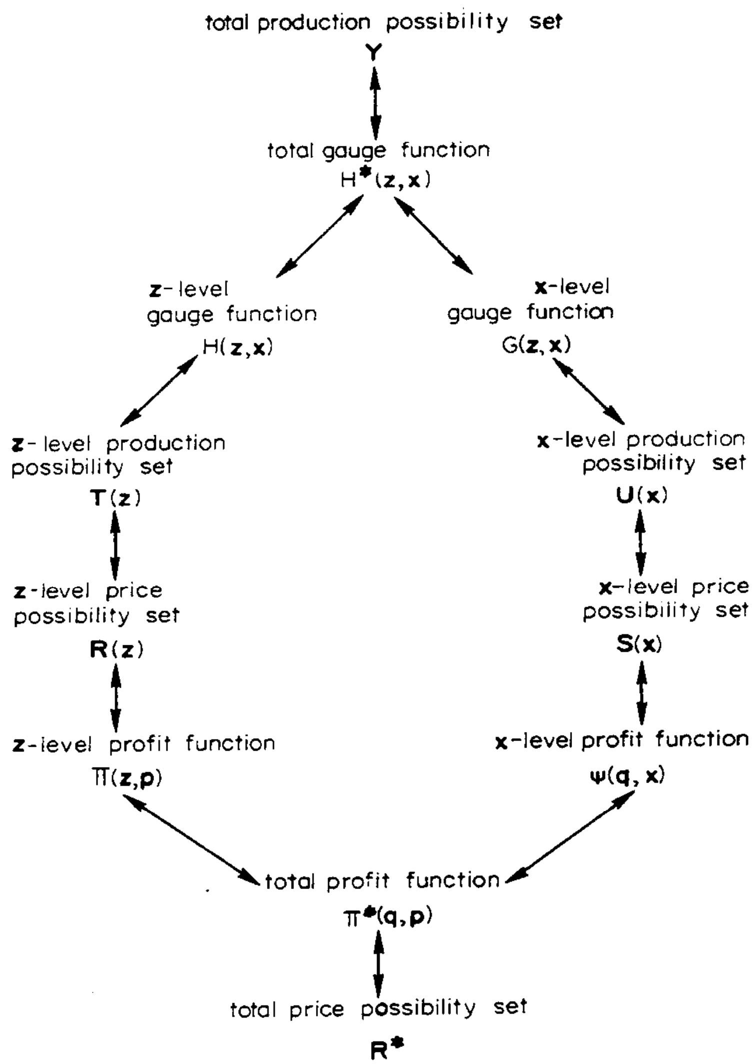


FIGURE 24

respect to either or both the netput vectors $\mathbf{z}$ and $\mathbf{x}$. These functions and the mappings between them provide a basis for the analysis of problems such as the relation of short and long run profit maximizing behavior, or the relation of value added to revenue and profit.

Figure 24 outlines the various possibility sets and gauge and profit functions we shall consider. These sets and functions will be assumed to lie in classes defined by the properties in Table 6. We shall establish that for members of these classes, the mappings in Table 7 hold.

Theorem 29. The mappings of Table 7 hold for members of the classes of functions and sets defined in Table 6, and for these classes are one to one onto. That is, for each member of a class in Table 6, the images of the mappings in Table 7 have the associated properties in Table 6; each member of a class in Table 6 is the image of a unique member of each of the remaining classes in Table 6 under the mappings of Table 7; and the mappings are invertible in the sense that any composition of Table 7 mappings which leads from a class in Table 6 back to the same class reduces to the identity mapping.

TABLE 6
Properties of possibility sets and profit functions.

1. Domains and centering functions

$\mathbf{Z}$ = a non-empty convex subset of $\mathbf{E}^M$	(60)
$\mathbf{X}$ = a non-empty convex subset of $\mathbf{E}^N$	(61)
$(\bar{\mathbf{z}},\bar{\mathbf{x}})$ = a point in $\mathbf{Z}\times\mathbf{X}$	(62)
$\mathbf{x}^*:\mathbf{Z}\to\mathbf{X}$ = a continuous function	(63)
$\mathbf{z}^*:\mathbf{X}\to\mathbf{Z}$ = a continuous function	(64)

2. The production possibility set $\mathbf{Y}\subseteq\mathbf{E}^M\times\mathbf{E}^N$

(i) $\mathbf{Y}$ is non-empty, convex, closed, and semi-bounded
(ii) $\mathbf{Z}=\{\mathbf{z}\in\mathbf{E}^M \mid (\mathbf{z},\mathbf{x})\in\mathbf{Y}$ for some $\mathbf{x}\in\mathbf{E}^N\}$
$\mathbf{X}=\{\mathbf{x}\in\mathbf{E}^N \mid (\mathbf{z},\mathbf{x})\in\mathbf{Y}$ for some $\mathbf{z}\in\mathbf{E}^M\}$; $(\bar{\mathbf{z}},\bar{\mathbf{x}})\in\mathbf{Y}$; $(\mathbf{z},\mathbf{x}^*(\mathbf{z}))\in\mathbf{Y}$ for $\mathbf{z}\in\mathbf{Z}$;
$(\mathbf{z}^*(\mathbf{x}),\mathbf{x})\in\mathbf{Y}$ for $\mathbf{x}\in\mathbf{X}$

3. The total gauge function $H^*:\mathbf{E}^M\times\mathbf{E}^N\to\bar{\mathbf{E}}$ [see equations (72) and (65)][a]

(i) H^* is non-negative, convex, and closed
(ii) H^* is finite on a non-empty convex cone with vertex at $(\bar{\mathbf{z}},\bar{\mathbf{x}})$
(iii) $\{(\mathbf{z},\mathbf{x})\in\mathbf{E}^M\times\mathbf{E}^N \mid H^*(\mathbf{z},\mathbf{x})=0\}$ is semi-bounded
(iv) For $\lambda\geqq 0$, $H^*(\bar{\mathbf{z}}+\lambda(\mathbf{z}-\bar{\mathbf{z}}),\ \bar{\mathbf{x}}+\lambda(\mathbf{x}-\bar{\mathbf{x}}))=\lambda H^*(\mathbf{z},\mathbf{x})$

TABLE 6 (*continued*)

4. The z-level gauge function $H:\mathbf{Z} \times \mathbf{E}^N \to \bar{\mathbf{E}}$ [see equations (80) and (66)][a,b]
   (i) For each $\mathbf{z} \in \mathbf{Z}$, $H(\mathbf{z},\mathbf{x})$ is non-negative, convex, and closed in $\mathbf{x}$
   (ii) For $\theta \geqq 0$, $H(\mathbf{z},\mathbf{x}^*(\mathbf{z}) + \theta(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))) = \theta H(\mathbf{z},\mathbf{x})$
   (iii) $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | H(\mathbf{z},\mathbf{x}) \leqq 1\}$ is semi-bounded
   (iv) $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | H(\mathbf{z},\mathbf{x}) \leqq 1\}$ is convex and closed[f]
   (v) $H(\bar{\mathbf{z}},\bar{\mathbf{x}}) \leqq 1$

5. The z-level production possibility set $\mathbf{T}(\mathbf{z}) \subseteq \mathbf{E}^N$ [see equations (92) and (67)][c]
   (i) $\mathbf{T}(\mathbf{z})$ is non-empty, convex, closed, and semi-bounded for each $\mathbf{z} \in \mathbf{Z}$
   (ii) If $\mathbf{z}^k \to \mathbf{z}^0$, $\mathbf{x}^k \in \mathbf{T}(\mathbf{z}^k)$, $\mathbf{x}^k \to \mathbf{x}^0$, then $\mathbf{z}^0 \in \mathbf{Z}$ and $\mathbf{x}^0 \in \mathbf{T}(\mathbf{z}^0)$
   (iii) $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | \mathbf{x} \in \mathbf{T}(\mathbf{z})\}$ is convex and semi-bounded
   (iv) $\bar{\mathbf{x}} \in \mathbf{T}(\bar{\mathbf{z}})$ and $\mathbf{x}^*(\mathbf{z}) \in \mathbf{T}(\mathbf{z})$

6. The z-level price possibility set $\mathbf{R}(\mathbf{z}) \subseteq \mathbf{E}^N$ [see equations (103) and (68)][d]
   (i) $\mathbf{R}(\mathbf{z})$ is convex and closed, with a non-empty interior, for each $\mathbf{z} \in \mathbf{Z}$
   (ii) $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{R}(\mathbf{z}))\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\}$ is convex, closed, and semi-bounded
   (iii) For all $\mathbf{p} \in \mathbf{R}(\bar{\mathbf{z}})$, $\mathbf{p}\cdot(\bar{\mathbf{x}} - \mathbf{x}^*(\bar{\mathbf{z}})) \leqq 1$

7. The z-level profit function $\Pi:\mathbf{Z} \times \mathbf{E}^N \to \bar{\mathbf{E}}$ [see equations (119) and (69)][a,e]
   (i) For each $\mathbf{z} \in \mathbf{Z}$, Π is convex, closed, and positively linear homogeneous in $\mathbf{p}$
   (ii) $\mathbf{P}(\mathbf{z}) = \{\mathbf{p} \in \mathbf{E}^N | \Pi(\mathbf{z},\mathbf{p}) < +\infty\}$ is a convex cone with a non-empty interior for each $\mathbf{z} \in \mathbf{Z}$
   (iii) For each $\mathbf{p} \in \mathbf{E}^N$, Π is concave and closed in $\mathbf{z}$
   (iv) $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{P}(\mathbf{z}))\mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p})\}$ is convex, closed, and semi-bounded
   (v) For all $\mathbf{p} \in \mathbf{P}(\bar{\mathbf{z}})$, $\mathbf{p}\cdot\bar{\mathbf{x}} \leqq \Pi(\bar{\mathbf{z}},\mathbf{p})$ and for $\mathbf{p} \in \mathbf{P}(\mathbf{z})$, $\mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) \leqq \Pi(\mathbf{z},\mathbf{p})$.

8. The total profit function $\Pi^*:\mathbf{E}^M \times \mathbf{E}^N \to \bar{\mathbf{E}}$ [see equations (126) and (70)][a]
   (i) Π^* is convex, closed, and positively linear homogeneous
   (ii) $\{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | \Pi^*(\mathbf{q},\mathbf{p}) < +\infty\}$ is a convex cone with a non-empty interior
   (iii) $\Pi^*(\mathbf{q},\mathbf{p}) \geqq \mathbf{q}\cdot\bar{\mathbf{x}} + \mathbf{p}\cdot\bar{\mathbf{x}}$ for all $(\mathbf{q},\mathbf{p})\mathbf{E}^M \times \mathbf{E}^N$

9. The total price possibility set $\mathbf{R}^*$ [see equations (134) and (71)]
   (i) $\mathbf{R}^*$ is convex and closed, with a non-empty interior
   (ii) $(\mathbf{0},\mathbf{0}) \in \mathbf{R}^*$

[a] $\bar{\mathbf{E}} = [-\infty, +\infty]$.
[b] The x-level gauge function $\mathbf{G}:\mathbf{E}^M \times \mathbf{X} \to \bar{\mathbf{E}}$ also has these properties with $\mathbf{z}$ and $\mathbf{x}$ interchanged.
[c] The x-level production possibility set $\mathbf{U}(\mathbf{x}) \subseteq \mathbf{E}^M$, $\mathbf{x} \in \mathbf{X}$, also has these properties with $\mathbf{z}$ and $\mathbf{x}$ interchanged.
[d] The x-level price possibility set $\mathbf{S}(\mathbf{x}) \subseteq \mathbf{E}^M$, $\mathbf{x} \in \mathbf{X}$, also has these properties with $\mathbf{z}$ and $\mathbf{x}$ interchanged.
[e] The x-level profit function $\Psi:\mathbf{E}^M \times \mathbf{X} \to \bar{\mathbf{E}}$ also has these properties with $\mathbf{z}$ and $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{q}$ interchanged.
[f] When $\mathbf{x}^*(\mathbf{z})$ is constant, say $\mathbf{x}^*(\mathbf{z}) = \bar{\mathbf{x}}$, condition (iv) can be replaced by the requirement that $H(\mathbf{z},\mathbf{x})$ be quasi-convex, with $\{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | H(\mathbf{z},\mathbf{x}) \leqq 1\}$ closed.

TABLE 7
Duality mappings.

1. Mappings yielding the total production possibility set $\mathbf{Y} \subseteq \mathbf{E}^M \times \mathbf{E}^N$

$$\mathbf{Y} = \{(\mathbf{z},\mathbf{x}) \in \mathbf{E}^M \times \mathbf{E}^N | H^*(\mathbf{z},\mathbf{x}) \leqq 1\} \qquad (65)$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | H(\mathbf{z},\mathbf{x}) \leqq 1\} \qquad (66)^{\mathrm{b}}$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | \mathbf{x} \in \mathbf{T}(\mathbf{z})\} \qquad (67)^{\mathrm{b}}$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{R}(\mathbf{z}))\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \qquad (68)^{\mathrm{b}}$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{Z} \times \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{R}^N)\mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p})\} \qquad (69)^{\mathrm{b}}$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{E}^N)(\forall \mathbf{q} \in \mathbf{E}^M)\mathbf{q}\cdot\mathbf{z} + \mathbf{p}\cdot\mathbf{x} \leqq \Pi^*(\mathbf{q},\mathbf{p})\} \qquad (70)$$

$$= \{(\mathbf{z},\mathbf{x}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall (\mathbf{q},\mathbf{p}) \in \mathbf{R}^*)\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq 1\} \qquad (71)$$

2. Mappings yielding the total gauge function $H^*: \mathbf{E}^M \times \mathbf{E}^N \to \bar{\mathbf{E}}$[a]

$$H^*(\mathbf{z},\mathbf{x}) = \inf\left\{\lambda > 0 | (\bar{\mathbf{z}},\bar{\mathbf{x}}) + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}},\mathbf{x} - \bar{\mathbf{x}}) \in \mathbf{Y}\right\} \qquad (72)$$

$$= \inf\left\{\lambda > 0 | \bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}}) \in \mathbf{Z} \ \& \ \mathbf{H}\left(\bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}}), \bar{\mathbf{x}} + \frac{1}{\lambda}(\mathbf{x} - \bar{\mathbf{x}})\right) \leqq 1\right\} \qquad (73)^{\mathrm{b}}$$

$$= \inf\left\{\lambda > 0 | \bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}}) \in \mathbf{Z} \ \& \ \bar{\mathbf{x}} + \frac{1}{\lambda}(\mathbf{x} - \bar{\mathbf{x}}) \in \mathbf{T}\left(\bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}})\right)\right\} \qquad (74)^{\mathrm{b}}$$

$$= \inf\left\{\lambda > 0 | \mathbf{z}' \equiv \bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}}) \in \mathbf{Z} \ \& \ (\forall \mathbf{p} \in \mathbf{R}(\mathbf{z}'))\mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq \lambda[1 + \mathbf{p}\cdot(\mathbf{x}^*(\mathbf{z}') - \bar{\mathbf{x}})]\right\} \qquad (75)^{\mathrm{b}}$$

$$= \inf\left\{\lambda > 0 | (\forall \mathbf{p} \in \mathbf{E}^N)\mathbf{z}' \equiv \bar{\mathbf{z}} + \frac{1}{\lambda}(\mathbf{z} - \bar{\mathbf{z}}) \in \mathbf{Z} \ \& \ \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq \lambda[\Pi(\mathbf{z}',\mathbf{p}) - \mathbf{p}\cdot\bar{\mathbf{x}}]\right\} \qquad (76)^{\mathrm{b}}$$

$$= \inf\{\lambda > 0 | (\forall \mathbf{q} \in \mathbf{E}^M)(\forall \mathbf{p} \in \mathbf{E}^N)\mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) \leqq \lambda[\Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\bar{\mathbf{z}} - \mathbf{p}\cdot\bar{\mathbf{x}}]\} \qquad (77)$$

$$= \sup\{\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) | \Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\bar{\mathbf{z}} - \mathbf{p}\cdot\bar{\mathbf{x}} \leqq 1\} \qquad (78)$$

$$= \sup\{\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) | (\mathbf{q},\mathbf{p}) \in \mathbf{R}^*\} \qquad (79)$$

3. Mappings yielding the z-level gauge function $H: \mathbf{Z} \times \mathbf{E}^N \to \bar{\mathbf{E}}$[a,b]

$$H(\mathbf{z},\mathbf{x}) = \inf\left\{\lambda > 0 | \left(\mathbf{z}, \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))\right) \in \mathbf{Y}\right\} \qquad (80)$$

TABLE 7 (*continued*)

$$= \inf\left\{\lambda > 0 \middle| H^*\left(\mathbf{z},\mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))\right) \leqq 1\right\} \quad (81)$$

$$= \inf\left\{\lambda > 0 \middle| \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \in \mathbf{T}(\mathbf{z})\right\} \quad (82)$$

$$= \sup\{\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) | \mathbf{p} \in \mathbf{R}(\mathbf{z})\} \quad (83)$$

$$= \inf\{\lambda > 0 | (\forall \mathbf{p} \in \mathbf{E}^N)\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda[\Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})]\} \quad (84)$$

$$= \sup\{\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) | \Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) \leqq 1\} \quad (85)$$

$$= \inf\{\lambda > 0 | (\forall \mathbf{p} \in \mathbf{E}^N)(\forall \mathbf{q} \in \mathbf{E}^M)\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda[\Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\mathbf{z} - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})]\} \quad (86)$$

$$= \inf\left\{\lambda > 0 \middle| (\forall(\mathbf{q},\mathbf{p}) \in \mathbf{R}^*)\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x}^*(\mathbf{z}) - \bar{\mathbf{x}}) + \frac{1}{\lambda}\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\right\} \quad (87)$$

$$= \inf\left\{\lambda > 0 \middle| \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \in \mathbf{X} \ \& \ \mathbf{G}\left(\mathbf{z},\mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))\right) \leqq 1\right\} \quad (88)$$

$$= \inf\left\{\lambda > 0 \middle| \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \in \mathbf{X} \ \& \ \mathbf{z} \in \mathbf{U}\left(\mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))\right)\right\} \quad (89)$$

$$= \inf\left\{\lambda > 0 \middle| \mathbf{x}' \equiv \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \in \mathbf{X} \ \& \ (\forall \mathbf{q} \in \mathbf{S}(\mathbf{x}'))\mathbf{q}\cdot(\mathbf{z} - \mathbf{z}^*(\mathbf{x}')) \leqq 1\right\} \quad (90)$$

$$= \inf\left\{\lambda > 0 \middle| \mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \in \mathbf{X} \ \& \ (\forall \mathbf{q} \in \mathbf{R}^M)\mathbf{q}\cdot\mathbf{z} \leqq \Psi\left(\mathbf{q},\mathbf{x}^*(\mathbf{z}) + \frac{1}{\lambda}(\mathbf{x} - \mathbf{x}^*(\mathbf{z}))\right)\right\} \quad (91)$$

4. Mappings yielding the z-level production possibility set $\mathbf{T}(\mathbf{z}) \in \mathbf{E}^{N\,b}$

$$\mathbf{T}(\mathbf{z}) = \{\mathbf{x} \in \mathbf{E}^N | (\mathbf{z},\mathbf{x}) \in \mathbf{Y}\} \quad (92)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | H^*(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (93)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | H(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (94)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{R}(\mathbf{z}))\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (95)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{E}^N)\mathbf{p}\cdot\mathbf{x} \leqq \Pi(\mathbf{z},\mathbf{p})\} \quad (96)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | (\forall \mathbf{p} \in \mathbf{E}^N)(\forall \mathbf{q} \in \mathbf{E}^M)\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{z} \leqq \Pi^*(\mathbf{q},\mathbf{p})\} \quad (97)$$

$$= \{\mathbf{x} \in \mathbf{E}^N | (\forall(\mathbf{q},\mathbf{p}) \in \mathbf{R}^*)\mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) \leqq 1\} \quad (98)$$

TABLE 7 (*continued*)

$$= \{\mathbf{x} \in \mathbf{X} | G(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (99)$$
$$= \{\mathbf{x} \in \mathbf{X} | \mathbf{z} \in \mathbf{U}(\mathbf{x})\} \quad (100)$$
$$= \{\mathbf{x} \in \mathbf{X} | (\forall \mathbf{q} \in \mathbf{S}(\mathbf{x}))\mathbf{q}\cdot(\mathbf{z} - \mathbf{z}^*(\mathbf{x})) \leqq 1\} \quad (101)$$
$$= \{\mathbf{x} \in \mathbf{X} | (\forall \mathbf{q} \in \mathbf{E}^M)\mathbf{q}\cdot\mathbf{z} \leqq \Psi(\mathbf{q},\mathbf{x})\} \quad (102)$$

5. Mappings yielding the z-level price possibility set $\mathbf{R}(\mathbf{z}) \subseteq \mathbf{E}^{N}$ [b]

$$\mathbf{R}(\mathbf{z}) = \{\mathbf{p} \in \mathbf{E}^N | (\forall (\mathbf{z},\mathbf{x}) \in \mathbf{Y})\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (103)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{E}^N) H^*(\mathbf{z},\mathbf{x}) \leqq 1 \Rightarrow \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (104)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{E}^N)\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq H(\mathbf{z},\mathbf{x})\} \quad (105)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{T}(\mathbf{z}))\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (106)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | \Pi(\mathbf{z},\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) \leqq 1\} \quad (107)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | \inf_{\mathbf{q}\in\mathbf{E}^M} [\Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\mathbf{z} - \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z})] \leqq 1\} \quad (108)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | \inf_{\mathbf{q}\in\mathbf{E}^M} \inf_{\lambda>0 \ni (\mathbf{q},\mathbf{p})/\lambda \in \mathbf{R}^*} [\lambda - \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) - \mathbf{p}\cdot(\mathbf{x}^*(\mathbf{z}) - \bar{\mathbf{x}})] \leqq 1\} \quad (109)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{X}) G(\mathbf{z},\mathbf{x}) \leqq 1 \Rightarrow \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (110)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{X})\mathbf{z} \in \mathbf{U}(\mathbf{x}) \Rightarrow \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (111)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | (\forall \mathbf{x} \in \mathbf{X})[(\forall \mathbf{q} \in \mathbf{S}(\mathbf{x}))\mathbf{q}\cdot(\mathbf{z} - \mathbf{z}^*(\mathbf{x})) \leqq 1] \Rightarrow \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq 1\} \quad (112)$$
$$= \{\mathbf{p} \in \mathbf{E}^N | \inf_{\mathbf{q}\in\mathbf{E}^M} \sup_{\mathbf{x}\in\mathbf{X}} [\Psi(\mathbf{q},\mathbf{x}) + \mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) - \mathbf{q}\cdot\mathbf{z}] \leqq 1\} \quad (113)$$

6. Mappings yielding the z-level profit function $\Pi: \mathbf{Z} \times \mathbf{E}^N \to \bar{\mathbf{E}}$ [a,b]

$$\Pi(\mathbf{z},\mathbf{p}) = \sup\{\mathbf{p}\cdot\mathbf{x} | (\mathbf{z},\mathbf{x}) \in \mathbf{Y}\} \quad (114)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x} | H^*(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (115)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x}\ H(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (116)$$
$$= \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \inf\{\lambda > 0 | (\forall \mathbf{x} \in \mathbf{E}^N)\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}^*(\mathbf{z})) \leqq \lambda H(\mathbf{z},\mathbf{x})\} \quad (117)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x} | H(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (118)$$
$$= \mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \inf\left\{\lambda > 0 | \frac{1}{\lambda}\mathbf{p} \in \mathbf{R}(\mathbf{z})\right\} \quad (119)$$
$$= \inf\{\Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\mathbf{z} | \mathbf{q} \in \mathbf{E}^M\} \quad (120)$$
$$= \inf_{\mathbf{q}\in\mathbf{E}^M}\left[\mathbf{p}\cdot\bar{\mathbf{x}} - \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \inf\left\{\lambda > 0 | \frac{1}{\lambda}(\mathbf{q},\mathbf{p}) \in \mathbf{R}^*\right\}\right] \quad (121)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x} | \mathbf{x} \in \mathbf{X}\ \&\ G(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (122)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x} | \mathbf{x} \in \mathbf{X}\ \&\ \mathbf{z} \in \mathbf{U}(\mathbf{x})\} \quad (123)$$
$$= \sup\{\mathbf{p}\cdot\mathbf{x} | \mathbf{x} \in \mathbf{X}\ \&\ (\forall \mathbf{q} \in \mathbf{S}(\mathbf{x}))\mathbf{q}\cdot(\mathbf{z} - \mathbf{z}^*(\mathbf{x})) \leqq 1\} \quad (124)$$
$$= \inf_{\mathbf{q}\in\mathbf{E}^M} \sup_{\mathbf{x}\in\mathbf{X}} [\Psi(\mathbf{q},\mathbf{x}) + \mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{z}] \quad (125)$$

TABLE 7 (*continued*)

7. Mappings yielding the total profit function $\Pi^*: \mathbf{E}^M \times \mathbf{E}^N \to \bar{\mathbf{E}}^a$

$$\Pi^*(\mathbf{q},\mathbf{p}) = \sup\{\mathbf{q}\cdot\mathbf{z} + \mathbf{p}\cdot\mathbf{x} | (\mathbf{z},\mathbf{x}) \in \mathbf{Y}\} \quad (126)$$
$$= \sup\{\mathbf{q}\cdot\mathbf{z} + \mathbf{p}\cdot\mathbf{x} | H^*(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (127)$$
$$= \mathbf{q}\cdot\bar{\mathbf{z}} + \mathbf{p}\cdot\bar{\mathbf{x}} + \inf\{\lambda > 0 | (\forall \mathbf{z} \in \mathbf{E}^M)(\forall \mathbf{x} \in \mathbf{E}^N)\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq \lambda H^*(\mathbf{z},\mathbf{x})\} \quad (128)$$
$$= \sup\{\mathbf{q}\cdot\mathbf{z} + \mathbf{p}\cdot\mathbf{x} | \mathbf{z} \in \mathbf{Z} \;\&\; H(\mathbf{z},\mathbf{x}) \leqq 1\} \quad (129)^b$$
$$= \sup\{\mathbf{q}\cdot\mathbf{z} + \mathbf{p}\cdot\mathbf{x} | \mathbf{z} \in \mathbf{Z} \;\&\; \mathbf{x} \in \mathbf{T}(\mathbf{z})\} \quad (130)^b$$
$$= \sup_{\mathbf{z}\in\mathbf{Z}}\left[\mathbf{p}\cdot\mathbf{x}^*(\mathbf{z}) + \mathbf{q}\cdot\mathbf{z} + \inf\left\{\lambda > 0 \Big| \frac{1}{\lambda}\mathbf{p} \in \mathbf{R}(\mathbf{z})\right\}\right] \quad (131)^b$$
$$= \sup_{\mathbf{z}\in\mathbf{Z}}[\Pi(\mathbf{z},\mathbf{p}) + \mathbf{q}\cdot\mathbf{z}] \quad (132)^b$$
$$= \mathbf{q}\cdot\bar{\mathbf{z}} + \mathbf{p}\cdot\bar{\mathbf{x}} + \inf\left\{\lambda > 0 \Big| \frac{1}{\lambda}(\mathbf{q},\mathbf{p}) \in \mathbf{R}^*\right\} \quad (133)$$

8. Mappings yielding the total price possibility set $\mathbf{R}^* \subseteq \mathbf{E}^M \times \mathbf{E}^N$

$$\mathbf{R}^* = \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall (\mathbf{z},\mathbf{x}) \in \mathbf{Y})\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq 1\} \quad (134)$$
$$= \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall \mathbf{z} \in \mathbf{R}^M)(\forall \mathbf{x} \in \mathbf{R}^N)\mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq H^*(\mathbf{z},\mathbf{x})\} \quad (135)$$
$$= \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | \mathbf{z} \in \mathbf{Z} \;\&\; H(\mathbf{z},\mathbf{x}) \leqq 1 \Rightarrow \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq 1\} \quad (136)^b$$
$$= \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | \mathbf{z} \in \mathbf{Z} \;\&\; \mathbf{x} \in \mathbf{T}(\mathbf{z}) \Rightarrow \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \mathbf{p}\cdot(\mathbf{x} - \bar{\mathbf{x}}) \leqq 1\} \quad (137)^b$$
$$= \left\{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall \mathbf{z} \in \mathbf{Z})\mathbf{p}\cdot(\mathbf{x}^*(\mathbf{z}) - \bar{\mathbf{x}}) + \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) + \inf\left\{\lambda > 0 \Big| \frac{1}{\lambda}\mathbf{p} \in \mathbf{R}(\mathbf{z})\right\} \leqq 1\right\} \quad (138)^b$$
$$= \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | (\forall \mathbf{z} \in \mathbf{Z})\Pi(\mathbf{z},\mathbf{p}) + \mathbf{q}\cdot(\mathbf{z} - \bar{\mathbf{z}}) - \mathbf{p}\cdot\bar{\mathbf{x}} \leqq 1\} \quad (139)^b$$
$$= \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | (\Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\bar{\mathbf{z}} - \mathbf{p}\cdot\bar{\mathbf{x}} \leqq 1\} \quad (140)$$

[a] $\bar{\mathbf{E}} = [-\infty, +\infty]$.

[b] A second class of mappings is obtained for x-level functions by making the substitutions

$\mathbf{z} \leftrightarrow \mathbf{x}$	$\mathbf{p} \leftrightarrow \mathbf{q}$
$H \leftrightarrow G$	$T \leftrightarrow U$
$\Pi \leftrightarrow \Psi$	$R \leftrightarrow S$

Proof: Many of the conclusions in this theorem summarize results established earlier in the chapter, or can be deduced as simple corollaries. We provide a broad outline of the argument, leaving details to the interested reader.

Consider first the class of production possibility sets Y satisfying the conditions of Table 6, and define the total gauge function H^*, the total profit function Π^*, and the total price possibility set R^* by equations (72), (126), and (134), respectively. The remaining mappings between these functions are given in equations (65), (70), (71), (77), (79), (128), (133), (135), and (140). The properties of these mappings follow from Theorem 24 and its corollary, using the following substitution of notation:

Theorem 24 ⟶	*Theorem 29*
$\mathbf{E}^N$	$\mathbf{E}^M \times \mathbf{E}^N$
$\mathbf{Z}$	none
$\mathbf{x}$	$(\mathbf{z},\mathbf{x})$
$\mathbf{T}(\mathbf{z})$	$\mathbf{Y}$
H	H^*
$\mathbf{p}$	$(\mathbf{q},\mathbf{p})$
Π	Π^*
$\mathbf{R}$	$\mathbf{R}^*$

The price possibility set is not treated explicitly in Theorem 24; however, its properties are an immediate corollary of the definition $\mathbf{R}^* = \{(\mathbf{q},\mathbf{p}) \in \mathbf{E}^M \times \mathbf{E}^N | \Pi^*(\mathbf{q},\mathbf{p}) - \mathbf{q}\cdot\bar{\mathbf{z}} - \mathbf{p}\cdot\bar{\mathbf{x}} \leqq 1\}$ and the properties of Π^*.

Consider next the z-level production possibility set $\mathbf{T}(\mathbf{z})$, and the z-level gauge function $H(\mathbf{z},\mathbf{x})$, profit function $\Pi(\mathbf{z},\mathbf{p})$, and price possibility set $\mathbf{R}'(\mathbf{z})$ defined by equations (82), (118), and (106), respectively. The remaining mappings between the functions are given in equations (83), (84), (94), (95), (96), (105), (107), (117), and (119). The properties of these mappings are a direct restatement of Theorem 24 and its corollary.

The properties and relations of the x-level gauge function $G(\mathbf{z},\mathbf{x})$, production possibility set $\mathbf{U}(\mathbf{x})$, price possibility set $\mathbf{S}(\mathbf{x})$, and profit function $\Psi(\mathbf{q},\mathbf{x})$ can be deduced from their formal duality to the functions H, $\mathbf{T}$, $\mathbf{R}$, and Π:

Primal ⟷	*Dual*
$\mathbf{x}$	$\mathbf{z}$
$\mathbf{p}$	$\mathbf{q}$
$\mathbf{T}(\mathbf{z})$	$\mathbf{U}(\mathbf{x})$
$H(\mathbf{z},\mathbf{x})$	$G(\mathbf{z},\mathbf{x})$
$\Pi(\mathbf{z},\mathbf{p})$	$\Psi(\mathbf{q},\mathbf{x})$
$\mathbf{R}(\mathbf{z})$	$\mathbf{S}(\mathbf{x})$

The mapping in equation (92) from **Y** to **T(z)** and conversely in (67) from **T(z)** to **Y** can immediately be seen to be mutual inverses, and to be one-to-one onto for the classes of **Y** and **T(z)** defined in Table 6. A dual relation holds between the classes of **Y** and **U(x)**.

The conclusions above provide a chain of mappings (via **Y**) between any two classes in Table 6 which have the properties claimed in the theorem. The remaining mappings in Table 7 are compositions of these chains.

For example, property (iii) of the **z**-level profit function, concavity and closure in **z**, can be deduced from the properties of the total profit function Π^* and the concave conjugate dual mapping (120); see Appendix A.3, Theorem 12.3. Verification of the formulae for these composite mappings is tedious, but straightforward, and is left to the reader. Q.E.D.

Several of the mappings in Table 7 deserve note. Equation (125) establishes that the **z**-level and **x**-level profit functions are conjugate saddle functions [see Rockafellar (1970, section 37)]. Equations (120) and (132) give the relation between short and long run profit functions. Equations (83) and (85) give simple dual mappings between the **z**-level gauge function and the profit function and price possibility set.

This chapter has set out the basic theory of duality in production economics, and developed the mathematical properties of dual functions. The remaining chapters of this book demonstrate the use of these methods in theoretical and empirical analysis.